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PREFIX CLASSES OF KROM FORMULAE WITH IDENTITY *

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Abstract

Two small classes of first order formulae without function symbols but with identity, in prenex conjunctive normal form with all disjunctions binary, are shown to have a recursively unsolvable decision problem, whereas for another such class an algorithm is developed which solves the decision problem of that class. This solves the prefix problem for such classes of formulae except for the Gödel-Kalmàr-Schütte case.

Zusammenfassung

Für zwei Klassen erststufiger Formeln in pränexer konjunktiver Normalform mit Identität aber ohne Funktionssymbole wird das Entscheidungsproblem als rekursiv unlösbar nachgewiesen. Für eine weitere solche Ausdrucksklasse wird ein Algorithmus zur Lösung des Entscheidungsproblems angegeben. Bis auf den Gödel-Kalmàr-Schütte-Fall löst dies das Präfixproblem für derartige Ausdrucksklassen.

Introduction and Statement of Results

We consider in this paper decision problems for subclasses of $\mathrm{Kr}^{=}$ – the class of all first order formulae in prenex conjunctive normal form with binary disjunctions of signed atomic formulae, including equalities – which are determined by the form of the prefixes of their elements.

For $Q^1, \ldots, Q^n \in \{\wedge, \vee\}$ let $Q^1 \ldots Q^n$ be the class of all closed prenex first order formulae without function symbols whose prefix has the form $\underset{x_1}{Q^1} \ldots \underset{x_n}{Q^n}$. Our results may then be stated as follows.

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Theorem 1. *The class* $\wedge\vee\wedge\vee\cap\mathrm{Kr}^{=}$ *is a conservative reduction class with respect to satisfiability*[1].

Theorem 2. *The class* $\wedge\wedge\wedge\vee\cap\mathrm{Kr}^{=}$ *is a reduction class with respect to satisfiability.*

Theorem 3. *The class* $\wedge\vee\wedge\cap\mathrm{Kr}^{=}$ *has a recursively solvable decision problem with respect to satisfiability.*

The decision problem for the classes $\wedge\vee\wedge\vee^{k}\cap\mathrm{Kr}$ seems still to be open – where Kr ("Krom" formulae) is the subclass of all formulae in $\mathrm{Kr}^{=}$ without identity –, the undecidability result of Theorem 2 is in contrast to the recursivity of the decision problem for the class $\vee\ldots\vee\wedge\ldots\wedge\vee\ldots\vee\cap\mathrm{Kr}$ (see Maslov, 1964). Theorem 3 extends the decidability proof for $\wedge\vee\wedge\cap\mathrm{Kr}$ given in Aanderaa and Lewis (1973).

Together with the decidability of $\vee\ldots\vee\wedge\ldots\wedge$ and of $\vee\ldots\vee\wedge\vee\ldots\vee$ [the latter case even with one monadic function symbol allowed, see Shelah (1977)] and the undecidability of the classes $\vee\wedge\vee\wedge\cap\mathrm{Kr}$, $\wedge\vee\vee\wedge\cap\mathrm{Kr}$ (see Aanderaa, 1971; Börger, 1971), $\wedge\wedge\vee\wedge\cap\mathrm{Kr}$, $\wedge\vee\wedge\wedge\cap\mathrm{Kr}$ (see Aanderaa and Lewis, 1973), the results of Theorems 1–3 yield a solution for the prefix problem for Krom classes with identity except for the restrictions of the Gödel-Kalmàr-Schütte case $\vee\ldots\vee\wedge\wedge\vee\ldots\vee\cap\mathrm{Kr}^{=}$ to Krom formulae.

Theorems 1 and 2 are proved by reducing to the two classes in question the undecidable classes $\vee\wedge\vee\wedge\cap\mathrm{Kr}$, resp. $\wedge\vee\wedge\wedge\cap\mathrm{Kr}$. This reduction is based on an "axiomatisation" of the number zero resp. the successor relation on the natural numbers. The algorithm exhibited for the proof of Theorem 3 uses and extends Aanderaa's and Lewis' algorithm for the same case without identity.

