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THE MONADIC THEORY OF ω_2 [1]

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Abstract. Assume ZFC + "There is a weakly compact cardinal" is consistent. Then:

(i) For every $S \subseteq \omega$, ZFC + "S and the monadic theory of ω_2 are recursive each in the other" is consistent; and

(ii) ZFC + "The full second-order theory of ω_2 is interpretable in the monadic theory of ω_2" is consistent.

Introduction. First we recall the definition of monadic theories. The monadic language corresponding to a first-order language L is obtained from L by adding variables for sets of elements and adding atomic formulas $x \in Y$. The monadic theory of a model M for L is the theory of M in the described monadic language when the set variables are interpreted as arbitrary subsets of M. Speaking about the monadic theory of an ordinal α we mean the monadic theory of $\langle \alpha, < \rangle$.

Formal theories of order were studied very extensively. We do not review that study here. Our attention is restricted to the monadic theory of ordinals. The pioneer here was Büchi. He proved decidability of the monadic theory of ω, the monadic theory of ω_1, and the monadic theory of ordinals $<\omega_2$. See the strongest result in [Bu]. Note that the last of these theories is not the monadic theory of ω_2, but the set of monadic statements true in every ordinal $<\omega_2$. As we will see below Büchi had a good reason to stop at ω_2.

Shelah studied the monadic theory of ω_2 in [Sh1]. We shall use some of his results. Let $U_i = \{\alpha < \omega_2 \colon \mathrm{cf}\, \alpha = \omega_i\}$ for $i \leq 1$, and I be the ideal of nonstationary sets. For $X \subseteq U_0$ let $D(X) = \{\alpha \in U_1 \colon \alpha \cap X$ is stationary in $\alpha\}$. We call $D(X)$ the *derivative* of X. It is easy to see that $D(X) = D(Y)$ modulo I if $X = Y$ modulo I, thus D can be considered as a relation on the Boolean algebra $\mathrm{PS}(\omega_2)/I$ of subsets of ω_2 modulo I. ($\mathrm{PS}(X)$ denotes here the power set of X and the corresponding Boolean algebra.) Shelah proved:

(i) the monadic theory of ω_2 and the first-order theory of $\langle \mathrm{PS}(\omega_2)/I, D\rangle$ are recursive each in the other;

(ii) the monadic theory of ω_2 is decidable if for every stationary $X \subseteq U_0$ and every Y_1, Y_2 with $D(X) = Y_1 \cup Y_2$ there are disjoint stationary X_1, X_2 such that $X_1 \cup X_2 = X$ and $D(X_i) = Y_i$ modulo I for $i = 1, 2$.

He noted also that (Baumgartner and Jensen's results imply that) "$\omega_2 \models (DX \neq 0$ for every $X \subseteq U_0)$" is independent in ZFC.

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Assuming ZFC + "There is a weakly compact cardinal" Magidor proved in [Ma] that ZFC + "$D(X) = U_1$ modulo I for every stationary $X \subseteq U_0$" is consistent. By (i) the monadic theory of ω_2 is decidable in Magidor's universe. In [Sh2] Shelah proved that a certain combinatorial principle (the uniformization property for ω_2) implies the premise of (ii), and that the uniformization is consistent with ZFC + CH. Later he proved that the uniformization property is consistent even with ZFC + GCH, see [St & Ki]. By (ii) the monadic theory of ω_2 is decidable in the corresponding universes of Shelah. It is different however from the monadic theory of ω_2 in Magidor's universe.

The first undecidability result on monadic theories of ordinals was presented by Magidor in Logic Colloquium 77 (Wroclaw, Poland, 1977). For $n \geq 2$ let E_n say that for every stationary $X \subseteq \omega_n$ consisting of ordinals of cofinality ω there is $\alpha < \omega_n$ of cofinality $> \omega$ such that $\alpha \cap X$ is stationary in α. Assuming consistency of ω supercompact cardinals Magidor proved that for every $S \subseteq \omega - \{0, 1\}$ there is a world with $\{n: E_n \text{ is true}\} = S$. Shelah proved in this direction the following. Assume consistency of ω Mahlo cardinals; then for every $S \subseteq \{2n: 1 \leq n < \omega\}$ there is a world with $\{2n: E_{2n} \text{ is true}\} = S$. And for every $S \subseteq \{2n: 1 \leq n < \omega\}$, it is consistent with ZFC + GCH that $\{2n: 1 \leq n < \omega$ and there is a stationary $X \subseteq \omega_{2n}$ such that for every $Y \subseteq X$ there is $Z \subseteq \omega_{2n}$ with $\{\alpha < \omega_{2n}: \mathrm{cf}(\alpha) > \omega$ and $\alpha \cap Z$ is stationary in $\alpha\} = Y$ modulo nonstationary sets$\}$ is equal to S. None of these three results (one of Magidor and two of Shelah) is published.

In Part I of this article we prove in detail:

THEOREM 1. *Assume there is a weakly compact cardinal. Then there is an algorithm $n \to \psi_n$ such that ψ_n is a sentence in the monadic language of order and for every $S \subseteq \omega$ there is a generic extension of the ground world with $\{n: \omega_2 \models \psi_n\} = S$.*

Thus there are continuum many possible monadic theories of ω_2 (in different universes) and for every $S \subseteq \omega$ there is a monadic theory of ω_2 (in some world) which is at least as complex as S.

The full second-order theory of a set X is the theory of X in the language with variables for elements, variables for arbitrary monadic predicates, variables for arbitrary dyadic predicates, etc. It depends on the cardinality of X only. It belongs more to set theory than to model theory and can be used as a standard of complexity. The monadic theory of ω_2 is easily interpretable in the full second-order theory of ω_2. Thus the full second-order theory of ω_2 gives an upper bound of complexity of the monadic theory of ω_2.

