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MODEST THEORY OF SHORT CHAINS. I

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Abstract. This is the first part of a two part work on the monadic theory of short orders (embedding neither ω_1 nor ω_1*). This part provides the technical groundwork for decidability results. Other applications are possible.

§0. Introduction. A *chain* is a linearly ordered set. A chain is short iff it embeds neither ω_1 nor ω_1^*. A chain is *Specker* iff it does not embed any uncountable subchain of the real line. It is easy to see that the property of shortness is expressible in the monadic (second-order) language of order.

Rabin proved in [Ra] decidability of the monadic theory of the chain Q of the rational numbers. Shelah gave in [Sh] a method (a kind of elimination of quantifiers) and proved by this method among other results that every short Specker chain without jumps and end-points is monadically equivalent to Q. It is well known that there exist uncountable chains without jumps and end-points which are short and Specker. By [Sh], there exists also a non-Specker chain monadically equivalent to Q. Thus the monadic theory of Q is not categoric in the monadic logic. Shelah conjectured in [Sh] that this theory is finitely axiomatizable in the monadic logic.

Working on this conjecture I come to the notion of *p-modest* chains where p is a positive integer. The exact definition may be found in [Gu] or [GS]. In the following it is important that p-modesty is expressible in the monadic language of order. By [Gu], a chain is monadically equivalent to Q iff it is short, has neither jumps nor end-points and is p-modest for every p. Under a consequence of the Continuum Hypothesis it is proved in [Gu] that the monadic theory of Q is not finitely axiomatizable in the monadic logic.

The notion of modesty led the authors to some nonimprovable decidability results about short chains. In order to obtain these results we elaborate here on Shelah's method mentioned above.

In particular we give two finite versions of the Feferman-Vaught Theorem. They and their variations seem to be very applicable.

§1. Finte fragments of theories. The subject of this section is the notion of $n-\xi$-theories. We extract this notion from [Sh]. It generalizes the notion of n-theories used by Läuchli in [Lä]. The original idea is due to Fraïsse (and was rediscovered by Taimanov).

Any theory in this paper is first-order if the contrary is not stated explicitly. With

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each theory T we associate a set of formulae in the language of T called *pseudo-atomic*.

DEFINITION 1.1. A theory T is *admissible* iff (i) the set of pseudo-atomic formulae in variables $v_1, \ldots, v_l$ (i.e. with free variables among $v_1, \ldots, v_l$) is finite and recursive in l, and (ii) there exists an algorithm associating a pseudo-atomic formula φ^* with each atomic formula φ in such a way that φ and φ^* are equivalent in T.

The following are some examples.

(1) Let T be a theory with only a finite number of predicate and individual constants and without functional constants. If the set of pseudo-atomic formulae coincides with the set of atomic formulae in the language of T then T is admissible.

(2) The theory of Boolean algebras with ordinary operations is admissible under an appropriate definition of pseudo-atomic formulae.

(3) Recall that a set ring is an arbitrary set of sets closed under union, intersection and subtraction. We regard set rings as models for the first-order language with the identity sign, individual constant 0 and dyadic functional symbols for union, intersection and subtraction. Each term s in variables $v_1, \ldots, v_l$ is equal in the theory of set rings to a term of the form $\bigcup_j \bigcap_i s_{ji}$ where s_{ji} is either v_i or $((v_1 \cup \cdots \cup v_l) - v_i)$. Thus the theory of set rings is admissible under an appropriate definition of pseudo-atomic formulae.

Below, T is an admissible theory, M is a model of T, $a = \langle a_1, \ldots, a_{\mathrm{lh}(a)} \rangle$ and $b = \langle b_1, \ldots, b_{\mathrm{lh}(b)} \rangle$ are finite sequences of elements of M, $\xi = \langle \xi n\colon 0 \le n < \mathrm{lh}(\xi) \rangle$ is a finite sequence of natural numbers and $\bar{\xi} = \langle \bar{\xi} n\colon n < \omega \rangle$ is an infinite sequence obtained from ξ by adding a tail of 0's. Of course here lh means length, so $\bar{\xi} n = \xi n$ if $n < \mathrm{lh}(\xi)$ and $\bar{\xi} n = 0$ otherwise. The sign $\frown$ will denote the concatenation of sequences. The power set of a set X will be denoted by PS(X).

DEFINITION 1.2 (FORMAL $n - \xi$-THEORIES). $T^0(l)$ is the set of pseudo-atomic formulae of T in $v_1, \ldots, v_l$. $T^0_\xi(l) = T^0(l)$, $T^{n+1}_\xi(l) = \mathrm{PS}(T^n_\xi(l + \bar{\xi} n))$. $T^n(\xi, l) = T^n_\xi(l)$.

DEFINITION 1.3. $\mathrm{Th}^0(M, a)$ is the set of formulae $\varphi(v_{i_1}, \ldots, v_{i_m})$ in $T^0(\mathrm{lh}(a))$ such that $\varphi(a_{i_1}, \ldots, a_{i_m})$ holds in M. $\mathrm{Th}^0_\xi(M, a) = \mathrm{Th}^0(M, a)$, $\mathrm{Th}^{n+1}_\xi(M, a) = \{\mathrm{Th}^n_\xi(M, a \frown b)\colon \mathrm{lh}(b) = \bar{\xi} n\}$. $\mathrm{Th}^n(\xi, M, a) = \mathrm{Th}^n_\xi(M, a)$.

$\mathrm{Th}^n_\xi(M, a)$ (respectively $\mathrm{Th}^0(M, a)$) will be called the *$n - \xi$-theory* (respectively *0-theory*) of the augmented model $\langle M, a \rangle$. One may call it also the *$n - \xi$-type* (respectively *0-type*) of a in M. Two augmented models $\langle M, a \rangle$ and $\langle N, c \rangle$ will be called *$n - \xi$-equivalent* (respectively *0-equivalent*) iff they have the same $n - \xi$-theory (respectively 0-theory).

