ISRAEL JOURNAL OF MATHEMATICS, Vol. 49, Nos. 1–3, 1984

THE MONADIC THEORY AND THE "NEXT WORLD"

BY

YURI GUREVICH AND SAHARON SHELAH[†]

Dedicated to the memory of Abraham Robinson on the tenth anniversary of his death

ABSTRACT

Suppose that V is a model of ZFC and $U \in V$ is a topological space or a richer structure for which it makes sense to speak about the monadic theory. Let B be the Boolean algebra of regular open subsets of U. If the monadic theory of U allows one to speak in some sense about a family of κ everywhere dense and almost disjoint sets, then the second-order V^B-theory of κ is interpretable in the monadic V-theory of U; this is our Interpretation Theorem. Applying the Interpretation Theorem we strengthen some previous results on complexity of the monadic theories of the real line and some other topological spaces and linear orders. Here are our results about the real line. Let r be a Cohen real over V. The second-order $V[r]$-theory of $\aleph_0$ is interpretable in the monadic V-theory of the real line. If CH holds in V then the second-order $V[r]$-theory of the real line is interpretable in the monadic V-theory of the real line.

TABLE OF CONTENTS

[†] The author thanks the United States–Israel Binational Science Foundation for supporting the research.

Received June 10, 1981

§0. Introduction

Our most important subject is the monadic (second-order) theory of the real line R. Relevant definitions of monadic and full second-order theories can be found in §1 below.

Recall that a topological space U is called *meager* if it is a union of $\aleph_0$ nowhere dense subsets. Let $c = 2^{\aleph_0}$. U will be called *pseudo-meager* if it is a union of $< c$ nowhere dense subsets. Assuming that R is not pseudo-meager Shelah interpreted in [6] the true first-order arithmetic (i.e. the first-order theory of the standard model of arithmetic) in the monadic theory of R. He conjectured in [6] that the monadic theory of R and the second-order theory of c are recursive in each other. (To prove the conjecture, he remarked, use the assumption "R is not pseudo-meager" or the Continuum Hypothesis.)

Under assumption "R is not pseudo-meager" Gurevich [2] interpreted the theory of c with quantification over constructible predicates (i.e. the theory of c in the language of second-order logic when n-place predicate variables range over constructible n-place predicates on c for $n = 1, 2, \cdots$) in the monadic theory of R. It implies in particular that the monadic theory of R and the second-order theory of c are interpretable in each other in the constructible universe.

Gurevich and Shelah [4] interpreted the true arithmetic in the monadic theory of R just in ZFC.

Let V be a model of ZFC and B be an atomless, complete, separable Boolean algebra in V. (There is a unique up to isomorphism atomless, complete, separable Boolean algebra in V, see §2 below.) Let V^B be the corresponding Boolean-valued model of ZFC. We speak about V-theories and V^B-theories meaning theories in V and theories in V^B respectively. Note that B is the Boolean algebra corresponding to the forcing notion for adding a Cohen real, and adding a Cohen real does not change the cardinality of continuum.

THEOREM 1. (i) *Assume that the real line is not pseudo-meager in V. Then the second-order V^B-theory of c is interpretable in the monadic V-theory of the real line.*

(ii) *The second-order V^B-theory of $\aleph_0$ is interpretable in the monadic V-theory of the real line.*

Theorem 1 follows from Theorem 2 in this introduction.

The main result of this article is the Interpretation Theorem proved in §4 below. It interprets the second-order $V^{B'}$-theory of a cardinal κ in the monadic

V-theory of U where U is a topological space or a richer structure for which it makes sense to speak about the monadic theory, and B' is the Boolean algebra of regular open subsets of U, and U, κ satisfy some conditions.

We use the Interpretation Theorem to strengthen interpretability results of [2] and [4], to interprete more in the monadic theories of the same topological spaces and the same linear orders. Linear orders are called *chains* here. In order to deal simultaneously with topological spaces and chains Gurevich [2] defines so-called *vicinity spaces.* In §5 below we repeat the definitions of vicinity spaces, of the vicinity space associated with a given T_1 topological space, of the vicinity space associated with a given chain, of p-modest vicinity spaces where $1 \leqq p < \omega$. A topological space is called *p-modest* if the associated vicinity space is p-modest. It is easy to see that no perfect subset of the real line is 1-modest.