THEOREM 2. *Assume there is a weakly compact cardinal. Then there is a generic extension of the ground world where the full second-order theory of ω_2 is interpretable (therefore recursive) in the monadic theory of ω_2.*

Theorem 2 is proved in Part II. It has actually the same proof as Theorem 1 with only a few alterations. Combining the technique of Part I and that of [Ma] we prove in Part III

THEOREM 3. *Assume GCH and existence of a weakly compact cardinal. For every $S \subseteq \omega_2$ there is a generic extension of the ground universe where S and the monadic theory of ω_2 are recursive each in the other.*

If κ is weakly compact in a world V then it is weakly compact in the constructible part of V (see [Je]) where GCH holds. Hence Theorem 3 gives

COROLLARY 4. *Assume* ZFC + "*There is a weakly compact cardinal*" *is consistent. Then for every* $S \subseteq \omega$, ZFC + "*S and the monadic theory of ω_2 are recursive each in the others*" *is consistent.*

We use the book [Je] and the article [Sho] as sources of notation, terminology and information.

PART I. CONTINUUM POSSIBLE MONADIC THEORIES OF ω_2

§1. **Coding.** Here a *graph* is a model $\langle X, R\rangle$ where R is a reflexive symmetric binary relation on X such that for every different x, y in X there is $z \in X$ with Rxz not equivalent to Ryz.

Claim 1. There is an algorithm $n \to \varphi_n$ such that φ_n is a first-order graph sentence and for every $S \subseteq \omega$ there is a graph $\langle \omega, R\rangle$ with $\{n: \langle \omega, R\rangle \text{ satisfies } \varphi_n\} = S$.

PROOF. For every $n \in \omega - S$ add to ω an auxiliary element n', for every $n \in S$ add two auxiliary elements n' and n''. Let R be the least reflexive symmetric relation on the resulting set containing pairs (n, n'), $(n, n+1)$ for $n < \omega$ and pairs (n, n'') for $n \in S$. It is easy to see that x is auxiliary iff it is R-connected with at most two elements including itself. Hence 0, 1, ... are definable. φ_n says that n is R-connected with two auxiliary elements. It remains only to replace all elements by natural numbers. $\square$

Given a graph $\langle \omega, R\rangle$ and assuming existence of a weakly compact cardinal we define in §2 a forcing notion P and prove in §6:

THEOREM 2. *Suppose G is a P-generic filter over the ground world V. Then in $V[G]$ there is a partition of $\{\alpha < \omega_2: \operatorname{cf} \alpha = \omega\}$ into stationary sets S_n, $n < \omega$, such that*

(i) *for every $\alpha < \omega_2$ of cofinality ω_1 there is a pair $(m, n) \in R$ such that $S_m \cup S_n$ includes a club subset of α, and*

(ii) *if $(m, n) \in R$, $A_m \subseteq S_m$, $A_n \subseteq S_n$ and A_m, A_n are stationary in ω_2 then there are stationarily many ordinals $\alpha < \omega_2$ of cofinality ω_1 with both $A_m \cap \alpha$ and $A_n \cap \alpha$ stationary in α.*

Note that clause (i) of Theorem 2 implies $D(S_m) \cap D(S_n) = \varnothing$ for $(m, n) \in (\omega \times \omega) - R$.

Claim 3. The first-order theory of graph $\langle \omega, R\rangle$ is interpretable in the monadic theory of ω_2 if there is a partition described in Theorem 2.

PROOF. U_0, U_1, the ideal I and the derivative D are defined in the Introduction. Here are some more definitions in the monadic theory of ω_2. Two subsets X, Y of U_0 are *connected* if $D(X) \cap D(Y)$ is stationary. (Note that $X \subseteq U_0$ is connected with itself iff X is stationary.) A stationary $X \subseteq U_0$ is an *atom* if there are no X_1, $X_2 \subseteq X$ and $Y \subseteq U_0$ such that X_1, X_2 are stationary and Y is connected with one of sets X_1, X_2 but not with the other. An atom X is *maximal* if $X = Y$ modulo I for every atom Y including X.

For every m every stationary $X \subseteq S_m$ is an atom. For, suppose X_1, X_2 are stationary subsets of X and $Y \subseteq U_0$. Let $Y_1 = \bigcup\{Y \cap S_n: (m, n) \in R\}$, $Y_2 = Y - Y_1$. If Y_1 is stationary then some $Y \cap S_n$ with $(m, n) \in R$ is stationary and Y is connected with both X_1 and X_2. Otherwise the derivative of Y coincides modulo I with the derivative Y_2 which avoids even the derivative of X, thus Y is connected with neither X_1 nor X_2.

If X is an atom then $X \subseteq S_m$ modulo I for some m. For, let $K = \{m: X \cap S_m$ is stationary$\}$. $K \neq 0$ because X is stationary. If different m, n belong to K there is l such that Rlm is not equivalent to Rln. Set $X_1 = X \cap S_m$, $X_2 = X \cap S_n$, $Y = S_l$ to contradict the assumption that X is an atom.

It is easy to see that an atom X is maximal iff $X = S_m$ modulo I for some m. Now interpret variables of the first-order graph language as maximal atoms (equality is equality modulo I) and R as connectedness. □

Claim 1, Theorem 2 and Claim 3 imply

THEOREM 4. *Assume there is a weakly compact cardinal. There is a recursive list $\psi_0, \psi_1, \ldots$ of monadic sentences such that for every $S \subseteq \omega$ there is a generic extension of the ground world with $\{n: \omega_2 \models \psi_n\} = S$.*

§2. Forcing notion. Suppose κ is a weakly compact cardinal and R is a reflexive symmetric binary relation on ω. We define a forcing notion P for collapsing κ onto ω_2 and creating stationary subsets S_m of ω_2 described in §1. A condition p is a triple $(p0, p1, p2)$ of countable functions. $p0$ gives a partial information about sets S_m, it is composed of pairs (α, m) where $\operatorname{cf} \alpha = \omega$, $m < \omega$; the intended meaning is $\alpha \in S_m$. $p1$ assigns a pair $(m, n) \in R$ to ordinals $\alpha < \kappa$ of cofinality $> \omega$; the intended meaning is: $S_l \cap \alpha$ is stationary in α iff $l \in \{m, n\}$. To assure our intentions $p2$ assigns a closed countable subset of α to each $\alpha \in \operatorname{dom} p1$ in such a way that $p2(\alpha) \subseteq \{\beta: p0(\beta)$ is equal to m or to $n\}$ where $(m, n) = p1(\alpha)$; the intended meaning is: $p2(\alpha)$ is an initial segment of a club subset of α included in $S_m \cup S_n$. A condition p refines a condition q $(p \leq q)$ if $q0 \subseteq p0$, $q1 \subseteq p1$ and for every $\alpha \in \operatorname{dom} p1$, $p2(\alpha)$ is an end extension of $q2(\alpha)$ and

$$\min(p2(\alpha) - q2(\alpha)) > \sup(\alpha \cap (\operatorname{dom} p0 \cup \operatorname{dom} p1)).$$

The last requirement is used to prove the following Claim 1. (It is convenient for us to treat elements of R as unordered pairs.)