$n - \xi$-equivalency can be characterized by a variant of the Eherenfeucht's Game Criterion given in [Eh].

LEMMA 1.1. (i) *$T^n_\xi(l)$ is hereditarily finite and computable from n, $\bar{\xi}|n$ and l. Conversely, n, $l + \Sigma_{i<n} \bar{\xi}|i$ are computable from $T^n_\xi(l)$.*

(ii) $\mathrm{Th}^n_\xi(M, a) \subset T^n_\xi(\mathrm{lh}(a))$.

(iii) $\mathrm{Th}^n_\xi(M, a)$ *is computable from each of* $\mathrm{Th}^n_\xi(M, a \frown b)$ *and* $\mathrm{Th}^{n+1}_\xi(M, a)$.

(iv) $\mathrm{Th}^n_\xi(M, \langle a_{f1}, \ldots, a_{fl} \rangle)$ *is computable from* $\mathrm{Th}^n_\xi(M, \langle a_1, \ldots, a_l \rangle)$ *and a permutation f of* $1, \ldots, l$.

The property $\mathrm{Th}^n_\xi(M, a) = t$ is elementary for every $t \subset T^n_\xi(\mathrm{lh}(a))$. For each formula φ in $v_1, \ldots, v_{\mathrm{lh}(a)}$ there exist n and ξ such that the truth value of $\varphi(a)$ in M can be computed from $\mathrm{Th}^n_\xi(M, a)$. We clarify these facts.

DEFINITION 1.4. $\Gamma_\xi^0(l)$ is the collection of Boolean combinations of formulae in $T^0(l)$. $\Gamma_\xi^{n+1}(l)$ is the collection of Boolean combinations of formulae $\exists v_{l+1} \cdots \exists v_{l+\bar{\xi}n}\psi$ where $\psi \in \Gamma_\xi^n(l + \bar{\xi}n)$. $\Gamma_\xi^n(M, a)$ is the collection of formulae $\varphi(v_{i_1}, \ldots, v_{i_m})$ in $\Gamma_\xi^n(\mathrm{lh}(a))$ such that $\varphi(a_{i_1}, \ldots, a_{i_m})$ holds in M.

THEOREM 1.2. (i) *The truth value of* "$\varphi \in \Gamma_\xi^n(M, a)$" *is computable from* $\mathrm{Th}_\xi^n(M, a)$.

(ii) *There exists an algorithm associating a formula* $\varphi_t \in \Gamma_\xi^n(l)$ *with each* $t \subset T_\xi^n(l)$ *in such a way that for every* M *and* $a = \langle a_1, \ldots, a_l\rangle$, $t = \mathrm{Th}_\xi^n(M, a)$ *iff* $\varphi_t(a)$ *holds in* M.

PROOF. (i) The case $n = 0$ is clear. Let $\mathrm{lh}(a) = l$ and $\varphi \in \Gamma_\xi^{n+1}(l)$. W.l.o.g., $\varphi = \exists v_{l+1} \cdots \exists v_{l+\bar{\xi}n}\psi$ where $\psi \in \Gamma_\xi^n(l + \bar{\xi}n)$. By the induction hypothesis, for each b with $\mathrm{lh}(b) = \bar{\xi}n$, the truth value of "$\psi \in \Gamma_\xi^n(M, a^\frown b)$" can be computed from $\mathrm{Th}_\xi^n(M, a^\frown b)$. Hence the truth value of "$\varphi \in \Gamma_\xi^{n+1}(l)$" can be computed from $\{\mathrm{Th}^n(a^\frown b)\colon \mathrm{lh}(b) = \bar{\xi}n\}$.

(ii) If $t \subset T^0(l)$ then φ_t is the conjuction of (a) the formulae in t and (b) the negations of the formulae in $T^0(l) - t$. Let $t \subset T_\xi^{n+1}(l)$. Then φ_t is the conjunction of (a) the formulae $\exists v_{l+1} \cdots \exists v_{l+\bar{\xi}n}\varphi_s$ where $s \in t$ and (b) the negations of the formulae $\exists v_{l+1} \cdots \exists v_{l+\bar{\xi}n}\varphi_s$ where $s \in T_\xi^{n+1}(l) - t$. Q.E.D.

THEOREM 1.3. *Suppose that for some n*, $\mathrm{Th}_\xi^{n+1}(M, a)$ *is computable from* $\mathrm{Th}_\eta^n(M, a)$ *where* $\eta = \eta(\xi, \mathrm{lh}(a))$ *is recursive. Then there exists a recursive function* $\zeta(m, \xi, l)$ *such that* $\mathrm{Th}_\xi^{n+m}(M, a)$ *is computable from* $\mathrm{Th}^n(\zeta(m, \xi, \mathrm{lh}(a)), M, a)$.

PROOF. Without loss of generality, $\eta(\xi, l)$ is always of length n. Define $\zeta(1, \xi, l) = \eta(\xi, l)$ and $\zeta(m + 1, \xi, l) = \eta(\zeta(m, \xi, l + \bar{\xi}(n + m))^\frown \langle\bar{\xi}(n + m)\rangle, l)$ if $m \geq 1$. By induction on m we construct an algorithm F_m computing $T_\xi^{n+m}(M, a)$ from $\mathrm{Th}^n(\zeta(m, \xi, \mathrm{lh}(a)), M, a)$. The construction is uniform in m and gives the desired algorithm.