0. Prerequisites and Notations

The *decision problem* of a class F of first order formulae refers to the question whether there exists an algorithm deciding for every formulae in F whether it is satisfiable or not; formally it is taken as subclass of all those formulae in F which are not contradictory. F is called a *(conservative) reduction class* – always with respect to satisfiability – iff there is a recursive function associating to every first order formulae α a formula $\bar{\alpha}$ in F such that α is satisfiable iff $\bar{\alpha}$ (and α is finitely satisfiable iff $\bar{\alpha}$ is). By Gurevich (1976a) it is already sufficient to have α satisfiable if $\bar{\alpha}$ is, and $\bar{\alpha}$ finitely satisfiable if α is finitely satisfiable, to show that F is a conservative reduction class. By $\alpha(x_1|t_1,\ldots,x_n|t_n)$ we denote the result of the simultaneous substitution of the variables x_i by the terms $t_i\cdot f^n(a)$ denotes

$$\underbrace{f(\ldots f}_{n\text{-times}}(a)\ldots).$$

[1] For unexplained terminology we refer the reader to Section 0.

1. Proof for Theorems 1 and 2

For the proof of Theorem 1 it is sufficient to reduce the decision problem of the conservative reduction class $\vee \wedge \vee \wedge \cap \mathrm{Kr}$ (see Aanderaa, 1971; Börger, 1971) to the decision problem of the class $\wedge \vee \wedge \vee \cap \mathrm{Kr}^{=}$. This reduction is given in the following

Lemma 1. *For an arbitrarily formula* $\bigvee_u \bigwedge_x \bigvee_v \bigwedge_y \alpha \in \vee \wedge \vee \wedge$ *let* Z *be a new monadic predicate letter and define* α' *by*

$$\alpha' \equiv \alpha \wedge Zu \wedge (Zx \rightarrow x = u) \wedge (Zy \rightarrow y = u).$$

Then the following properties hold:

a) *if* $\bigwedge_x \bigvee_v \bigwedge_y \bigvee_u \alpha'$ *is satisfiable, then also* $\bigvee_u \bigwedge_x \bigvee_v \bigwedge_y \alpha$ *is satisfiable;*

b) *if* $\bigvee_u \bigwedge_x \bigvee_v \bigwedge_y \alpha$ *is finitely satisfiable, then also* $\bigwedge_x \bigvee_v \bigwedge_y \bigvee_u \alpha'$ *is finitely satisfiable;*

c) $\alpha \in \mathrm{Kr}$ *iff* $\alpha' \in \mathrm{Kr}$.

Proof of Lemma 1. Property c) is obvious from the definition of α'. The reduction property a) follows from the fact that α' implies $((Zx \wedge Zy) \rightarrow x = y) \wedge Zu$, showing u to be unique and independent from x or y.
Reduction property b) is obvious.

This concludes the proof of Lemma 1 and by Gurevich (1976a) therefore also of Theorem 1. For the proof of Theorem 2 we proceed in the same way by reducing to the class under consideration the class $\wedge \vee \wedge \wedge \cap \mathrm{Kr}$ which is known to be a reduction class, see Aanderaa and Lewis (1973). The reduction is effected by the following

Lemma 2. *For an arbitrary formula* $\bigwedge_x \bigvee_v \bigwedge_y \bigwedge_z \alpha \in \wedge \vee \wedge \wedge$ *let* S *be a new binary predicate symbol and define* α' *by*

$$\alpha' \equiv \alpha \wedge Sxv \wedge (Sxy \rightarrow y = v) \wedge (Sxz \rightarrow z = v).$$

Then the following properties hold:

a) $\bigwedge_x \bigwedge_y \bigwedge_z \bigvee_v \alpha'$ *is satisfiable iff* $\bigwedge_x \bigvee_v \bigwedge_y \bigwedge_z \alpha$ *is satisfiable;*

b) $\alpha' \in \mathrm{Kr}$ *iff* $\alpha \in \mathrm{Kr}$.

Proof of Lemma 2. Property b) is obvious from the definition of α'. The reduction property a) from right to left follows from the fact that by the axiom of choice for every x a *unique* v can be chosen so that $Sxv \wedge (Sxy \rightarrow y = v) \wedge (Sxz \rightarrow z = v)$ is fulfilled by interpreting S as graph of the choice function. The reduction property a) from left to right follows from the fact that α' implies

$$Sxv \wedge (Sxy \wedge Sxz \rightarrow y = z)$$

expressing that v depends only from x and not really also from y or z. This concludes the proof for Lemma 2 and therefore for Theorem 2 (see Gurevich, 1976a).