Claim 1. P is ω_1-closed.

PROOF. Given $p_0 > p_1 > \cdots$ set $x = \bigcup\{p_n0: n < \omega\}$, $y = \bigcup\{p_n1: n < \omega\}$ and $z(\alpha)$ be the closure of $\bigcup\{p_n2(\alpha): \alpha \in \operatorname{dom} p_n1\}$ for $\alpha \in \operatorname{dom} y$. Let $d = \{\alpha \in \operatorname{dom} y: \sup z(\alpha)$ does not belong to any $p_n2(\alpha)\}$. If $d \neq \varnothing$ then $(x, y, z) \notin P$ because $\sup z(\alpha) \notin \operatorname{dom} x$ for $\alpha \in d$. Select a function $f = \{(\sup z(\alpha), m_\alpha): \alpha \in d, m_\alpha \in y(\alpha)\}$; this is possible because $\alpha, \beta \in d$, $\alpha < \beta$ imply $\sup z(\alpha) < \alpha < \sup z(\beta)$. $(x \cup f, y, z)$ belongs to P and refines any p_n. □

If $p \in P$ then $\operatorname{dom} p0 \cup \operatorname{dom} p1$ will be called the domain of p and denoted $\operatorname{dom}(p)$. $\sup \operatorname{dom}(p)$ will be called the height of p and denoted $h(p)$.

Claim 2. P satisfies the κ-chain condition.

PROOF. By contradiction suppose that $\{p_\alpha: \alpha < \kappa\}$ is a set of pairwise incompatible conditions. Define $f(\alpha) = \sup(\alpha \cap \operatorname{dom} p_\alpha)$. f is regressive on $\{\alpha: \operatorname{cf} \alpha > \omega\}$. By Fodor's Lemma there is a stationary $A \subseteq \kappa$ with $f(\alpha) = \delta$ for some δ and any $\alpha \in A$. Let $\lambda = |\delta|^\omega$. There are at most λ possibilities for $p0|\delta$, $p1|\delta$, $p2|\delta$. Hence there is α such that $B = \{\beta \in A: p_\alpha i$ and $p_\beta i$ coincide on δ for $i \leq 2\}$ is of cardinality κ. There is $\beta \in B$ exceeding $h(p_\alpha)$. Then p_α, p_β are compatible, thus we have a contradiction. □

COROLLARY 3. *For every P-name a there is a function* $f\colon \kappa \to \kappa$ *such that for every* $\alpha < \kappa$ *and every p forcing* $\alpha \in a$ *there is q such that* $h(q) \le f\alpha$, *q is compatible with p and q forces* $\alpha \in a$.

PROOF. Consider a maximal antichain $C \subseteq P$ such that every element of C either forces $\alpha \in a$ or forces $\alpha \notin a$. Set $f\alpha = \sup\{h(p)\colon p \in C\}$. □

In the rest of this section G is a P-generic ultrafilter over the ground world V, $Gi = \bigcup\{pi\colon p \in G\}$ for $i \le 1$. It is easy to see that dom $G0 = \{\alpha < \kappa\colon \operatorname{cf} \alpha = \omega\}$ and dom $G1 = \{\alpha < \kappa\colon \operatorname{cf}\alpha > \omega\}$. As P is ω_1-closed, $\omega_1^{V[G]} = \omega_1^V$. Thus cf $\alpha > \omega$ in V iff cf $\alpha > \omega$ in $V[G]$. For $\alpha < \kappa$ with cf $\alpha > \omega$ let $G2(\alpha) = \bigcup\{p2(\alpha)\colon p \in G\}$.

Claim 4. Suppose $\alpha < \kappa$ and cf $\alpha > \omega$. Then $G2(\alpha)$ is a club subset of α of cofinality ω_1.

PROOF. If $\beta \in G2(\alpha)$ then $\beta \in p2(\alpha)$ for some $p \in G$, hence cf $\beta = \omega$. Let $c = \{(\beta, p)\colon \alpha \in \operatorname{dom} p$ and $\beta \in p2(\alpha)\}$, so that $G2(\alpha)$ is the denotation of c in $V[G]$. For $\beta < \alpha$ the set $\{p\colon p$ forces $\gamma \in c$ for some $\gamma > \beta\}$ is dense, hence $G2(\alpha)$ is unbounded. Suppose $\beta < \kappa$ is a limit point for $G2(\alpha)$. There are $\gamma > \beta$ and $p \in G$ forcing $\gamma \in c$. Then $G2(\alpha) \cap \gamma = p2(\alpha) \cap \gamma$. As $p2(\alpha)$ is closed, $\beta \in p2(\alpha) \subseteq G2(\alpha)$. Thus $G2(\alpha)$ is a club subset of α consisting of ordinals of cofinality ω, hence cf $G2(\alpha) = \omega_1$. □

COROLLARY 5. κ *is* ω_2 *in* $V[G]$.

PROOF. In $V[G]$: $\kappa > \omega_1$ because $\omega_1^{V[G]} = \omega_1^V$, $\kappa \le \omega_2$ by Claim 4, κ is a cardinal because P satisfies the κ-chain condition. □

Claim 6. Every new club subset of κ in $V[G]$ includes an old club subset of κ.

Proof is well known and uses only the κ-chain condition. Suppose the empty condition $(\varnothing, \varnothing, \varnothing)$ forces "c is a club subset of κ". Let $C' = \{\alpha\colon$ the empty condition forces $\alpha \in c\}$. It is obvious that C' is closed. We prove that C' is unbounded. For $\alpha < \kappa$ let $A(\alpha)$ be the set of ordinals $\beta < \alpha$ such that some p forces "β is the least element of c above α". $|A(\alpha)| < \kappa$ because P satisfies the κ-chain condition. Let $f\alpha = \sup A(\alpha)$. Now given α_0 let $\alpha_{n+1} = f\alpha_n$, $\alpha = \sup\{\alpha_n\colon n < \omega\}$. The empty condition forces that c meets every interval $(\alpha_n, \alpha_{n+1}]$, hence it forces $\alpha \in c$, i.e., $\alpha \in C'$. □

§3. Decomposition of forcing.