By the condition of the theorem F_1 is given. Let $l = \mathrm{lh}(a)$, $\alpha = \zeta(m, \xi, l + \bar{\xi}(n + m))$ and $\beta = \alpha^\frown\langle\bar{\xi}(n + m)\rangle$. Then

$$\begin{aligned}\mathrm{Th}_\xi^{n+m+1}(M, a) &= \{\mathrm{Th}_\xi^{n+m}(M, a^\frown b)\colon \mathrm{lh}(b) = \bar{\xi}(n + m)\}\\ &= \{F_m(\mathrm{Th}_\alpha^n(M,\ a^\frown b)\colon\ \mathrm{lh}(b) = \bar{\xi}(n + n)\} = \{F_m(t)\colon\\ &\quad t \in \mathrm{Th}_\beta^{n+1}(M, a)\}\\ &= \{F_m(t)\colon\ t \in F_1(\mathrm{Th}^n(\eta(\beta, l), M, a))\}\\ &= F_{m+1}(\mathrm{Th}^n(\zeta(m + 1, \xi, l), M, a)). \qquad \text{Q.E.D.}\end{aligned}$$

If the universal fragment of T is decidable it is convenient to use the following definition.

DEFINITION 1.5. $\mathrm{Tr}^0(T, l)$ is the set of $\mathrm{Th}^0(M, a)$ where M is a model of T and a is a sequence of elements of M of length l. $\mathrm{Tr}_\xi^0(T, l) = \mathrm{Tr}^0(T, l)$ and $\mathrm{Tr}_\xi^{n+1}(T, l) = \mathrm{PS}(\mathrm{Tr}_\xi^n(T,\ l + \bar{\xi}n))$. An element s of $\mathrm{Tr}^0(T, l)$ is a *trace* (of length l) of $t \in \mathrm{Tr}_\xi^n(T, l + m)$ iff $n = 0$ and $s = t \cap T^0(l)$ or $n > 0$ and s is a trace of every $t' \in t$.

LEMMA 1.4. *Suppose that* $\mathrm{Tr}^0(T, l)$ *is computable from l. Then* $\mathrm{Tr}_\xi^n(T, l)$ *is computable from n*, ξ, *and l; and the trace of length l of* $t \in \mathrm{Tr}_\xi^n(T, l + m)$ *is computable from l and t.*

PROOF. Clear.

In the remaining part of this paper we suppose that the nonlogical constants of the theory T in question are linearly ordered. This imposes a lexicographic order on every $T^0(l)$ and every $T_\xi^n(l)$, $\mathrm{Tr}_\xi^n(T, l)$.

§2. Finite versions of the Feferman-Vaught Theorem. Let T, T', T'' be admissible theories. Suppose that T has an individual constant 0 and T' extends the theory of set rings.

Let $\{M_i: i \in I\}$ be a nonempty set of models of T, J be a model of T' expanding a ring of subsets of I (to the language of T'), and M be a model of T'' whose universe is a subset of the cartesian product Π of universes of models M_i. Suppose that for every $x \in \Pi$, $x \in M$ iff $\{i: x(i) \neq 0\} \in J$.

Let f, g be finite sequences of elements of M, $l = \mathrm{lh}(f)$ and $f(i) = \langle f_1(i), \ldots, f_l(i)\rangle$ for $i \in I$.

First version. Suppose that J contains every subset of I.

DEFINITION 2.1. $[n, \xi, f]$ is the sequence $\langle [n, \xi, f]_t: t \subset T^n_\xi(l)\rangle$ where each $[n, \xi, f]_t = \{i: \mathrm{Th}^n_\xi(M_i, f(i)) = t\}$.

LEMMA 2.1. $\{[n, \xi, f\hat{\ }g]: \mathrm{lh}(g) = \bar{\xi}n\}$ *coincides with the set of sequences* $R = \langle R_s: s \subset T^n_\xi(l + \bar{\xi}n)\rangle$ *of elements of J such that R is pairwise disjoint and* $s \in t$ *whenever* R_s *meets* $[n + 1, \xi, f]_t$.

PROOF. Given R we build g such that $R = [n, \xi, f\hat{\ }g]$. If $i \in R_s \cap [n + 1, \xi, f]_t$ take $g(i)$ satisfying $\mathrm{Th}^n_\xi(M_i, f(i)\hat{\ }g(i)) = s$. Q.E.D.

THEOREM 2.2. *There exists a recursive function* $F(n, \xi, l)$ *such that* $\mathrm{Th}^n_\xi(M, f)$ *is computable from* $\mathrm{Th}^n F((n, \xi, l), J, [n, \xi, f])$ *providing that* $\mathrm{Th}^0(M, g)$ *is computable from* $\mathrm{Th}^0(J, [0, 0, g])$.

PROOF. Let $F(0, \xi, l)$ be the empty sequence and

$$F(n + 1, \xi, l) = F(n, \xi, l + \bar{\xi}n)\hat{\ } \langle m\rangle$$

where $m = |T^n_\xi(l + \bar{\xi}n)|$.

Let $\eta = F(n, \xi, l + \bar{\xi}n)$ and $\zeta = F(n + 1, \xi, l)$. It is sufficient to check that $\{\mathrm{Th}^n_\eta(J, [n, \xi, f\hat{\ }g]: \mathrm{ln}(g) = \bar{\xi}n\}$ is computable from $\mathrm{Th}^{n+1}_\xi(J, [n + 1, \xi, f])$. The latter is the set of $\mathrm{Th}^n_\eta(J, [n + 1, \xi, f]\hat{\ }R)$ where $R = \langle R_s: s \subset T^n_\xi(l + \bar{\xi}n)\rangle$. Now use Lemma 2.1. Q.E.D.

Second version. We write $\mathrm{Tr}^n_\xi(m)$ instead of $\mathrm{Tr}^n_\xi(T, m)$. Suppose that $\mathrm{Tr}^0(m)$ is computable from m.

Suppose that T has the identity sign and that there exists a distinguished model of T comprising exactly two elements. A model of T will be called *trivial* iff it is isomorphic to the distinguished model. If N is a model of T and a is a finite sequence of elements of N then the augmented model $\langle N, a\rangle$ is *trivial* iff N is trivial and a is a sequence of zeros. An element of $\mathrm{Tr}^n_\xi(m)$ is *trivial* iff it is $n - \xi$-theory of a trivial structure.