THEOREM 2. (i) *There is an algorithm interpreting the second-order V^B-theory of c in the monadic V-theory of any first-countable, non-scattered, regular topological space having no subspace which is nowhere dense, perfect, separable and pseudo-meager.*

(ii) *There is an algorithm interpreting the second-order V^B-theory of $\aleph_0$ in the monadic theory of any metrizable non-p-modest topological space.*

Theorem 2(i) is proved in §6. It uses [2] and strengthens the result of [2] about interpreting the theory of c with quantification over constructible predicates in monadic theories of essentially the same topological spaces. Theorem 2(ii) is proved in §7. It uses [4] and strengthens the result of [4] about interpreting the true first-order arithmetic in the monadic theories of the same topological spaces.

Recall that a chain is short if it does not embed either ω_1 or the order dual to ω_1. A chain is p-modest if the vicinity spaces associated with its subchains are p-modest. A chain is modest if it is p-modest for every $1 \leqq p < \omega$. The monadic theory of modest short chains is decidable, see [1] or [3]. If every pseudo-meager subspace of R is meager then there is a uniform in p algorithm, interpreting the true arithmetic in the monadic theory of any non-p-modest short chain, see [3]. If every pseudo-meager subspace of R is meager, then there is a uniform in p algorithm interpreting the theory of c with quantification over constructible predicates in the monadic theory of any non-p-modest short chain, see [2]. There is a uniform in p algorithm interpreting the true arithmetic in the monadic theory of any non-p-modest short chain, see [4]. Using [2] and [4] we prove in sections 6 and 7 the following theorem.

THEOREM 3. (i) *Assume that in* V *every pseudo-meager subspace of the real line is meager. There is a uniform in* p *algorithm interpreting the second-order* V^B*-theory of* c *in the monadic* V*-theory of any non-*p*-modest short chain.*

(ii) *There is a uniform in* p *algorithm interpreting the second-order* V^B*-theory of* $\aleph_0$ *in the monadic* V*-theory of any non-*p*-modest short chain.*

A few words about notation and terminology. Given point sets X, Y in a topological space we say that X is dense in Y if $X \cap Y$ is everywhere dense in the subspace Y. Speaking about regular topological spaces we mean regular T_1 spaces. We use [5] as a source of notation, terminology and results in set theory, but the cardinality of continuum is denoted by c. Claim $i.k$ means the kth claim in the ith section when we speak about [2], [4] or this paper.

§1. Second-order theories

The second-order theory of a set S is the theory of S in the language without non-logical constants having variables ranging over elements of S, variables ranging over arbitrary monadic predicates on S, variables ranging over arbitrary dyadic predicates on S, etc. Atomic formulas have a form $P(v_1, \cdots, v_n)$ where P is an n-adic predicate variable and $v_1, \cdots, v_n$ are individual variables. Usual propositional connectives and usual quantifiers $\forall$, $\exists$ are used to build other formulas. Predicate variables can be quantified as well as individual ones.

The second-order theory of an order $(S, <)$ is a natural extension of the second-order theory of S; just consider $<$ as a non-logical constant. The second-order theories of other first-order structures are defined in a similar way. The second-order theory of a topological space (S, Top) is another natural extension of the second-order theory of S; it is obtained by introducing new atomic formulas "X is open". The second-order theory of a set is, however, so powerful that in many cases the second-order theory of an S-based structure is easily interpretable in the second-order theory of S. For example, if κ is a non-zero cardinal then the second-order theory of the natural order $(\kappa, <)$ is interpretable in the second-order theory of the set κ. The second-order theory of the real line $(R, <)$ and the second-order theory of the usual topological space (R, Top) are easily interpretable in the second-order of the set R.

The monadic theory of a structure is the fragment of its second-order theory whose formulas do not use n-place predicate variables for $n \geqq 2$. Instead of $P(v)$ we write $v \in P$. Thus the monadic theory of an order $(S, <)$ is the first-order theory of the two-sorted structure $(S, \mathrm{PS}(S); \in, <)$ where $\mathrm{PS}(S)$ is the collection of subsets (the power set) of S and $\in$ is the containment relation between elements of S and elements of $\mathrm{PS}(S)$. The monadic theory of a topological space (S, Top) is the first-order theory of the two-sorted structure $(S, \mathrm{PS}(S); \in, \mathrm{Top})$ where Top is a monadic predicate on $\mathrm{PS}(S)$. Generally, the first-order theory of a two-sorted first-order structure $(S, \mathrm{PS}(S); \in, \cdots)$, where $\in$ is the containment relation between elements of S from one side and elements of $\mathrm{PS}(S)$ from the other side, will be called the monadic theory of the structure $(S, \cdots)$.