First we recall the notion of quotient forcing. Suppose B is a partial ordering and A is a submodel of B satisfying the following conditions: if $p \in A$ and $p \le q$ then $q \in A$; if two elements of A are compatible in B then they are compatible in A; and for each $q \in B$ there is a unique $p \in A$ (the projection of q) such that $q \le p$ and p, q are compatible with exactly the same elements of A. Let $c = \{(q, p)\colon q \in B$ and p is the projection of $q\}$. For every A-generic filter G over the ground world V, $c^{V[G]}$ (i.e., the denotation of c in $V[G]$) is equal to $\{q \in B\colon q$ is compatible with any $p \in G\}$. This denotation is the quotient forcing in the following sense: B is isomorphic to a dense subset of $A * c$ and forcing with B is a composition of forcing with A and subsequent forcing with the denotation of c.

For $0 < \alpha \le \kappa$ let $P(<\alpha)$ be the submodel of P comprising conditions of height $<\alpha$. For $p \in P$ let $p(<\alpha) = (p0|\alpha, p1|\alpha, p2|\alpha)$, it is the projection of p into $P(<\alpha)$. For every $\alpha < \beta \le \kappa$ with cf $\alpha > \omega$ we get the quotient forcing completing forcing with $P(<\alpha)$ to a forcing with $P(<\beta)$. Two cases are of special interest for us. Let $\lambda < \kappa$ be of cofinality $>\omega$ and $P(\le\lambda) = P(<\lambda + 1)$.

Case 1. $\alpha = \lambda$, $\beta = \lambda + 1$. Suppose G is a $P(<\lambda)$-generic over V filter and $G0 = \bigcup\{p0: p \in G\}$. The quotient forcing notion is the submodel of $P(\leq\lambda)$ comprising conditions $p \in P(\leq\lambda)$ such that $p(<\lambda) \in G$ and $p2(\lambda) \subset G0^{-1}\{m, n\}$ provided $p1(\lambda) = \{m, n\}$. Let $P(\lambda, m, n, G)$ be the following forcing notion in $V[G]$ (for shooting a club subset of λ in $G0^{-1}\{m, n\}$): conditions are closed countable subsets of λ included in $G0^{-1}\{m, n\}$, a stronger condition means an end extension.

Claim 1. Suppose J is a $P(\leq\lambda)$-generic filter over V, $H = J \cap P(<\lambda)$, $\{m, n\} = p1(\lambda)$ for some $p \in J$. Then H is a $P(<\lambda)$-generic filter over V and there is a $P(\lambda, m, n, H)$-generic filter I over $V[H]$ such that $V[H][I] = V[J]$.

Case 2. $\alpha = \lambda + 1$, $\beta = \kappa$. Suppose G is a $P(\leq\lambda)$-generic filter over V, and $G0 = \bigcup\{p0: p \in G\}$. The quotient forcing comprises conditions $p \in P$ such that $p(\leq\lambda) \in G$ and for each $\gamma > \lambda$, if $\gamma \in \operatorname{dom} p1$ and $p1(\gamma) = \{m, n\}$ then $\lambda \cap p2(\gamma) \subseteq G0^{-1}\{m, n\}$. Let $P(>\lambda)$ be the submodel of P comprising conditions $p \in P$ such that $p(\leq\lambda) = \emptyset$ and $\lambda \cap p2(\lambda) = \emptyset$ for $\gamma \in \operatorname{dom} p1$.

Claim 2. Suppose J is a P-generic filter over V and $H = J \cap P(\leq\lambda)$. Then H is a $P(\leq\lambda)$-generic filter over V and there is a $P(>\lambda)$-generic filter I over $V[H]$ such that $V[H][I] = V[J]$.

Proof is easy.

Let us remark that one can˜go from H and I to J in Claims 1 and 2.

§4. Preserving stationarity. I. Suppose λ is a regular cardinal and A is a stationary subset of λ comprising ordinals of cofinality ω. Let Q be the following forcing notion (for shooting a club subset of λ in A): conditions are closed countable subsets of λ included in A, a stronger condition means an end extension. Suppose $B \subseteq A$ is stationary in λ.

THEOREM 1. *Forcing with Q does not destroy stationarity of B.*

We prove Theorem 1 in the rest of this section. Suppose $\Vdash_Q$ (c is a club subset of λ). It suffices to prove that for every $p_0 \in Q$ there are $p \leq p_0$ and $\mu \in B$ with $p \Vdash \mu \in c$. Let M be a model (i.e., a relational system) with universe $H(\lambda)$ (that is the collection of sets hereditary of cardinality $<\lambda$), unary relation $x \in Q$, and binary relations $x \in y$, $x \leq y$ (as conditions in Q), $x \Vdash y \in c$. Let M' be a model obtained from M by adding Skolem functions. For $\mu < \lambda$ let $M(\mu)$ be the least submodel of M' including μ (i.e., containing any $\alpha < \mu$).

LEMMA 2. *$\{\mu: \lambda \cap M(\mu) = \mu\}$ is club in λ.*

PROOF OF LEMMA 2. It is obvious that the set is closed. To prove unboundedness suppose $\mu_0 < \lambda$ and consider the sequence $\mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots$ where $\mu_{n+1} = \sup(\lambda \cap M(\mu_n))$. If $\mu = \sup\{\mu_n: n < \omega\}$ then $\lambda \cap M(\mu) = \mu$. $\square$

Given p_0 take $\mu \in B$ such that $p_0 \in M(\mu)$ and $\lambda \cap M(\mu) = \mu$. Let $\alpha_0 < \alpha_1 < \cdots$ converge to μ. Build $p_0 > p_1 > \cdots$ such that $p_n \in M(\mu)$ and $p_{n+1} \Vdash \beta_n \in c$ for some $\alpha_n < \beta_n < \mu$; it is possible because $M(\mu)$ is an elementary submodel of M. Then $p = \bigcup\{p_n: n < \omega\} \cup \{\mu\} \in Q$ and $p \Vdash \mu \in c$.