We suppose that J contains every subset of $I^* = \{i: M_i \text{ is not trivial}\}$.

DEFINITION 2.2. $[n, \xi, f]$ is the sequence $\langle [n, \xi, f]_t: t \in \mathrm{Tr}^n_\xi(l)\rangle$ where $[n, \xi, f]_t = \{i: \mathrm{Th}^n_\xi(M_i, f(i)) = t\}$ if t is not trivial and $[n, \xi, f]_t = 0$ if t is trivial.

LEMMA 2.3. *Every* $[n, \xi, f]_t$ *belongs to J.*

PROOF. It is sufficient to check that $X = [n, \xi, f]_t - I^*$ belongs to J. There exists $S \subset \{1, \ldots, l\}$ such that for each $i \in I - I^*$, $\mathrm{Th}^n_\xi(M_i, f(i)) = t$ iff $\{k: f_k(i) \neq 0\} = S$. If $X_k = \{i: f_k(i) \neq 0\} - I^*$ then $X = \bigcap\{X_k: k \in S\} - \bigcup\{X_k: k \notin S\}$. Q.E.D.

LEMMA 2.4. $\{[n, \xi, f\hat{\ }g]: \mathrm{lh}(g) = \bar{\xi}n\}$ *coincides with the set of sequences* $R = \langle R_s: s \in \mathrm{Tr}^n_\xi(l + \bar{\xi}n)\rangle$ *of elements of J such that*:

(i) *R is pairwise disjoint*,

(ii) $R_s = 0$ *if s is trivial,*

(iii) *each* $[n + 1, \xi, f]_t \subset \bigcup\{R_s : s \in t\}$,

(iv) *if* R_s *does not meet any* $[n + 1, \xi, f]_t$ *then s belongs to the trivial element of* $\mathrm{Tr}_\xi^{n+1}(l)$.

PROOF. It is clear that every $[n, \xi, f^\frown g]$ with $\mathrm{lh}(g) = \bar{\xi}n$ satisfies (i)–(iv).

Given R satisfying (i)–(iv) we build g such that $[n, \xi, f^\frown g] = R$. If $i \in R_s$ take $g(i)$ satisfying $\mathrm{Th}_\xi^n(M_i, f(i)^\frown g(i)) = s$; if i does not belong to any R_s let $g(i)$ be the sequence of zeros in M_i of length $\bar{\xi}n$. We have to check only that for each k, $X_k = \{i : g_k(i) \neq 0\} \in J$. It is easy to see that X_k is a union of sets R_s. Q.E.D.

DEFINITION 2.3. $T(0, \xi, l)$ is the empty sequence

$$T(n + 1, \xi, l) = T(n, \xi, l + \bar{\xi}n)^\frown \langle m \rangle$$

where $m = |\mathrm{Tr}_\xi^n(l + \bar{\xi}n)|$.

THEOREM 2.5. *Suppose that* $\mathrm{Th}^0(M, g)$ *is computable from* $\mathrm{Th}^0(J, [0, 0, g])$. *Then* $\mathrm{Th}_\xi^n(M, f)$ *is computable from* $\mathrm{Th}_\eta^n(J, [n, \xi, f])$ *where* $\eta = T(n, \xi, l)$.

The proof is similar to that of Theorem 2.2.

§3. Relevancy.

Terminology. A chain is a linearly ordered set. Each chain is regarded to be equipped with the interval topology. An interval is always open and nonempty. *A* subset of a chain is *convex* iff it is not empty and together with every two points it contains all points between them. Two points form a *jump* iff they are different and there is no point between them. A chain is *complete* iff every bounded point set has a supremum and an infimum. An equivalence relation E on a chain M is a *congruence* iff every equivalence class of E is convex. The corresponding quotient chain is denoted by M/E.

DEFINITION 3.1. *A* chain is *short* iff it embeds neither ω_1 nor ω_1^*.

DEFINITION 3.2. Let K be a class of short chains and Rl associate a ring $\mathrm{Rl}(M)$ of subsets of M with each chain M in K. The pair $\langle K, \mathrm{Rl}\rangle$ is *nice* iff it satisfies the following conditions (N1)–(N7).

(N1) K is closed under convex subchains and homomorphic images. There exists a chain in K without jumps and endpoints.

(N2) Either all chains in K without jumps and endpoints are complete or none of them are.

(N3) $\mathrm{Rl}(M)$ contains every finite subset of M. If X is a convex subchain of M then $\mathrm{Rl}(X) = \{X \cap Y : Y \in \mathrm{Rl}(M)\}$.

(N4) Let h be a homomorphism from a chain $M \in K$ onto a chain $N \in K$. If $X \in \mathrm{Rl}(M)$ then $h(X) \in \mathrm{Rl}(N)$. If $X \subset M$ and $h(X) \in \mathrm{Rl}(N)$ and $X \cap h^{-1}(y) \in \mathrm{Rl}(M)$ for every $y \in N$ then $X \in \mathrm{Rl}(M)$.

(N5) If E is a congruence on $M \in K$ then $\mathrm{Rl}(M/E)$ contains every subset of $\{X \in M/E : |X| > 1\}$.

(N6) If $M \in K$ has neither jumps nor endpoints, $X \subset M$ is everywhere dense and either X or $M - X$ belongs to $\mathrm{Rl}(M)$, then there exists $Y \in \mathrm{Rl}(M)$ such that $Y \subset X$ and both Y and $X - Y$ are everywhere dense.

(N7) $M \in \mathrm{Rl}(M)$ either for every chain M in K without jumps and endpoints or for no one of them.

Note. Rl is actually a functor from a category of chains to the category of set rings (in each case morphisms are the corresponding homomorphisms onto).

In the rest of this paper $\langle K, \mathrm{Rl}\rangle$ is a nice pair. *A* subset X of $M \in K$ will be called *relevant* iff $X \in \mathrm{Rl}(M)$.