It is an easy exercise to check that the second-order theory of an infinite set S is interpretable in its dyadic fragment (whose formulas do not use n-adic predicate variables for $n \neq 2$). A 3-adic predicate P on a non-empty set S will be called a *pairing predicate* if for evey ordered pair (x, y) of elements of S there is exactly one $z \in S$ with $(x, y, z) \in P$.

CLAIM 1. *If P is a pairing predicate on a non-empty set S then the second-order theory of S is interpretable in the monadic theory of structure (S, P).*

The proof is clear.

§2. Boolean-valued model

As usual, ZFC is the Zermelo–Fraenkel set theory with axiom of choice. Given a model V of ZFC and a complete Boolean algebra B in V one builds a Boolean-valued model V^B of ZFC, see [5].

CLAIM 1. *If $\{a_i : i \in I\}$ is an antichain in B and $\sigma_i \in V^B$ for $i \in I$ then there is $\tau \in V^B$ such that $a_i \leqq \| \sigma_i = \tau \|$ for $i \in I$.*

PROOF. See lemma 18.5 in [5]. □

CLAIM 2. *Let $\psi(v)$ be a formula in the language of ZFC with only one free variable and perhaps some parameters from V^B. There is $\sigma \in V^B$ such that*

$$\| \psi(\sigma) \| = \| \exists v \psi(v) \|$$

PROOF. See lemma 18.6 in [5]. □

CLAIM 3. *Let $\psi(v)$ be as in Claim 2 and $\sigma \in V^B$. Suppose $\| \exists v (v \in \sigma) \| = 1$. Then there is $\tau \in V^B$ such that $\| \tau \in \sigma \| = 1$ and*

$$\| \psi(\tau) \| = \| (\exists v \in \sigma) \psi(v) \| .$$

PROOF. By Claim 2 there are τ_0, τ_1 and $a \in B$ such that $\| \tau_0 \in \sigma \| = 1$ and

$$\| \tau_1 \in \sigma \wedge \psi(\tau_1) \| = \| (\exists v \in \sigma) \psi(v) \| = a.$$

By Claim 1 there is $\tau \in V^B$ such that $a \leqq \|\tau = \tau_1\|$ and $(-a) \leqq \|\tau = \tau_0\|$. Thus $a \leqq \|\tau \in \sigma\|$ and $(-a) \leqq \|\tau \in \sigma\|$ hence $\|\tau \in \sigma\| = 1$. Also

$$a \leqq \|\psi(\tau)\| \leqq \|\tau \in \sigma \wedge \psi(\tau)\| \leqq \|(\exists v \in \sigma)\psi(v)\| \leqq a. \qquad \square$$

Recall that an open subset G of a topological space is called regular if the interior of the closure of G coincides with G. It is easy to check that the interior of the closure of any open set is regular. The following claim is well known and easy to check.

CLAIM 4. *The regular open subsets of any topological space U form a complete Boolean algebra with 0 being the empty set, $1 = U$, $G \leqq H$ iff $G \subseteq H$, $G \cdot H = G \cap H$, $G + H$ being the interior of the closure of $G \cup H$, and $-G$ being the interior of $U - G$.*

The Boolean algebra of regular open subsets of a topological space U is denoted $\mathrm{RO}(U)$.

An infinite Boolean algebra B is called *separable* if there is a countable dense subset of $B^+ = B - \{0\}$. The following claim is well known.

CLAIM 5. *Every two atomless, complete, separable Boolean algebras are isomorphic.*

PROOF. Let T be the set of words in the alphabet $\{0, 1\}$ ordered as follows: $x < y$ iff x is a proper initial segment of y. By lemma 17.2 in [5] there is a unique up to isomorphism complete Boolean algebra embedding T in such a way that T is dense in it.