§5. Preserving stationarity. II.

Claim 1. Let ε be an ordinal of cofinality ω_1 and A be a stationary subset of ε. Let Q be an ω_1-closed forcing notion. Then forcing with Q does not destroy stationarity of A i.e. the empty condition Q-forces "A is a stationary subset of ε".

This claim is a corollary of a stronger result of Baumgartner: ω_1-closed forcing does not destroy stationarity of sets comprising ordinals of cofinality ω, see [Ba]. For the reader's convenience we prove our claim.

PROOF. W.l.o.g. $\varepsilon = \omega_1$: there is club $X \subseteq \varepsilon$ of type ω_1 and it suffices to prove that $A \cap X$ remains stationary in X.

By contradiction suppose $p \Vdash$ (c is a club subset of ω_1 avoiding A). Build (in the ground world) a sequence $\langle p_\alpha : \alpha < \omega_1 \rangle$ of conditions and a sequence $\langle \delta_\alpha : \alpha < \omega_1 \rangle$ of countable ordinals such that $p_0 \leq p$, $p_0 \Vdash \delta_0 \in c$, $\delta_{\alpha+1} > \delta_\alpha$, $p_{\alpha+1} \leq p_\alpha$, $p_{\alpha+1} \Vdash \delta_{\alpha+1} \in c$, and if α is limit then $\delta_\alpha = \sup\{\delta_\beta : \beta < \alpha\}$ and p_α is stronger than p_β for $\beta < \alpha$. $\{\delta_\alpha : \alpha < \omega_1\}$ is a club subset of ω_1 avoiding A, which is impossible. $\square$

§6. Standard stationary sets. Suppose G is a P-generic filter over the ground world V. Let $Gi = \bigcup\{pi : p \in G\}$ for $i \leq 2$, $G2(\alpha) = \bigcup\{p2(\alpha) : p \in G\}$ for $\alpha < \kappa$ with $\operatorname{cf} \alpha > \omega$, and $S_m = \{\alpha : G0(\alpha) = m\}$. The set $s_m = \{(\alpha, p); p0(\alpha) = m\}$ is a name of S_m.

Claim 1. Every S_m is a stationary subset of κ.

PROOF. Given $p_0 \Vdash$ (c is a club subset of κ) build sequences $p_0 > p_1 > \cdots$ and $\alpha_0 < \alpha_1 < \cdots$ such that $\alpha_i > h(p_i)$ and $p_{i+1} \Vdash \alpha_i \in c$. Let $\alpha = \sup\{\alpha_i : i < \omega\}$. By the proof of Claim 1 in §2 there is a condition p of height α refining any p_i. Let $q = (p0 \cup \{(\alpha, m)\}, p1, p2)$. Then $p_0 > q \Vdash (\alpha \in c \cap s_m)$. $\square$

Claim 2. Sets S_m partition κ.

PROOF. For every p and ordinal α of cofinality ω there is $q \leq p$ with $\alpha \in \operatorname{dom} q^0$. $\square$

By the same token every $\alpha < \kappa$ with $\operatorname{cf} \alpha > \omega$ belongs to $\operatorname{dom}(G1)$.

Claim 3. Suppose $\alpha < \omega_2$ and $\operatorname{cf} \alpha = \omega_1$ in $V[G]$. Then $G2(\alpha)$ is a club subset of α. If $G1(\alpha) = (m, n)$ then $G2(\alpha) \subseteq S_m \cup S_n$.

Proof is clear.

THEOREM 4. *Suppose $(m, n) \in R$, $A_m \subseteq S_m$, $A_n \subseteq S_n$, and A_m, A_n are stationary subsets of ω_2 in $V[G]$. Then there are stationarily many ordinals $\alpha < \omega_2$ of cofinality ω_1 such that both $A_m \cap \alpha$ and $A_n \cap \alpha$ are stationary in α.*

We prove Theorem 4 in the rest of this section. Without loss of generality $m = 0$, $n = 1$. Suppose $C \in V$ is the part of a club subset of κ comprising ordinals of cofinality $> \omega$. Since every new club subset of κ includes an old one it suffices to find $\alpha \in C$ such that both $A_0 \cap \alpha$, $A_1 \cap \alpha$ are stationary in α.

Let a_i' be a P-name for A_i and $a_i = \{(\alpha, p) : p\ P\text{-forces } \alpha \in a_i'\}$ for $i \leq 1$. There is $p_0 \in G$ forcing "a_i is a stationary subset of κ" for $i \leq 1$. It suffices to find $\lambda \in C$ and $p \leq p_0$ such that p P-forces "$a_i \cap \lambda$ is stationary in λ" for $i \leq 1$.

By Corollary 3 of §2 there is in V a function $f: \kappa \to \kappa$ such that for every $\alpha < \kappa$ and $p \in P$ forcing $\alpha \in a_i$ for some $i \leq 1$ there is $q \in P$ of height $\leq f\alpha$ compatible with p and forcing the same statement. Let M be the model with universe V_κ (the collection of sets of rank $< \kappa$), distinguished element p_0, unary predicates $x \in C$, $x \in P$, and binary predicates $x \in y$, $x \leq y$ (with respect to P), $x \Vdash y \in a_0$, $x \Vdash y \in a_1$, $(x, y) \in f$. Let σ be a sentence in the language of M saying: κ is inaccessible and closed under f, and C is unbounded in κ, and for every club subset C' of κ and $p_1 \leq p_0$ there are $\alpha_0, \alpha_1 \in C'$ and $p_2 \leq p_1$ with $p_2 \Vdash \alpha_i \in a_i$ for $i \leq 1$. It is easy to see that σ is Π_1^1.

Any weakly compact cardinal is Π_1^1-indescribable. Hence there is $\lambda < \kappa$ such that V_λ forms a submodel of M satisfying σ. Thus λ is inaccessible, $h(p_0) < \lambda$, $C \cap \lambda$ is unbounded in λ (therefore $\lambda \in C$), λ is closed under f and for every $p_1 \leq p_0$ of height $<\lambda$ and every club subset C' of λ there are $\alpha_0, \alpha_1 \in C'$ and $p_2 \leq p_1$ of height $<\lambda$ such that p_2 P-forces $\alpha_i \in a_i$ for $i \leq 1$.