LEMMA 3.1. (a) *K contains all finite chains, ω(the chain of natural numbers) and ω^* (the chain dual to ω).*

(b) *Every subset of ω (respectively, ω^*) is relevant.*

PROOF. (a) Use (N1).

(b) Use (N5).

In the rest of this paper our subject is the theory of chains in K with quantification over relevent subsets. For technical reasons (see Lemma 3.4) we use the following language.

Let L be the language of set rings enriched by a dyadic predicate constant $\leq$ and individual constants c_0, c_1, c_2. Each $M \in K$ forms a *natural model* for L in the following way: the variables range over the relevant subsets of M, $\leq$ is the induced order of singletons, $c_0 = M$ if $M \in \mathrm{Rl}(M)$ and $c_0 = 0$ otherwise, $c_1 = \{\min M\}$ (respectively $c_2 = \{\max M\}$) if M has a minimal (respectively maximal) point and $c_1 = 0$ (respectively $c_2 = 0$) otherwise. Let T be the theory of the above described natural models.

LEMMA 3.2. *T is admissible under an appropriate definition of pseudo-atomic formulae.*

PROOF. See the third example in §1.

We write $\mathrm{Tr}^n_\xi(l)$ instead of $\mathrm{Tr}^n_\xi(T, l)$.

DEFINITION 3.3. An *augmented chain* (or an *ac*) is a structure $\tilde{M} = \langle M, P\rangle$ where $M \in K$ and P is a finite sequence of relevant subsets of M. The length of P is the *weight* of $\tilde{M}$. An ac $\langle N, Q\rangle$ is a *subac* (an *interval*) of $\tilde{M}$ iff N is a convex subset (an interval) of M and $Q = P|N$. An ac $\langle M, P\rangle$ is *trivial* iff M is one-point and every member of P is empty. An element t of $\mathrm{Tr}^n_\xi(l)$ is *trivial* iff t is the $n - \xi$-theory of a trivial ac.

LEMMA 3.3. (a) *$\{\mathrm{Th}^0(\tilde{M})$: $\tilde{M}$ in an ac of the weight $l\}$ is computable from l.*

(b) *$\mathrm{Tr}^n_\xi(l)$ is computable from n, ξ and l. The trace of length l of $t \in \mathrm{Tr}^n_\xi(l + m)$ is computable from l and t.*

(c) *The trivial element of $\mathrm{Tr}^n_\xi(l)$ is computable from n, ξ and l.*

PROOF. Clear.

DEFINITION 3.4. An ac $\tilde{M} = \langle M, P\rangle$ is the *sum* of acs $\tilde{M}_i$ with respect to a chain I iff there exist a congruence E on M and an order isomorphism $f\colon I \to M/E$ such that $\tilde{M}_i \cong \langle f(i), P|f(i)\rangle$. If every $\tilde{M}_i$ is isomorphic to an ac $\tilde{N}$ then $\tilde{M}$ is the *product* of I and $\tilde{N}$. Let $t_0, t_1, t_2 \in \mathrm{Tr}^n_\xi(l)$. $t_0 = t_1 + t_2$ iff there exist acs $\tilde{M}_0$, $\tilde{M}_1$, $\tilde{M}_2$ such that $\tilde{M}_0 = \tilde{M}_1 + \tilde{M}_2$ and $t_i = \mathrm{Th}^n_\xi(\tilde{M}_i)$. $t_0 = \omega \cdot t_1$ (respectively $t_0 = \omega^* \cdot t_1$) iff there exist acs $\tilde{M}_0$, $\tilde{M}_1$ such that $\tilde{M}_0 = \omega \cdot \tilde{M}_1$ (respectively $\tilde{M}_0 = \omega^* \cdot \tilde{M}_1$) and $t_i = \mathrm{Th}^n_\xi(M_i)$.

DEFINITION 3.5. Let $\tilde{M} = \langle M, P\rangle$ be an ac and E be a congruence on M. Then $E(n, \xi, P)$ is the sequence $\langle E_t(n, \xi, P)$: $t \in \mathrm{Tr}^n_\xi(\mathrm{lh}(P))\rangle$ where $E_t(n, \xi, P) = \{X \in M/E : \mathrm{Th}^n_\xi(X, P \mid X) = t\}$ if t is not trivial and $E_t(n, \xi, P) = 0$ if t is trivial.

DEFINITION 3.6. An element s of $\mathrm{Tr}^n_\xi(l)$ is a *prototype* of $t \in \mathrm{Tr}^n_\eta(m)$ iff $\eta = T(n, \xi, l)$ and there exist an ac $\tilde{M} = \langle M, P\rangle$ and a congruence E on M such that $s = \mathrm{Th}^n_\xi(\tilde{M})$ and $t = \mathrm{Th}^n_\eta(M/E, E(n, \xi, P))$. (About $T(n, \xi, l)$ see Definition 2.3.)

LEMMA 3.4. $\mathrm{Th}^0(\tilde{M})$ *is computable from* $\mathrm{Th}^0(M/E, E(0, 0, P))$.

PROOF. Clear.

THEOREM 3.5. (a) *The* $n - \xi$*-theory* $\mathrm{Th}^n_\xi(\tilde{M})$ *of an ac* $\tilde{M} = \langle M, P\rangle$ *of weight* l *is computable from* $\mathrm{Th}^n_\eta(M/E, E(n, \xi, P))$ *where* $\eta = T(n, \xi, l)$.

(b) *There exists an algorithm satisfying the following condition. If* t, t_1, t_2 *belong to* $\mathrm{Tr}^n_\xi(l)$ *and* $t = t_1 + t_2$ *then the algorithm computes* t *from* t_1 *and* t_2.

(c) *There exists an algorithm satisfying the following condition. If* t_0, t_1 *belong to* $\mathrm{Tr}^n_\xi(l)$ *and* $t_0 = \omega \cdot t_1$ *(respectively* $t_0 = \omega^* \cdot t_1$*) then the algorithm computes* t_0 *from* t_1.