Given an atomless, complete, separable Boolean algebra B we construct an order-preserving map $f: T \to B^+$ such that fT is dense in B. Let $\{a_n : n < \omega\}$ be a dense subset of B^+. If x is the empty word set $f(x) = 1$. Suppose that $x \in T$ is of length n and $f(x)$ is already defined. Choose $f(x0)$ and $f(x1)$ such that $f(x0) \cdot f(x1) = 0$, $f(x0) + f(x1) = f(x)$ and if $a_n \cdot f(x) \neq 0$ then $f(x0) \leqq a_n$. $\square$

CLAIM 6. *Suppose that U is a first-countable, regular, separable topological space without isolated points. Then $\mathrm{RO}(U)$ is atomless and separable, and the cardinality of U is at most c.*

PROOF. For any $x \in u$ choose regular open sets $G(n, x)$ such that the closure of $G(n+1, x)$ is included into $G(n, x)$ and $\{G(n, x) : n < \omega\}$ is a basis for neighborhoods of x. Let E be a countable everywhere dense subset of U. Then

$$S = \{G(n, x) : n < \omega, x \in E\}$$

is dense in the Boolean algebra of regular open subsets of U.

For $x \in U$ let

$$f_x(n) = \{H \in S : H \subseteq G(n, x)\}.$$

If $x \neq y$ then $G(n, x) \cap G(n, y) = 0$ for some n, hence $f_x \neq f_y$. But there are only c functions from ω to PS(S). □

§3. Setting

Let V be a model of ZFC and $U \in V$ be a topological space or a richer structure for which it makes sense to speak about the monadic theory. We are interested in the monadic theory of U in V. Thus speaking about a point set X in U we mean that $X \in V$.

Let $B = \mathrm{RO}(U)$ and $B^+ = \{a \in B : a \neq 0\}$.

We suppose that there are formulas Premise($\bar{u}$), Share($\bar{u}, v_0$), Pairing($\bar{u}, v_0, v_1, v_2, v_3$) in the monadic language of U and there is an infinite cardinal κ satisfying the following conditions. $\bar{u}$ and (v_0, v_1, v_2, v_3) are sequences of set variables, the formulas Premise, Share, Paring do not have free variables except those shown. Premise ($\bar{u}$) is satisfiable in U. If t is a sequence of point sets and Premise (t) holds then there is a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of point sets and there is a pairing predicate P on κ such that the conditions (C0)–(C3) below hold.

(C0) Every A_α is everywhere dense and $A_\alpha \cap A_\beta$ is nowhere dense for $\alpha \neq \beta$.

(C1) Share(t, X) holds iff for every $a \in B^+$ there are $\alpha < \kappa$ and $a \geqq b \in B^+$ such that $X \cap b = A_\alpha \cap b$.

If Share(t, X) holds we say that X is a *t-share*.

(C2) Suppose X, Y, Z are t-shares and $a \in B$. Pairing (t, X, Y, Z, a) holds iff for every $a \geqq a' \in B^+$ there are $(\alpha, \beta, \gamma) \in P$ and $a' \geqq b \in B^+$ such that $X \cap b = A_\alpha \cap b$ and $Y \cap b = A_\beta \cap b$ and $Z \cap b = A_\gamma \cap b$.

(C3) For every function $f : \kappa \to B$ there is a point set X such that

$$f(\alpha) = \sum \{b \in B : A_\alpha \cap b \subseteq X\} \qquad \text{for } \alpha < \kappa.$$

An element σ of V^B will be called a *quasi-element* (respectively a *quasi-set*) if $\|\sigma \in \kappa\| = 1$ (respectively $\|\sigma \subseteq \kappa\| = 1$). (Here and on we ignore the difference between an element of V and the canonical name for it in V^B.) We'll say that a t-share X *represents* a quasi-element σ if

$$\|\sigma = \alpha\| = \sum \{b \in B : X \cap b = A_\alpha \cap b\} \qquad \text{for } \alpha < \kappa.$$

We'll say that a point set X *represents* a quasi-set σ if

$$\|\alpha \in \sigma\| = \sum \{b \in B : A_\alpha \cap b \subseteq X\} \qquad \text{for } \alpha < \kappa.$$

CLAIM 1. *Every t-share represents some quasi-element and every quasi-element is represented by some t-share.*

PROOF. Given a t-share X let

$$a_\alpha = \sum \{b \in B : X \cap b = A_\alpha \cap b\} \qquad \text{for } \alpha < \kappa.$$

By Claim 2.1 there is $\sigma \in V^B$ such that $a_\alpha \leqq \|\sigma = \alpha\|$ for $\alpha < \kappa$. Evidently σ is a quasi-element represented by X.