2. Proof for Theorem 3

The algorithm designed by Aanderaa and Lewis (1973) for the case $\wedge \vee \wedge \cap \mathrm{Kr}$ cannot be used directly for formulae with equality because it is based on Krom's chain criterion which does not hold for formulae with identity. Indeed the set of formulae

$$x = y \qquad x = u \vee x = w \qquad x \neq y \vee x \neq u \qquad u = w$$

is contradictory but admits no pair of chains as requested in Krom's chain criterion for identity free Krom formulae. In the following we show an argument which enables to use the Aanderaa-Lewis decision procedure also for the case with identity.

The idea of the proof is to isolate first if the given formulae $\alpha \in \wedge \vee \wedge \cap \mathrm{Kr}^{=}$ has infinite models by reducing this problem to the satisfiability question for a variant α_1 of α in the same class but without equality (Lemma 1). It is then shown that if α admits no infinite models, then one can calculate a number $c(\alpha)$ such that if α is satisfiable at all, then it is satisfiable over a domain with at most $c(\alpha)$ elements (Lemma 2). This yields the desired decision procedure for $\wedge \vee \wedge \cap \mathrm{Kr}^{=}$.

Definition. For arbitrary $\alpha \equiv \bigwedge_x \bigvee_v \bigwedge_y \beta \in \wedge \vee \wedge \cap \mathrm{Kr}^{=}$ define:

$$\alpha_0 \equiv \bigwedge_x \bigvee_v \bigwedge_y \beta_0 \qquad \beta_0 \equiv \beta(s = t / Ist) \wedge Ixx \wedge (Ixy \rightarrow Iyx)$$

with a new binary predicate symbol I and $\beta(s = t/Ist)$ resulting from β by substituting every equality $s = t$ by Ist;

$$\alpha_1 \equiv \bigwedge_x \bigvee_v \bigwedge_y \beta_1 \qquad \beta_1 \equiv \beta_0 \wedge (Iyx \rightarrow Kyv) \wedge (Kyx \rightarrow Kyv) \wedge \neg Kxx$$

for a new binary predicate symbol K (intended as substitute of the relation $<$). Clearly $\alpha_1 \in \wedge \vee \wedge \cap \mathrm{Kr}$.

Lemma 1. *If α_1 is satisfiable, then α is satisfiable.*

Lemma 2. *There is a partial recursive function associating to every* $\alpha \in \wedge \vee \wedge \cap \mathrm{Kr}^{=}$ *with* α_1 *contradictory a number* $c(\alpha)$ *such that if* α *is satisfiable, then it can be satisfied over a domain with not more than* $c(\alpha)$ *elements.*

From Lemmas 1 and 2 a *decision procedure for* $\wedge \vee \wedge \cap \mathrm{Kr}^{=}$ can be described as follows: first decide if α_1 is satisfiable by using the algorithm from Aanderaa and Lewis (1973) for $\wedge \vee \wedge \cap \mathrm{Kr}$. If α_1 is satisfiable, then by Lemma 1 also α is. If α_1 is contradictory, calculate $c(\alpha)$ and check if α can be fulfilled over a domain with at most $c(\alpha)$ elements. If this is not the case, then α is contradictory by Lemma 2.

Proof of Lemma 1. If α_1 is satisfiable, then it has a cannonical model $\langle N; \mathfrak{I}, \mathfrak{K}, \ldots\rangle$ i.e. with domain N of all natural numbers, where for every choice a for x, v can be chosen as $a+1 \in N$ (see for Example Kreisel and Krivine, 1967, pp. 18–20). The interpretation $\mathfrak{I}$ of I must be the equality – consequently α is satisfied by the same model –. In fact the assumption that $\mathfrak{I}(m,n)$ is true for some $m \neq n$ yields the truth of $\mathfrak{I}(m,n)$ for some $n<m$ (by symmetry of I), therefore the truth of $\mathfrak{K}(m,n+1)$ and consequently of $\mathfrak{K}(m,n+1)$, $\mathfrak{K}(m,n+2), \ldots, \mathfrak{K}(m,m)$ contradicting $\neg Kxx$ in β_1.