(d) *There exists an algorithm* Pr *such that* $s = \mathrm{Pr}(t)$ *whenever* s *is a prototype of* t.

PROOF. (a) follows from Theorem 2.5.

(d) is just a reformulation of (a).

(b) and (c) follow from (a) because the monadic second-order theory of two-point chains is decidable as well as the monadic second-order theory of ω. The latter result proved in (Bü). Q.E.D.

§4. Uniformity. Let $\tilde{M} = \langle M, P\rangle$ be an arbitrary ac, $l = \mathrm{lh}(P)$, $\tilde{X} = \langle X, P \mid X\rangle$ be a subac of $\tilde{M}$ and $\tilde{I} = \langle I, P \mid I\rangle$ be an interval of $\tilde{M}$.

DEFINITION 4.1. $\tilde{M}$ is $n - \xi$-uniform (respectively 0-*uniform*) iff M has neither jumps nor endpoints, P is pairwise disjoint and $\tilde{M}$ is $n - \xi$-equivalent (respectively 0-equivalent) to any interval of itself.

THEOREM 4.1. *Suppose that* $\tilde{M}$ *is* 0-*uniform. For every* n *and* ξ *there exists an* $n - \xi$-uniform interval of $\tilde{M}$.

PROOF. For every points $x < y$ in M define $f(x, y) = \mathrm{Th}^n_\xi([x, y), P \mid [x, y))$. If $x < y < z$ then $f(x, y) + f(y, z) = f(x, z)$. In terms of §1 in [Sh], f is an additive coloring of a densely ordered set M by a finite number of colors. By Theorem 1.3 in [Sh], there exist an interval I of M, a subset $Y \subset I$ and a "color" t such that Y is dense in I and $f(x, y) = t$ for every $x < y$ in Y.

Let $\tilde{J} = \langle J, P \mid J\rangle$ be an arbitrary interval of $\tilde{I}$. Since M is short there exists a subchain $Z \subset J \cap Y$ isomorphic to $\omega^* + \omega$ and cofinal in J in both directions. Clearly $\mathrm{Th}^n_\xi(\tilde{J}) = (\omega^* + \omega)\cdot t$. Thus all intervals of $\tilde{I}$ are $n - \xi$-equivalent and $\tilde{I}$ is $n - \xi$-uniform. Q.E.D.

If K' is a class of acs let $\mathrm{Th}^n_\xi(K') = \{\mathrm{Th}^n_\xi(N) \colon N \in K'\}$. We use also the function $T(n, \xi, m)$ defined in §2.

LEMMA 4.2. *Let* $S \subset \mathrm{Tr}^n_\xi(l)$ *and* K' *be a class of acs of weight* l *closed under subacs.* $\mathrm{Th}^n_\xi(K) \subset S$ *if* S *satisfies the following conditions*:

(Sl) S *contains the* $n - \xi$*-theory of every one-point ac in* K';

(S2) $t_1 + t_2 \in S$ *if* $t_1, t_2 \in S$ *and there exists an ac* $N_1 + N_2$ *in* K' *such that* $\mathrm{Th}^0(N_i)$ *and* t_i *have the same trace of length* 0;

(S3) $\omega\cdot t$ *(respectively* $\omega^* \cdot t$*) belongs to* S *if there exists an ac* $\omega \cdot N$ *in* K' *such that* $\mathrm{Th}^0(N)$ *and* t *have the same trace of length* 0; *and*

(S4) *let* $\bar{N} = \langle N, Q\rangle \in K'$ *where* N *is the underlying chain of* $\bar{N}$. $\mathrm{Th}^n_\xi(\bar{N}) \in S$ *if there exists a congruence* E *on* N *such that* $\langle N/E, E(n, \xi, Q)\rangle$ *is* $n - T(n, \xi, l)$*-uniform and* $\{t\colon E_t(n, \xi, Q) \neq 0\} \subset S$.

PROOF. Assuming $\tilde{M} \in K'$, we prove that $\mathrm{Th}^n_\xi(\tilde{M}) \in S$.

A convex subset X of M will be called *good* iff $\mathrm{Th}^n_\xi(\tilde{X}) \in S$. Define a relation

E on M as follows: xEy iff every convex subset between x and y is good. By (S1) and (S2), E is a congruence and M/E has no jumps.

We show now that a convex subset X of M is good if every two points of X are E-equivalent. The general case is easily reducible to the case when X has exactly one end-point a. Suppose $a = \min X$. (The proof is dual if $a = \max X$.) Since M is short there exists an increasing sequence $\{x_n : n \in \omega\}$ cofinal in X. By Ramsey's Theorem (which may be found in [CK]), there exist t and a subsequence $\{x_{n_k} : k \in \omega\}$ such that every $\{x_{n_k}, x_{n_{k+1}}]$ gives a subac whose $n - \xi$-theory is equal to t. By (S3) and (S2), X is good.

By contradiction suppose that M is not good. By Theorem 4.1, there exists an interval I of M giving an $n - \eta$-uniform interval of $\langle M/E, E(n, \xi, P)\rangle$. Use (S4) to check that every two points of I are E-equivalent which is impossible. Q.E.D.

DEFINITION 4.2 (UNIFORM $n - \xi$-THEORIES). $U^0(\tilde{M}) = U_\xi^0(\tilde{M}) = \mathrm{Th}^0(\tilde{M})$. $U_\xi^{n+1}(\tilde{M})$ is the set of $U_\eta^n(I/E, R)$ such that $\eta = T(n, \xi, l)$, I is an interval of M, E is a congruence on I, $R = \langle R_t : t \in \mathrm{Tr}_\xi^n(l + \bar{\xi} n)\rangle$, $\langle I/E, R\rangle$ is $n - \eta$-uniform, $R_t = 0$ if t is the trivial element of $\mathrm{Tr}_\xi^n(l)$ and $\mathrm{Th}^0(\tilde{X})$ is a trace of t whenever $x \in R_t$. The set $\{t : R_t \neq 0\}$ is the *spectrum* of $U_\eta^n(I/E, R)$ (and of $\mathrm{Th}_\eta^n(I/E, R)$).