Let $b_i = \{(\alpha, p)\colon \alpha < \lambda, h(p) < \lambda, p\ P\text{-forces } \alpha \in a_i\}$.

Claim 5. Suppose I is a P-generic filter over V, $p_0 \in I$, $H = I \cap P(<\lambda)$. Then

$$a_i^{V[I]} \cap \lambda = b_i^{V[H]} = b_i^{V[I]} \quad \text{for } i \leq 1.$$

PROOF. If $\alpha \in b_i^{V[H]}$ then $\alpha < \lambda$ and there is $p \in H$ P-forcing $\alpha \in a_i$, hence $\alpha \in a_i^{V[I]}$.

Suppose $\alpha \in a_i^{V[I]} \cap \lambda$. There is $p \in I$ P-forcing $\alpha \in a_i$. It suffices to prove that $p(<\lambda)$ P-forces $\alpha \in a_i$. Suppose the contrary. Then there is $q \leq p(<\lambda)$ P-forcing $\alpha \notin a_i$. As λ is closed under f there is $q' \in P(<\lambda)$ compatible with q and forcing $\alpha \notin a_i$. But q' is compatible with $p(<\lambda)$ hence with p, which is impossible. □

Let τ be the statement "b_0, b_1 are stationary subsets of λ".

Claim 6. p_0 $P(<\lambda)$-forces τ.

PROOF. $P(<\lambda)$ satisfies the λ-chain condition and the empty condition $P(<\lambda)$-forces that every new club subset of λ includes an old club subset of λ. (Just repeat the proof of the corresponding statements about P and κ.) For every $C' \in V$ and $p_1 \in P(<\lambda)$, if C' is a club subset of λ and $p_1 \leq p_0$ then there are $\alpha_0, \alpha_1 \in C'$ and $p_2 \in P(<\lambda)$ such that $p_2 \leq p_1$ and p_2 P-forces $\alpha_0 \in a_0$, $\alpha_1 \in a_1$. By the previous claim p_2 $P(<\lambda)$-forces $\alpha_i \in b_i$ for $i \leq 1$. Thus p_0 $P(<\lambda)$-forces τ. □

Let $p = (p_0 0, p_0 1 \cup \{(\lambda, \{0, 1\})\}, p_0 2 \cup \{(\lambda, \varnothing)\})$.

Claim 7. p forces τ with respect to $P(\leq\lambda)$.

PROOF. Suppose J is a $P(\leq\lambda)$-generic filter over V containing p, $H = J \cap P(<\lambda)$. By §3 there is a $P(\lambda, 0, 1, H)$-generic filter I over $V(H)$ such that $V[J] = V[H][I]$. By §4 (with $Q = P(\lambda, 0, 1, H)$, $A = \{\alpha < \lambda\colon q0(\alpha) \in 2 \text{ for some } q \in H\}$) forcing with $P(\lambda, 0, 1, H)$ does not destroy stationarity of $b_i^{V[H]}$. □

Claim 8. p forces τ with respect to P.

PROOF. Use (Claim 2 of) §3 and §5. □

By Claims 5 and 8 p P-forces "$a_i \cap \lambda$ is a stationary subset of λ" for $i \leq 1$.

PART II. INTERPRETING THE FULL SECOND-ORDER THEORY OF ω_2 IN THE MONADIC THEORY OF ω_2

§7. **Coding.** Recall that the full second-order theory of a set X is the theory of X in the language with variables for elements, monadic predicates, dyadic predicates, etc. This theory depends on the cardinality of X only. Speaking about the full second-order theory of ω_2 we mean the full second-order theory of the underlying set. Fix a pairing function $\gamma = nu(\alpha, \beta)$ on ω_2 (so that nu gives a one-to-one correspondence from $\omega_2 \times \omega_2$ onto ω_2). Define left(α, γ) if $\gamma = nu(\alpha, \beta)$ for some β, and right(β, γ) if $\gamma = nu(\alpha, \beta)$ for some α.

Claim 1. The full second-order theory is interpretable in the monadic theory of $\langle \omega_2, \text{left}, \text{right} \rangle$.

Proof is obvious.

As in §1, a graph is a model $\langle X, R \rangle$ where R is a reflexive symmetric binary relation on X such that $x \neq y \rightarrow \exists z\,(Rxz$ is not equivalent to $Ryz)$.

Claim 2. There is a graph $\langle \omega_2, R\rangle$ such that the monadic theory of $\langle \omega_2$, left, right$\rangle$ is interpretable in the monadic theory of $\langle \omega_2, R\rangle$.

PROOF. 5α is the αth ordinal divisible by 5. Let R be the least reflexive symmetric binary relation on ω_2 containing the following pairs:

(i) $(5\alpha + i, 5\alpha + i + 1)$ for $\alpha < \omega_2$, $i < 4$,

(ii) $(5\alpha, 5\beta + 2)$ for $(\alpha, \beta) \in$ left,

(iii) $(5\alpha, 5\beta + 3)$ for $(\alpha, \beta) \in$ right.

Equality is defined in $\langle \omega_2, R\rangle$ as indistinguishability. Let $w(x)$ be the cardinality of $\{y\colon Rxy \text{ and } x \neq y\}$. The statements $w(x) = n$, $n < \omega$, are expressible in the monadic theory of $\langle \omega_2, R\rangle$. It is easy to see that $\alpha = 0$ modulo 5 iff $w(\alpha) > 3$, $\alpha = 1$ modulo 5 iff $w(\alpha) = 2$, $\alpha = 4$ modulo 5 iff $w(\alpha) = 1$, $\alpha = 3$ modulo 5 iff $w(\alpha) = 3$ and α is R-connected with some $\beta = 4$ modulo 5, $\alpha = 2$ modulo 5 iff $w(\alpha) = 3$ and α is not R-connected with any $\beta = 4$ modulo 5. Furthermore, $5\alpha + 1$ is the only element which equals 1 modulo 5 and is R-connected with 5α, $5\alpha + 2$ is the only element which equals 2 modulo 5 and is R-connected with $5\alpha + 1$, $5\alpha + 3$ is the only element which equals 3 modulo 5 R-connected with $5\alpha + 2$. $\langle \omega_2$, left, right$\rangle$ is now interpretable in $\langle \omega_2, R\rangle$, and the interpretaticn of the model gives rise to an interpretation of its monadic theory. $\square$