Given a quasi-element σ form

$$X = \bigcup \{A_\alpha \cap \|\alpha = \sigma\| : \alpha < \kappa\}.$$

Evidently X is a t-share representing σ. □

CLAIM 2. *Suppose that t-shares X_0, X_1, X_2 represent quasi-elements $\sigma_0, \sigma_1, \sigma_2$, and $a \in B$. Then* Pairing(t, X_0, X_1, X_2, a) *is equivalent to*

$$a \leqq \|(\sigma_0, \sigma_1, \sigma_2) \in P\|.$$

PROOF is clear.

CLAIM 3. *Every point set represents some quasi-set, and every quasi-set is represented by some point set.*

PROOF. The first statement is obvious, the second follows from (C3). □

CLAIM 4. *Suppose a t-share X represents a quasi-element σ, and a point set Y represents a quasi-set τ. Then*

$$\|\sigma \in \tau\| = \sum \{a \in B : X \cap a \subseteq Y\}.$$

PROOF. Let $a^* = \Sigma\{a \in B : X \cap a \subseteq Y\}$ and $a \in B^+$. First suppose that $a \leqq a^*$. For every $a \geqq a' \in B^+$ there are $\alpha < \kappa$ and $a' \geqq b \in B^+$ such that $X \cap b = A_\alpha \cap b \subseteq Y$, so that $b \leqq \|\sigma = \alpha\|$, $b \leqq \|\alpha \in \tau\|$, $b \leqq \|\sigma \in \tau\|$. Hence $a \leqq \|\sigma \in \tau\|$.

Now suppose that $a \leqq \|\sigma \in \tau\|$. For every $a \geqq a' \in B^+$ there are $\alpha < \kappa$ and $a' \geqq b \in B^+$ such that $X \cap b = A_\alpha \cap b$, so that $b \leqq \|\sigma = \alpha\|$, $b \leqq \|\alpha \in \tau\|$, $b \leqq a^*$. Hence $a \leqq a^*$. □

§4. Interpretation Theorem

We work in the setting of §3 and suppose Premise(t). In particular $B = \mathrm{RO}(U)$. We speak about V-theories and V^B-theories meaning theories in V and theories in V^B respectively.

THEOREM 1. *There is an algorithm, depending only on formulas* Premise, Share *and* Pairing, *which interprets the second-order V^B-theory of κ in the monadic V-theory of U.*

PROOF. According to §3 there is a pairing predicate P on κ satisfying the condition (C2). By Claim 1.1 it suffices to construct an algorithm interpreting the monadic V^B-theory of (κ, P) in the monadic V-theory of U.

In order to speak about the monadic theory of (κ, P) in V^B we write $(u_0, u_1, u_2) \in P$, "there is $u \in \kappa$" and "there is $v \subseteq \kappa$" instead of $P(u_0, u_1, u_2)$, "there is an element u" and "there is a set v" respectively.

Let ϕ be a sentence in the monadic language of (κ, P) with parameters: quasi-elements for individual variables and quasi-sets for set variables. By induction on ϕ we express the statement $b \leqq \|\phi\|$ by an equivalent statement $(b \leqq \|\phi\|)_t$ in the monadic theory of U with as many parameters as in ϕ.

Case 1. ϕ is $(\sigma_0, \sigma_2, \sigma_2) \in P$. By Claim 3.1 there are t-shares X_0, X_1, X_2 representing the quasi-elements σ_0, σ_1, σ_2 respectively. Let $(b \leqq \|\phi\|)_t$ be Pairing(t, X_0, X_1, X_2, b). It is equivalent to $b \leqq \|\phi\|$ by Claim 3.2.

Case 2. ϕ is $\sigma \in \tau$. By Claims 3.1 and 3.3 there are a t-share X and point set $Y \subseteq U$ representing the quasi-element σ and the quasi-set τ respectively. Let $(b \leqq \|\phi\|)_t$ say that $X \cap b - Y$ is nowhere dense. It is equivalent to $b \leqq \|\phi\|$ by Claim 3.4.

Case 3. ϕ is $\phi_1 \wedge \phi_2$. Let $(b \leqq \|\phi\|)_t$ be $(b \leqq \|\phi_1\|)_t \wedge (b \leqq \|\phi_2\|)_t$. It is obviously equivalent to $b \leqq \|\phi\|$.