THEOREM 4.3. *There exists an algorithm computing* $\mathrm{Th}_\xi^n(\tilde{M})$ *from* $U_\xi^n(\tilde{M})$ *whenever* $\tilde{M}$ *is* $n - \xi$*-uniform.*

PROOF. Assuming that $\tilde{M}$ is $(n + 1) - \xi$-uniform we compute $\mathrm{Th}_\xi^{n+1}(\tilde{M})$ from the set U^* of $\mathrm{Th}_\eta^n(I/E, R)$ where $\eta = T(n, \xi, l)$, $\langle I/E, R\rangle$ is $n - \eta$-uniform and $U_\eta^n(I/E, R) \in U_\xi^{n+1}(\tilde{M})$.

Let K' be the set of acs $\langle \tilde{X}, Q\rangle = \langle X, (P|X)^\frown Q\rangle$ where $\mathrm{lh}(Q) = \bar{\xi} n$. It is enough to compute $S' = \mathrm{Th}_\xi^n(K')$. We first prove the following lemma.

LEMMA 4.4. (a) *The set* $S_0 = \{\mathrm{Th}_\xi^n(\bar{N}) : \bar{N}$ *is a one-point ac in* $K'\}$ *is computable from* U^*.

(b) *Two subacs of* $\tilde{M}$ *are* $(n + 1) - \xi$*-equivalent if they are* 0*-equivalent.*

(c) *Let* $t_1, t_2 \in S'$ *and* $M_1, M_2 \in K$ *and* $\mathrm{Th}^0(M_i)$ *be a trace of* t_i. $t_1 + t_2 \notin S'$ *iff* (i) M_1 *has a maximal point and* M_2 *has a minimal point, or* (ii) M *is complete and* M_1 *does not have a maximal point and* M_2 *does not have a minimal point.*

(d) *Let* $t \in S'$. *Then* $\omega \cdot t \in S'$ *iff* $\omega^* \cdot t \in S'$ *iff* $t + t \in S'$.

(e) *If* $t \in U^*$ *and* S' *includes the spectrum of* t *then* t *has a prototype and it belongs to* S'.

PROOF OF LEMMA 4.4. (a) S_0 is computable from $\{k : P_k \neq 0\}$ which is computable from any $t \in U^*$. By Lemma 4.1, $U^* \neq 0$.

(b) Use the $(n + 1) - \xi$-uniformity.

(c) and (d) are easy to check.

(e) Let $t = \mathrm{Th}_\eta^n(I/E, R)$. If $X \in R_s$ then $s \in S'$ and $\mathrm{Th}_\xi^n(\tilde{X}, Q(X)) = s$ for some $Q(X)$. All sequences $Q(X)$ together form a sequence Q of relevant subsets of I such that $\mathrm{Th}_\xi^n(\tilde{I}, Q)$ is the prototype of t.

To complete the proof of the theorem let S be the least subset of $\mathrm{Tr}_\xi^n(l + \bar{\xi} n)$ such that (i) $S_0 \subset S$, (ii) S satisfies (S2), (S3), and (iii) for each $t \in U^*$, if S includes the spectrum of t then t has a prototype and it belongs to S. By Lemmas 4.2 and 4.4, $S = \mathrm{Th}_\xi^n(K')$. By Theorem 3.5, S is computable from U^*. Q.E.D.

THEOREM 4.5. *Suppose that an arbitrary chain* N *belongs to* K *if there exists a congruence* E *on* N *such that* (i) $N/E \in K$, (ii) *every* $X \in N/E$ *forms a chain belonging* *to K and* (iii) *every subset of* $\{X \in N/E\colon |X| > 1\}$ *is relevant in* N/E. *Then* $\mathrm{Th}^n_\xi(K)$ *is computable from* $\bigcup\{U^{n+1}_{\xi|n}(N)\colon N$ *is an* $(n+1) - \xi|n$*-uniform ac of weight* $|\mathrm{Tr}^n_\xi(0)|\}$.

PROOF. Use Lemma 4.2. (We do not detail the proof since Theorem 4.5 is not used in what follows.)

§5. Elimination of quantifiers. Let $M \in K$, $\tilde{M} = \langle M, P\rangle$ be an arbitrary 0-uniform ac, $l = \mathrm{lh}(P)$, $\tilde{X} = \langle X, P|X\rangle$ be a subac of $\tilde{M}$, $\tilde{I} = \langle I, P|I\rangle$ be an interval of $\tilde{M}$.

DEFINITION 5.1. If E is a congruence on M then $\tilde{M}/E = \langle M/E, E(0, 0, P)\rangle$.

DEFINITION 5.2. $\mathrm{Lh}(1, \xi, l) = l$ and for each $n \geq 1$, $\mathrm{Lh}(n+1, \xi, l) = \mathrm{Lh}(n, T(n, \xi, l), |\mathrm{Tr}^n_\xi(l + \bar{\xi}n)|)$.

About $T(n, \xi, l)$ see Definition 2.3.

THEOREM 5.1. *There exists an algorithm* $F(n, \xi, t, f)$ *such that if* $f(\mathrm{Th}^0(N)) = U^1_0(N)$ *for every* 0*-uniform ac* N *of weight* $\leq \mathrm{Lh}(n, \xi, l)$ *then* $F(n, \xi, \mathrm{Th}^0(\tilde{M}), f) = U^n_\xi(\tilde{M})$.