Let R be as in Claim 2. Assuming existence of a weakly compact cardinal we define in the next section a forcing notion P and prove

THEOREM 3. *Suppose G is a P-generic filter over the ground universe V. Then in $V[G]$ there is a partition of $\{\alpha < \omega_2\colon \mathrm{cf}\,\alpha = \omega\}$ into stationary sets S_α, $\alpha < \omega_2$, such that*:

(0) *$\beta \in S_\alpha$ implies $\alpha < \beta$*;

(1) *for every ordinal α of cofinality ω_1 there is a pair $(\beta, \gamma) \in R$ such that $S_\beta \cup S_\gamma$ includes a club subset of α*; *and*

(2) *if $R\alpha\beta$ holds, $A_\alpha \subseteq S_\alpha$, $A_\beta \subseteq S_\beta$ and A_α, A_β are stationary in ω_2 then there are stationarily many ordinals γ of cofinality ω_1 such that $A_\alpha \cap \gamma$ and $A_\beta \cap \gamma$ are stationary in γ.*

Claim 4. The monadic theory of the graph $\langle \omega_2, R\rangle$ is interpretable in the monadic theory of ω_2 if there is a partition described in Theorem 2.

PROOF. We use definitions of I, U_0, U_1, connected, atom, maximal atom given in §1.

If X is an atom then $X \subseteq S_\alpha$ modulo I for some α. For, suppose X is an atom. Then X is stationary and, by (0) and Fodor's Lemma $X \cap S_\alpha$ is stationary for some α. If $X - S_\alpha$ is stationary then there are $\beta \neq \alpha$ with stationary $X \cap S_\beta$ and γ such that $R\alpha\gamma$ holds and $R\beta\gamma$ fails (or conversely), set $X_1 = X \cap S_\alpha$, $X_2 = X \cap S_\beta$, $Y = S_\gamma$ to get a contradiction.

Every stationary $X \subseteq S_\alpha$ is an atom. For suppose X_1, X_2 are stationary subsets of X, $Y \subseteq U_0$, $K_0 = \{\beta\colon R\alpha\beta \text{ holds}\}$, $K_1 = \omega_2 - K_0$ and $Y_i = \{\beta \in Y\colon \beta \in S_\gamma \text{ for some } \gamma \in K_i\}$ for $i \leq 1$. If Y_0 is stationary then, by (0) and Fodor's Lemma, $Y_0 \cap S_\beta$ is stationary for some $\beta \in K_0$, in this case Y is connected with both X_1 and X_2. Suppose $Y_0 \in I$. Then $D(Y_0) \in I$. If $\alpha \in D(X) - D(Y_0)$ then, by (1), there is a club subset of α avoiding Y_1. Hence $D(X) \cap D(Y) \subseteq D(Y_0) \in I$ and Y is not connected even with X.

It is easy to see that $X \subseteq U_0$ is a maximal atom iff $X = S_\alpha$ modulo I for some α. A subset X of U_0 will be called *regular* if for every maximal atom Y either $Y \subseteq X$ modulo I or $Y \subseteq U_0 - X$ modulo I. Now, given a monadic graph sentence φ

interpret individual variables as maximal atoms, set variables as regular subsets of U_0, equality as equality over I, R as connectedness, and the containment relation as inclusion modulo I. The resulting monadic sentence ψ holds in ω_2 iff φ holds in $\langle \omega_2, R\rangle$. □

Claims 1, 2, 4 and Theorem 3 imply

THEOREM 5. *Assume there is a weakly compact cardinal. Then there is a generic extension of the ground universe where the full second-order theory of ω_2 is interpretable in the monadic theory of ω_2.*

§**8. Forcing.** Suppose κ is a weakly compact cardinal and R is as in §7. We define a forcing notion p for collapsing κ onto ω_2 and creating a partition of $\{\alpha < \omega_2: \operatorname{cf} \alpha = \omega\}$ into stationary sets S_α, $\alpha < \omega_2$, described in §7. A condition $p = (p0, p1, p2)$ where:

$p0$ is a countable function from a part of $\{\alpha < \kappa: \operatorname{cf} \alpha = \omega\}$ into ω_2 such that $\beta = p0(\alpha)$ implies $\beta < \alpha$ (the intended meaning of $\beta = p0(\alpha)$ is: $\alpha \in S_\beta$),

$p1$ is a countable function from a part of $\{\alpha < \kappa: \operatorname{cf} \alpha > \omega\}$ into R (the intended meaning of $(\beta, \gamma) \in p1(\alpha)$ is $\alpha \cap S_\delta$ is stationary in α iff $\delta = \beta$ or $\delta = \gamma$),

$p2$ is a countable function with dom $p2$ = dom $p1$, if $\alpha \in \operatorname{dom} p1$ and $p1(\alpha) = (\beta, \gamma)$ then $p2(\alpha)$ is a closed countable subset of α included in $\{\delta: p0(\delta) = \beta$ or $p0(\delta) = \gamma\}$ (the intended meaning $p2(\alpha)$ is an initial segment of a club subset of α included in $S_\beta \cup S_\gamma$).

By definition, p refines q ($p \le q$) if $q0 \subseteq p0$, $q1 \subseteq p1$ and for every $\alpha \in \operatorname{dom} q1$, $p2(\alpha)$ is an end extension of $q2(\alpha)$ in such a way that

$$\min(p2(\alpha) - q2(\alpha)) > \sup(\alpha \cap (\operatorname{dom} p0 \cup \operatorname{dom} p1)).$$

Suppose G is a P-generic filter over the ground universe V, and $S_\alpha = \{\beta: p0(\beta) = \alpha$ for some $p \in G\}$ for $\alpha < \kappa$. The clause (0) of Theorem 2 in §7 is obvious, the rest of the theorem is proved exactly as in Part I.

PART III. MONADIC THEORY OF A GIVEN COMPLEXITY

§**9. Coding.** Given a sequence $s_0 < s_1 < \cdots$ of positive integers and assuming GCH and existence of a weakly compact cardinal, we define in the next section a forcing notion P and prove the following theorem. (U_0, U_1, D are defined in the introduction.)