Case 4. ϕ is $\sim \psi$. Let $(b \leqq \|\phi\|)_t$ say that there is no $b \geqq a \in B^+$ satisfying $(a \leqq \|\psi\|)_t$. It holds iff there is no $b \geqq a \in B^+$ with $a \leqq \|\psi\|$ iff $b \cdot \|\psi\| = 0$ iff $b \leqq \|\phi\|$.

Case 5. ϕ is $(\exists u \in \kappa)\psi(u)$. Let $(b \leqq \|\phi\|)_t$ say that there is a t-share u satisfying $(b \leqq \|\psi(u)\|)_t$. First suppose that $b \leqq \|\phi\|$. By Claim 2.3 there is a quasi-element σ with $\|\psi(\sigma)\| = \|\phi\| \geqq b$. By Claim 3.1 there is a t-share X representing σ. By the induction hypothesis $(b \leqq \|\psi(u)\|)_t$ holds for $u = X$. Now suppose that there is a t-share X such that $(b \leqq \|\psi(u)\|)_t$ holds for $u = X$. By

Claim 3.1 X represents some quasi-element σ. By the induction hypothesis $b \leqq \|\psi(\sigma)\|$. Hence $b \leqq \|\phi\|$.

Case 6. ϕ is $(\exists v \subseteq \kappa)\psi(v)$. Let $(b \leqq \|\phi\|)_t$ say that there is a point set v satisfying $(b \leqq \|\psi(v)\|)_t$. The equivalence is established as in Case 5.

In particular $1 \leqq \|\phi\|$ iff $(1 \leqq \|\phi\|)_t$ holds in U. □

§5. Vicinity spaces

For the reader's convenience we present here the notion of vicinity spaces. It was introduced in [2] in order to deal simultaneously with topological spaces and linear orders.

A *vicinity space* is a non-empty set (of *points*) together with a relation "*a point set X is a vicinity of a point x*" satisfying the following conditions:

(V1) x does not belong to any vicinity of x,

(V2) if the intersection of two vicinities of x is not empty then it includes another vicinity of x,

(V3) the relation "X meets Y" on the vicinities of x is transitive, and

(V4) if x belongs to a vicinity Y of another point and X is a vicinity of x then Y includes a vicinity of x meeting X.

For each vicinity X of a point x the union of all vicinities of x meeting X will be called a *direction* around x. By (V3) different directions around x are disjoint. (V4) can be reformulated as follows: if x belongs to a vicinity Y of another point that Y includes a vicinity of x in every direction around x.

EXAMPLE 1. U is a T_1 topological space. Isolated points of U have no vicinities. If x is not isolated then $\{G - \{x\} : G$ is an open neighborhood of $x\}$ is the collection of vicinities of x. Thus there is at most one direction around any point.

EXAMPLE 2. U is a chain (i.e. a linear order). A point set X is a vicinity of a point x iff X is not empty, and X is open in the interval topology of U, and x does not belong to X, and either $x = \sup(X)$ or $x = \inf(X)$. Thus there are at most two directions around any point.

The vicinity space of Example 1 (respectively 2) will be called *associated* with the original top space (respectively chain).

The *monadic theory* of a vicinity space is defined in an obvious way. In Examples 1 and 2 the monadic theory of the vicinity space is easily interpretable in the monadic theory of the original topological space and chain respectively.

The *natural topology* of a vicinity space is defined as follows: a point set X is open iff it includes a vicinity of each point $x \in X$ in every direction around x. This definition restores the original topology in Example 1 and the interval topology in Example 2.

The number of directions around a point is called the *degree* of this point. The *degree* of a vicinity space is the supremum of the degrees of its points. We restrict our attention to vicinity spaces satisfying the following conditions: the natural topology is regular and first-countable, and the degree of the space is finite but positive.

We'll need a few more definitions. Let U be a vicinity space of degree r. The *repletion* of a subset X of U is the set $\mathrm{rp}(X)$ of points x such that the degree of x is equal to r and every vicinity of x meets X. If $X = \mathrm{rp}(X) \neq 0$ we say that X is *replete*. If $0 \neq X \subseteq \mathrm{rp}(X)$ we say that X is *coherent*. If X meets every vicinity of every one of its points it forms a *subspace* of X in the following natural way: a point set $Y \subseteq X$ is a vicinity of a point $x \in X$ in X iff there is $Z \subseteq U$ such that Z is a vicinity of x in U and $Y = X \cap Z$.