PROOF. The desired algorithm is built by recursion on n. Let F_n be an algorithm such that if $f(\mathrm{Th}^0(N)) = U^1_0(N)$ for every 0-uniform ac N of weight $\leq \mathrm{Lh}(n, \eta, m)$ then $F_n(\eta, \mathrm{Th}^0(N), f) = U^n_\eta(N)$ for every 0-uniform ac N of weight m. Assuming that $f(\mathrm{Th}^0(N)) = U^1_0(N)$ for every 0-uniform ac N of weight $\leq \mathrm{Lh}(n+1, \xi, l)$ we compute $U^n_\xi(\tilde{M})$ from n, ξ, $\mathrm{Th}^0(\tilde{M})$, f and F_n.

If $u \in U^1_0(\tilde{M})$ then there exist an interval $\tilde{I}$ of $\tilde{M}$ and a congruence E on I such that $\tilde{I}/E$ is 0-uniform and $u = \mathrm{Th}^0(\tilde{I}/E)$. Let $S(u)$ be the set of $\mathrm{Th}^0(I/E, R)$ such that $R = \langle R_t\colon t \in \mathrm{Tr}^n_\xi(l + \bar{\xi}n)\rangle$, $\langle I/E, R\rangle$ is 0-uniform, $R_t = 0$ if t is trivial, $R_t \subset E_s(0, 0, P|I)$ if s is a trace of t and s is not trivial, and R_t is disjoint from any $E(0, 0, P|I)$ if the trace of t of length l is trivial.

$S(u)$ is computable from u (use the condition (N6) of Definition 3.2). Let $\eta = T(n, \xi, l)$ and $N = \langle I/E, R\rangle$ be as above. By Lemma 4.1, N as an $n - \xi$-uniform interval J. $U^n_\eta(J) = F_n(\eta, \mathrm{Th}^0(J), f) = F_n(n, \mathrm{Th}^0(N), f)$. Thus $U^{n+1}_\xi(\tilde{M}) = \{F_n(\eta, s, f)\colon s \in S(u)$ and $u \in U^1_0(\tilde{M})\}$. Q.E.D.

THEOREM 5.2. $\tilde{M}$ *is* $n - \xi$*-uniform if* $U^1_0(N)$ *is computable from* $\mathrm{Th}^0(N)$ *for every* 0*-uniform ac* N *of weight* $\leq \mathrm{Lh}(n+1, \xi|n, l)$.

PROOF. Let S be the set of $\mathrm{Th}^n_\xi(\tilde{X})$ such that either X is one-point or $\tilde{I}$ is $n - \xi$-uniform where I is the interior of X, By Theorems 4.3 and 5.1, all $n - \xi$-uniform intervals of M are $n - \xi$-equivalent. By Lemma 4.2, it is sufficient to check that S satisfies conditions (S1)–(S4) where K' is the class of subacs of $\tilde{M}$.

The cases of (S1)–(S3) are easy to verify. We verify (S4). Let $\eta = T(n, \xi, l)$, E be a congruence on a convex $X \subset M$; $\langle X/E, E(n, \xi, P|X)\rangle$ be $n - \eta$-uniform and $\{t\colon E_t(n, \xi, P|X) \neq 0\} \subset S$. We have to prove that $\mathrm{Th}^n_\xi(\tilde{X}) \in S$.

Without loss of generality $\mathrm{lh}(\xi) = n$. Then $u = U^n_\xi(X/E, E(n, \xi, P|X)) \in U^{n+1}_\xi(\tilde{M})$. Let $\tilde{I}$ be an $n - \xi$-uniform interval of $\tilde{M}$. By Theorem 5.1, $U^{n+1}_\xi(\tilde{M}) = U^{n+1}_\xi(\tilde{I})$. Hence $u = U^n_\eta(J/E', R)$ where J is an interval of I, E' is a congruence on J and R is a sequence $\langle R_t\colon t \in \mathrm{Tr}^n_\xi(l)\rangle$ such that $\mathrm{Th}^0(Y, P|Y)$ is a trace of t whenever $Y \in R_t$. It is easy to see that $R = E'(n, \xi, P|J)$. By Theorems 4.3 and 3.5, $\mathrm{Th}^n_\xi(\tilde{X}) = \mathrm{Th}^n_\xi(\tilde{I})$. But $\mathrm{Th}^n_\xi(\tilde{I}) \in S$. Q.E.D.

THEOREM 5.3. *Suppose that* $K_1, K_2, \ldots$ *are subclasses of* K *and there exists an algorithm computing* $U^1_0(N, Q)$ *from* $\mathrm{Th}^0(N, Q)$ *whenever there exists* p *such that* $N \in K_p$ *and* $\langle N, Q\rangle$ *is a* 0*-uniform ac of weight less than* p. *Then there exists an algorithm* *associating a pair* (p, φ') *with each sentence* φ *in L in such a way that* φ' *is either* φ *or the negation of* φ *and for every* $q \geq p$ *and every chain* $N \in K_q$ *without jumps and endpoints,* N *satisfies* φ'.

PROOF. By §1, it is sufficient to construct an algorithm associating a pair (p, t) with arbitrary n and ξ in such a way that $\mathrm{Th}^n_\xi(N) = t$ for every $q \geq p$ and every $N \in K_q$ without jumps and end-points.

Let $p = \mathrm{Lh}(n + 1, \xi|n, 0)$, $q \geq p$ and N be a chain in K_q without jumps and end-points. The algorithm of Theorem 5.1 computes $U^n_\xi(N)$. By Theorem 5.2, N is $n - \xi$-uniform. The algorithm of Theorem 4.3 computes $\mathrm{Th}^n_\xi(N)$ from $U^n_\xi(N)$. Q.E.D.

COROLLARY 5.4. *Suppose that there exists an algorithm computing* $U^1_0(N)$ *from* $\mathrm{Th}^0(N)$ *whenever* N *is a* 0*-uniform ac. Then there exists an algorithm associating a sentence* φ' *with each sentence* φ *in L in such a way that* φ' *is either* φ *or the negation of* φ *and for every chain N in K without jumps and end-points,* (*the natural model for L formed by*) *N satisfies* φ'.

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