THEOREM 1. *Suppose G is a P-generic filter over the ground universe V. Then in $V[G]$ there are*

(i) *a partition of U_0 into stationary sets A_{ni} where $n < \omega$, $i < s_n$ and*

(ii) *a partition of U_1 into stationary sets B_n, C_{ni} where $n < \omega$, $i < s_n$ such that $D(X) = B_n \cup C_{ni}$ modulo I for every $n < \omega$, $i < s_n$ and stationary $X \subseteq A_{ni}$.*

It is clear that the sequence $s_0, s_1, \ldots$ is computable from the monadic theory of ω_2 in $V[G]$. The converse is true too.

Claim 2. Suppose ω_2 can be partitioned in the way described in Theorem 1. Then the monadic theory of ω_2 is computable from the sequence $s_0, s_1, \ldots$.

PROOF. As was mentioned in the introduction the monadic theory of ω_2 is recursive in the first-order theory of $M = \langle \mathrm{PS}(\omega_2)/I, D\rangle$. For every n let

$$E_n = \left(\bigcup\{A_{ni}: i < s_n\}\right) \cup B_n \cup \left(\bigcup\{C_{ni}: i < s_n\}\right)$$

and I_n, D_n be the restrictions of I, D, respectively, on PS(E_n). It is easy to see that M is the direct product of structures $M_n = \langle \mathrm{PS}(E_n)/I_n, D_n \rangle$. The first-order theory of M_n is computable from s_n. Hence the first-order theory of $\{M_n: n < \omega\}$ is computable from the sequence $s_0, s_1, \ldots$. By the Feferman-Vaught Theorem (see [FV]) the first-order theory of M is computable from the first-order theory of $\{M_n: n < \omega\}$. $\square$

Theorem 1 and Claim 2 imply

THEOREM 3. *Assume GCH holds and there is a weakly compact cardinal. Then for every sequence $s_0 < s_1 < \cdots$ of positive integers there is a generic extension of the ground world where the sequence $s_0, s_1, \ldots$ and the monadic theory of ω_2 are recursive each in the other.*

§10. Forcing. Suppose GCH holds and κ is a weakly compact cardinal and $s_0 < s_1 < \cdots$ is a sequence of positive integers. First we define a partial ordering Q for collapsing κ onto ω_2 and creating stationary sets A_{ni}, B_n, C_{ni} behaving in the way described in Theorem 1 of §9 with only one exception: instead of $D(X) = B_n \cup C_{ni}$ we shall have only $B_n \subset D(X) \subseteq B_n \cup C_{ni}$ for every $n < \omega$, $i < s_n$ and stationary $X \subseteq A_{ni}$. Q is just a version of the forcing notion in §2.

A condition $p = (p0, p1, p2)$ where

$p0$ is a countable function from a part of U_0 into $K = \{(n, i): n < \omega, i < s_n\}$ (the intended meaning of $p0(\alpha) = (n, i)$ is $\alpha \in A_{ni}$),

$p1$ is a countable function from a part of U_1 into $\omega \cup K$ (the intended meaning of $p1(\alpha) = n$ is $\alpha \in B_n$; the intended meaning of $p1(\alpha) = (n, i)$ is $\alpha \in C_{ni}$),

$p2$ is a countable function with $\mathrm{dom}\, p2 = \mathrm{dom}\, p1$, if $\alpha \in \mathrm{dom}\, p1$ and $p1(\alpha) = n$, $p2(\alpha)$ is a closed countable subset of α included into $\{\beta: p0(\beta) = (n, i)$ for some $i < s_n\}$ (the intended meaning is: $p2(\alpha)$ is an initial segment of a club subset of α included in $\bigcup\{A_{ni}: i < s_n\}$); if $\alpha \in \mathrm{dom}\, p1(\alpha)$ and $p1(\alpha) = (n, i)$ then $p2(\alpha)$ is a closed countable subset of α included in $\{\beta: p0(\beta) = (n, i)\}$ (the intended meaning is $p2(\alpha)$ is an initial segment of a club subset of α included in A_{ni}).

p refines q if $q0 \subseteq p0$, $q1 \subseteq p1$ and for every $\alpha \in \mathrm{dom}\, p1$, $p2(\alpha)$ is an end extension of $q2(\alpha)$ in such a way that

$$\min(p2(\alpha) - q2(\alpha)) > \sup(\alpha \cap (\mathrm{dom}\, p0 \cup \mathrm{dom}\, p1)).$$

THEOREM 1. *Suppose G is a Q-generic filter over the ground world V, $A_{ni} = \{\alpha: p0(\alpha) = (n, i)$ for some $p \in G\}$, $B_n = \{\alpha: p1(\alpha) = n$ for some $p \in G\}$, $C_{ni} = \{\alpha: p1(\alpha) = (n, i)$ for some $p \in G\}$ for $n < \omega$, $i < s_n$. In $V[G]$: $\kappa = \omega_2$, and every A_{ni}, B_n, C_{ni} is stationary in ω_2, the sets A_{ni} partition U_0, the sets B_n, C_{ni} partition U_1 and $D(X) \subseteq B_n \cup C_{ni}$ for every $n < \omega$, $i < s_n$ and every stationary $X \subseteq A_{ni}$.*

Theorem 1 is proved in the same way Theorem 2 of §1 was proved.

We would like to have $D(X) = B_n \cup C_{ni}$ for $n < \omega$, $i < s_n$ and stationary $X \subseteq A_n$. In [Ma] Magidor, assuming GCH and existence of a weakly compact cardinal, forced a universe where the derivative of every stationary subset of U_0 is equal to U_1 modulo the ideal I of nonstationary subsets of ω_2. First he collapsed κ onto ω_2, then he started to shoot club subsets of ω_2 through $U_0 \cup D(X)$ where $X \subseteq U_0$ is stationary. We do here almost the same.

Let G, A_{ni}, B_n, C_{ni} be as in Theorem 1. Over $V[G]$ start shooting club subsets of ω_2 through $\omega_2 - (D(A_{ni}) - D(X))$ where $X \subseteq A_{ni}$ is stationary. Iterate the procedure ω_3 times with support of cardinality ω. The proof that the resulting model is the required one is just like that in [Ma, §2] with only one modification: use §§4,5 of our Part I instead of Magidor's Lemma 5 in the proof of claim which appears in the proof of Lemma 4 in §2 of [Ma].

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