Let $p < \omega$. A vicinity space U is *perfunctorily p-modest* if for every coherent everywhere dense sets $X_1, \cdots, X_p$ there is $Y \subseteq X_1 \cup \cdots \cup X_p$ such that Y is replete in U and $X_1, \cdots, X_p$ are dense in Y. U is *p-modest* if every coherent subspace of U is perfunctorily p-modest.

§6. First application

The first application of the Interpretation Theorem uses the article [2].

In this section B is an atomless, complete, separable Boolean algebra in the model V of ZFC. For each $p < \omega$ let K_p be the collection of structures $(U, X_1, \cdots, X_p)$ where U is a coherent separable vicinity space, and $X_1, \cdots, X_p$ are countable everywhere dense subsets of U, and there is no pseudo-meager replete subspace Y of U such that $X_1, \cdots, X_p$ are dense in Y.

THEOREM 1. *There is a uniform in p algorithm interpreting the second-order V^B-theory of c in the monadic V-theory of any space in K_p.*

PROOF. Let $U \in K_p$. By Claim 2.6 $\mathrm{RO}(U)$ is separable. By Claim 2.5 $\mathrm{RO}(U)$ is isomorphic to B hence $V^{\mathrm{RO}(U)}$ is isomorphic to V^B. By Theorem 4.1 it suffices to construct formulas Premise, Share and Pairing in the monadic language of U in such a way that the construction is uniform in p and the conditions of §3 are satisfied for $B = \mathrm{RO}(U)$ and $\kappa = c$.

We use notation, terminology and results of [2]. Evidently U is a guard space. By Claim 2.6 the cardinality of U is at most c.

Let Premise($\bar{u}$) say that $\bar{u}$ is a constructible stable tower of height c whose arena is the whole space. It is the conjunction of some formulas explicitly written in sections 4–6 of [2]. The proof of lemma 5.4 in [2] provides an appropriate tower, thus Premise($\bar{u}$) is satisfiable in U. Let

$$t = (D, D^0, D^1, D^2, D^3, W)$$

be a tower such that Premise(t) holds. By theorem 4.3 in [2] t has a skeleton $\langle X_\alpha : \alpha < c \rangle$. Set $A_\alpha = D^0 \cap X_\alpha$ for $\alpha < c$. Condition (C0) of §3 is evidently satisfied.

Let Share(t, X) say that $X = D^0 \cap Y$ for some limit t-storey Y. Limit t-storeys are defined in 5 of [2]. Use theorem 4.3 in [2] to check condition (C1) of §3.

Let Pairing(t, X_0, X_1, X_2, b) say that there are limit t-storeys Y_0, Y_1, Y_2 such that $X_i = D^0 \cap Y_i$ for $i \leqq 2$ and

$$b \subseteq \mathrm{do}(Y_2 \approx nu_0(Y_0, Y_1)).$$

Use lemmas 5.1 and 5.2 in [2] to check condition (C2) of §3. Just repeat the proof of lemma 6.2 in [2] to check condition (C3) of §3. □

THEOREM 2. *There is an algorithm interpreting the second-order V^B-theory of c in the monadic V-theory of any first-countable non-scattered regular topological space U having no separable, perfect, nowhere dense subspace which is pseudo-meager in itself.*

PROOF. Let K be the class of topological spaces described in Theorem 2. Theorem 1 with $p = 0$ provides an algorithm f interpreting the second-order V^B-theory of c in the monadic theory of any topological space $U \in K$ whose associated vicinity space belongs to K_0.

If ϕ is a sentence in the monadic language of topology let $g\phi$ be a sentence in the monadic language of topology saying that there is a perfect subspace X such that every perfect subspace $Y \subseteq X$ satisfies ϕ. The superposition of f and g is the required algorithm. □

THEOREM 3. *Assume that in V every pseudo-meager subspace of the real line is meager. There is a uniform in p algorithm interpreting the second-order theory of c in the monadic V-theory of any non-p-modest short chain.*

Theorem 3 corresponds to corollary 5 in [2]; it is more convenient however to use [4]. In order to prove Theorem 3 just repeat the proofs of theorems 6.2 and 7.2 in [4]. (We may not worry here about zero-dimensionality and simplify somewhat the proof.) □

§7. Second application

The second application of the Interpretation Theorem uses article [4]. Here B is once again an atomless, complete, separable Boolean algebra, p is a positive natural number, and K_p is a class of structures $(U, X_1, \cdots, X_p)$ where U is a coherent, second-countable, zero-dimensional vicinity space, and $X_1, \cdots, X_p$ are countable, disjoint, everywhere dense subsets. If a structure $(U, X_1, \cdots, X_p) \in K_p$ then $X_1, \cdots, X_p$ will be called *guardians* of the structure.