Remark. α_1 cannot be satisfied over a finite domain. It is satisfiable (and therefore over infinite domains) iff α can be satisfied over an infinite domain.

Proof of Lemma 2. Let $\alpha_0^f \equiv \bigwedge_x \bigwedge_y \beta_0(v/fx)$ be the Skolem normal form for α_0. Assume that α_1 is contradictory.
Then the set

$$\{\alpha_0^f\} \cup \left\{\bigwedge_x \neg Ixf^n x \,|\, n \in N,\, n \neq 0\right\},$$

where $f^n x \equiv \underbrace{ff\ldots f}_{r\text{-times}}x$ is contradictory; for otherwise it could be satisfied by a canonical model over N with f interpreted as successor function and I as equality so that α_1 could be satisfied by the same model with K interpreted as $<$-relation. By the compactness theorem of predicate logic there exists then a number n_0 such that already

$$\alpha_0^f \wedge \bigwedge_x \neg Ixfx \wedge \bigwedge_x \neg Ixf^2 x \wedge \ldots \wedge \bigwedge_x \neg Ixf^{n_0} x$$

is contradictory. It follows from the completeness theorem that

$$\alpha_0^f \rightarrow \bigvee_x Ixfx \vee \bigvee_x Ixf^2 x \vee \ldots \vee \bigvee_x Ixf^{n_0} x$$

is a theorem of predicate logic and therefore also

$$\alpha_0^f(Ist/s=t) \rightarrow \bigvee_x x=fx \vee \bigvee_x x=f^2 \vee \ldots \vee \bigvee_x x=f^{n_0} x$$

and

$$\alpha^f \to \bigvee_x x = fx \vee \bigvee_x x = f^2 x \vee \ldots \vee \bigvee_x x = f^{n_0} x$$

with the Skolem normal form α^f of α. Such a number n_0 can be calculated effectively from α^f. The deducibility of the last mentioned formula means that every model of α^f contains a submodel of cardinality $\leq n_0$. Since α^f is satisfiable iff α is, this proves the claim of Lemma 2 with $c(\alpha) := n_0$.

Remark to the Proof for Theorem 3. Another proof for Theorem 3 follows from Lemma 1 of this paragraph and the following obvious

Lemma 3. *If α is satisfiable in an infinite domain, then α_1 is satisfiable.*
Indeed by Lemma 1, α is satisfiable if α_1 is.
If α_1 is contradictory, then by Lemma 3 α either is contradictory too or it is finitely satisfiable. Since both these properties are recursively enumerable we can decide recursively which of these two cases occurs.

Concluding Remarks. The idea to formalize, using identity, the role of the natural number zero and of the successor relation among natural numbers as far as they are used for and together with the description of 2-register machine programs by Krom formulae to get small undecidable Krom classes grew out from discussions between the last two authors in Münster in February 1978. It is not known if $\wedge \vee \wedge \wedge \cap \mathrm{Kr}$ is conservative or not, whereas all other prefix classes are known to be conservative reduction classes if they are known to be undecidable.
The undecidability of $\wedge \vee \wedge \wedge \cap \mathrm{Kr}$ depends from the fact that ternary predicates have been used. In fact the restriction of $\wedge \vee \wedge \ldots \wedge \cap \mathrm{Kr}$ to formulae with only binary predicate symbols can be reduced to $\wedge \vee \wedge \cap \mathrm{Kr}$ and is therefore decidable (see Börger, 1973). By using this latter reduction, Theorem 3 – which came out from discussions between the first two authors in Oslo in June 1978 – shows also the decidability of the restriction of $\wedge \vee \wedge \ldots \wedge \cap \mathrm{Kr}^=$ to formulae with only binary predicate symbols.

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Added in proof: The above mentioned question about conservativity of the class $\bigwedge\bigvee\bigwedge\bigwedge\cap$Kr has been solved in the meanwhile, see S.O. Aanderaa, E. Börger, and H.R. Lewis: Conservative Reduction Classes of Krom Formulas. in: The Journal of Symbolic Logic (to appear).

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