THEOREM 1. *There is a uniform in p algorithm interpreting the second-order V^B-theory of $\aleph_0$ in the monadic V-theory of any $U \in K_p$.*

PROOF. Let $U \in K_p$. By Claim 2.5 RO(U) is isomorphic to B hence $V^{\mathrm{RO}(U)}$ is isomorphic to V^B. By Theorem 4.1 it suffices to construct formulas Premise, Share and Pairing in the monadic language of U such that the construction is uniform in p and the conditions of §3 are satisfied with $B = \mathrm{RO}(U)$ and $\kappa = \omega$. We use notation, terminology and results of [4].

Let Premise($\bar{u}$) say that $\bar{u}$ is an arithmetical tower. It is the conjunction of some formulas written explicitly in sections 4 and 5 of [4]. Claim 4.2 in [4] provides an arithmetical tower thus Premise($\bar{u}$) is satisfiable in U. Let t be an arithmetical tower in U, and D^0 be the union of guardians of U. By Theorem 5.1 t has a skeleton $\langle X_n : n < \omega\rangle$. Set $A_n = D^0 \cap X_n$ for $n < \omega$. Condition (C0) of §3 is evidently satisfied.

Let Share(t, X) say that $X = D^0 \cap Y$ for some t-storey Y; condition (C1) of §3 is evidently satisfied.

There is a formula ϕ (v_0, v_1, v_2) in the first-order language of $\langle \omega, \mathrm{Add}, \mathrm{Mlt}\rangle$ defining a pairing predicate on ω. Let Pairing(t, X_0, X_1, X_2, b) say that there are t-storeys Y_0, Y_1, Y_2 such that $X_i = D^0 \cap Y_i$ for $i \leqq 2$ and

$$b \subseteq \mathrm{do}_t(\phi(Y_0, Y_1, Y_2)).$$

Condition (C2) is evidently satisfied.

The skeleton of t can be chosen in such a way that the $\{A_n : n < \omega\}$ is pairwise disjoint. In this case condition (C3) of §3 becomes trivial. □

CLAIM 2. *There is a uniform in p algorithm interpreting the second-order V^B-theory of $\aleph_0$ in the monadic V-theory of any second-countable, zero-dimensional, non-p-modest vicinity space.*

PROOF. See the proof of theorem 6.2 in [4]. □

THEOREM 3. *There is a uniform in p algorithm interpreting the second-order V^B-theory of $\aleph_0$ in the monadic V-theory of any second-countable, zero-dimensional, non-p-modest short chain.*

PROOF. Just repeat the proof of theorem 7.2 in [4]. □

Recall that a T_1 topological space is called *p-modest* if the associated vicinity space is p-modest.

THEOREM 4. *There is a uniform in p algorithm interpreting the second-order V^B-theory of $\aleph_0$ in the monadic V-theory of any non-p-modest, regular, second-countable top. space.*

PROOF. Just repeat the proof of theorem 8.3 in [4].

THEOREM 5. *There is an algorithm interpreting the second-order V^B-theory of $\aleph_0$ in the monadic theory of any metizable non-p-modest top. space.*

PROOF. See the proof of theorem 8.5 in [4].

REFERENCES

1. Y. Gurevich, *Monadic theory of order and topology I*, Isr. J. Math. **27** (1977), 299–319.
2. Y. Gurevich, *Monadic theory of order and topology II*, Isr. J. Math. **34** (1979), 45–71.
3. Y. Gurevich and S. Shelah, *Modest theory of short chains II*, J. Symb. Logic **44** (1979), 491–502.
4. Y. Gurevich and S. Shelah, *Monadic theory of order and topology in* ZFC, Ann. Math. Logic **23** (1982), 179–182.
5. T. J. Jech, *Set Theory*, Academic Press, 1978.
6. S. Shelah, *The monadic theory of order*, Ann. of Math. **102** (1975), 379–419.

INSTITUTE FOR ADVANCED STUDIES
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL

Current address of first author
DEPARTMENT OF COMPUTER SCIENCE
THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109 USA

Current address of second author
INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL