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MONADIC THEORY OF ORDER AND TOPOLOGY II

BY

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ABSTRACT

Assuming the Continuum Hypothesis we interpret the theory of the cardinal $2^{\aleph_0}$ with quantification over the constructible monadic, dyadic, etc. predicates in the monadic (second-order) theory of the real line, in the monadic theory of any other short non-modest chain, in monadic topology of Cantor's Discontinuum and some other monadic theories. We build monadic sentences defining the real line up to isomorphism under some set-theoretic assumptions. There are some other results.

§0. Introduction

This paper is a continuation of [2] called "Part I" below. Speaking about monadic theories we always mean monadic second-order theories.

The monadic theory MT(R) of the real line R is our most important subject. The sign " $<$ " is its only non-logical constant. We are interested in the following questions about MT(R): what is the complexity of it, what can be expressed in it, is it finitely axiomatizable and/or categoric in the monadic fragment of second-order logic?

Assuming a consequence of the Continuum Hypothesis CH Shelah interpreted the true first-order arithmetic (i.e. the first-order theory of the standard model of arithmetic) in MT(R). Let c be the cardinality of the continuum. According to Part I, a topological space is called pseudo-meager iff it is a union of less than c nowhere dense point sets. The assumed consequence of CH is that R is not pseudo-meager. Shelah conjectured that MT(R) and the second-order theory of c (call it T2(c)) are recursive each in the other, see Conjecture 7D in [7].

Let CT2(c) be the theory of the ordinal c in the language of second-order logic when for each $n = 1, 2, \cdots$, the n-place predicate variables range over the

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constructible n-place relations on c. One may say that CT2(c) is the second-order theory of the ordinal c computed in the constructive universe. Of course, c may be not the cardinality of continuum in the constructive universe. In §6 we prove

THEOREM 1. *Assume that R is not pseudo-meager. Then* CT2(c) *is interpretable in* MT(R).

COROLLARY 2 ($V = L$). MT(R) *and* T2(c) *are interpretable each in the other.*

We use the notion "A theory T_1 is interpretable in a theory T_2" in the usual sense. In particular, T_1 is recursive in T_2 if it is interpretable in T_2. So Corollary 2 approves Shelah's Conjecture 7D in the constructive universe. It reduces in the constructive universe the model-theoretic problem about the complexity of MT(R) to the set-theoretic problem about the complexity of T2(c).

The expressive power of MT(R) was investigated in Part I. In particular we proved there (refuting Shelah's Conjecture 7G in [7]) that under CH there exists a formula $\varphi(X)$ in the monadic language of order such that a point set D satisfies $\varphi(D)$ in R iff D is countable. Let $\exp \kappa = 2^\kappa$ for every cardinal κ and ω^*_α be the cardinal $\aleph_\alpha$ of the constructive universe. In §10 we prove the following Theorems 3 and 4.

THEOREM 3. *Assume* CH *and* $\exp \aleph_1 > \aleph_2$. *Then there exists a sentence in the monadic language of order defining the real line R up to isomorphism.*

THEOREM 4. *Assume $c = \omega^*_1$. Then there exists a sentence in the monadic language of order defining the real line R up to isomorphism.*

COROLLARY 5 ($V = L$). MT(R) *is finitely axiomatizable and categoric in the monadic fragment of second-order logic.*

Theorems 3 and 4 approve Conjecture 0.4 in [7] (under corresponding set-theoretic assumptions) and disprove Conjecture 0.5 there.

Theorem 4 remains true if the assumption $c = \omega^*_1$ is replaced by CH together with $c = \omega^*_\alpha$ where α is simply definable in the constructive universe e.g. $\alpha = 5$ or $\alpha = 3\omega^2 + \omega + 7$. But (correcting [3]) we do not know whether CH alone is sufficient for finite satisfiability and categoricity of MT(R). Recall that a chain is a linearly ordered set.

CONJECTURE. *It is consistent with* ZFC + CH *that there exists a chain monadically equivalent but not isomorphic to the real line.*

If U is a chain monadically equivalent but not isomorphic to R and CH holds then U is not Suslin and there exists an interval I of U such that every countable subset of I is nowhere dense in I and there exists an everywhere dense subset D of U such that D does not embed any uncountable subchain of R, see §10.

THEOREM 6 (CH). *There are two chains with the same monadic theory whose completions are not monadically equivalent.*

Theorem 6 is proved in §10. It approves Conjecture 0.6 in [7].

The real line is not our only subject. Generalizing topological T_1 spaces we define in §1 *vicinity spaces* in such a way that (i) the notion of vicinity spaces of degree $\leqq 1$ and the notion of top. T_1 spaces are essentially the same, and (ii) each chain forms in some standard way a vicinity space of degree $\leqq 2$. In §7 we interprete CT2(c) in the monadic theories of some vicinity spaces which implies Theorems 7 and 9 below. Note that the mentioned interpretation theorem (Theorem 7.2) and its consequences remain true if CT2(c) is replaced by $\bigcap\{\mathrm{CT2}(\alpha) : \alpha < c^+\}$ where c^+ is the least cardinal bigger than c and CT2(α) is the second order theory of ordinal α computed in the constructive universe.

The (monadic) *topological language* is the language having variables for points, variables for point sets and non-logical constants for the membership relation and the closure operation. The *monadic theory* of a top. space U (or the *monadic topology* of U) is the theory of U in the top. language when the set variables range over all point sets in U.

THEOREM 7. *There exists an algorithm interpreting* CT2(c) *in the monadic theory of any* top. T_1 *space U satisfying the following conditions: U is regular, first countable, dense, of cardinality c and such that no separable, perfect and nowhere dense subset of X is pseudo-meager in itself.*

Under CH or the Martin Axiom, the Baire Category Theorem gives many top. spaces satisfying the conditions stated in Theorem 7.

COROLLARY 8. *Assume that the real line is not pseudo-meager. Then* CT2(c) *is interpretable in the monadic topology of the Cantor's Discontinuum.*

Let us recall that a chain is short iff it embeds neither ω_1 nor ω_1^*. A definition of p-modest chains (where $p = 1, 2, \cdots$) can be found in §7. A chain is modest iff it is p-modest for every p. According to [4], a chain is monadically equivalent to the rational chain iff it is short, modest and has neither jumps nor endpoints. By [6], the monadic theory of the rational chain is decidable. Hence the monadic theory of short, modest chains is decidable. The authors sketch in [4] the proof that the true arithmetic is uniformly in p interpretable in the monadic theory of any short chain which is not p-modest if each pseudo-meager subset of R is meager. They refer to this paper for details.

THEOREM 9. *Assume that every pseudo-meager subset of the real line R is meager. Then there exists a uniform in p algorithm interpreting* CT2(c) *in the monadic theory of any short chain which is not p-modest.*

In §9 we prove undecidability of some restricted monadic theories of vicinity spaces.

The towers defined in Part I (with use of Shelah's "wonder sets") remain our main tool. We use them here in more general situation. In order to ease understanding we define here the towers in an independent way and repeat some features of Part I. Corollary 6.3 (i.e. Corollary 3 in §6) below corrects a slip in Corollary 5.7 of Part I.

§1. Vicinity spaces

We prove in this paper parallel results about top. spaces and chains. In order to treat both cases simultaneously we introduce vicinity spaces. It seems to us that this notion is interesting by itself.

Let U be a non-empty set and f be a function associating a family of non-empty subsets of U with each element a of U. f is a *vicinity function* iff it satisfies the following conditions:

(V1) If $X \in f(a)$ then $a \not\in X$,

(V2) If $X_1, X_2, X_3 \in f(a)$ and both $X_1 \cap X_2$ and $X_2 \cap X_3$ are not empty then there exists $Y \in f(a)$ such that $Y \subset X_1 \cap X_2 \cap X_3$, and

(V3) If $b \in X \in f(a)$ and $Y \in f(b)$ then there exists $Z \in f(b)$ such that $Z \subset X \cap Y$.

A non-empty set U together with a vicinity function f on it is a *vicinity space*. X is a *vicinity* of a iff $X \in f(a)$. It is easy to see that the relation $X \cap Y \neq 0$ is an equivalence relation on each $f(a)$. The corresponding equivalence classes are called *directions* around a. The number of directions around a is the *degree* of a. The *degree* of a vicinity space U is the supremum of the degrees of points in U.

We define topology in vicinity spaces as follows: X is open iff it includes a vicinity of each point a of it in each direction around a. Topological T_1 spaces are converted into vicinity spaces by the following definitions: X is a vicinity of a iff $X \cup \{a\}$ is open and $a \in \bar{X} - X$. The notion of vicinity spaces of degree $\leqq 1$ and that of T_1 spaces are essentially the same. Chains are converted into vicinity spaces by the following definitions: X is a vicinity of a iff it is a non-empty open interval of the form (a, b) or (b, a). Of course there exist other vicinity functions in T_1 spaces and chains but the described functions will be considered to be standard. One may easily invent many other examples of vicinity spaces.

In the rest of this paper we restrict our attention to the (topologically) regular vicinity spaces which are first-countable, of finite degree and of cardinality at most c. In particular, for each point a there exists a countable *vicinity basis* $\{B_n : n < \omega\}$ around a, each vicinity of a includes a vicinity B_n for some n.

The *repletion* $\mathrm{rp}(X)$ of a point set X in a vicinity space U is the collection of points a in U such that the degree of a is that of U and X meets every vicinity of a. X is *coherent* iff it is non-empty and is a part of its repletion, X is *replete* iff it is non-empty and coincides with its repletion. If X meets every vicinity of every one of its points it forms a *subspace* of U in the following natural way: Y is a vicinity of a point a in X iff there exists a vicinity Z of a in U such that $Y = X \cap Z$.

The monadic language of vicinity spaces has variables for points, variables for point sets and non-logical constants for the membership relation and the vicinity function. It will be called the *vicinity language*, its formulas will be called *vicinity formulas*. The *monadic theory* of a vicinity space is the theory of it in the vicinity language when the set variables range over all point sets in it.

THEOREM 1 (Run-away Theorem). *Let A, X, X_0, $X_1, \cdots$ be point sets in a vicinity space and $X = \bigcup\{X_n : n < \omega\}$. Suppose that A is meager, X is coherent and disjoint from A, and each X_n is* ewd. *Then there exist $B \subset X$ and a family S of subsets of B satisfying the following conditions: B is countable and coherent; and $\bar{B}$ is disjoint from A; and $|S| = c$; and each $Y \in S$ is coherent, closed in B and* nwd *in B; and each X_n is dense in each $Y \in S$; and $\bar{Y} \cap \bar{Z}$ is a scattered subset of B for every different Y, Z in S.*

Theorem 1.1 in Part I is essentially the top. version of Theorem 1 (it generalizes a statement on page 413 in [7]). Theorem 1.1 in [4] is essentially the chain version of Theorem 1. For the reader's convenience we present here the following

PROOF. Let A be a union of nwd sets $A(0), A(1), \cdots$. W.l.o.g., every $A(n)$ is closed. Let $f:\{0,1\}\times\omega\times\omega\times\omega\to\omega$ be one-one and onto, and $(\alpha n, \beta n, \gamma n, \delta n) = f^{-1}(n)$. Let s, t range over the finite sequences of natural numbers, $\mathrm{lh}\, s$ be the length of s. s is regarded as a function from $\mathrm{lh}\, s$ to ω. $t = s^\frown n$ means that t extends s by $t(\mathrm{lh}\, s) = n$. For each point x let $\{V_n(x): n < \omega\}$ be a vicinity basis around x.

LEMMA 2. *There exist open sets $G(s)$ and points $x(s)$ such that*:

(i) *$\bar G(s)$ is disjoint from $A(\mathrm{lh}\, s)$, $\bar G(s^\frown n)\subset G(s)$ and $\bar G(s^\frown m)$ is disjoint from $\bar G(s^\frown n)$ if $m \neq n$*;

(ii) $x(s)\in G(s)$, $\lim_{n\to\infty} x(s^\frown n) = x(s)$, $x(s^\frown n)\in V_{\beta n}(x(s))\cap X_{\gamma n}$;

and

(iii) *If $x = \lim_{n\to\infty} x(s_n)$ then either $x = x(t)$ for some t or there exists a strictly increasing sequence $t_0\subset t_1\subset\cdots$ such that $x\in\bigcap \bar G(t_n)$.*

PROOF OF LEMMA 2. Let $G(0)$ be the complement of $A(0)$ and $x(0)$ be an arbitrary point in $G(0)\cap X$. Suppose that $G(t)$ and $x(t)$ are chosen for every t with $\mathrm{lh}(t)\leqq l$ and that the relevant cases of (i) and (ii) hold. Let $\mathrm{lh}(s) = l$. Pick consecutively

$$x(s^\frown n)\in G(s)\cap V_{\beta n}(x(s))\cap X_{\gamma n} - (A(l+1)\cup\{x(s^\frown m): m < n\}).$$

Using regularity of the space choose consecutively disjoint open sets $G(s^\frown n)$ and $H(s^\frown n)$ such that $x(s^\frown n)\in G(s^\frown n)$ and $H(s^\frown n)$ includes the complement of $G(s)$ and $A(l+1)$, $x(s)$, $\{x(s^\frown m): m\neq n\}$ and $\bigcup\{\bar G(s^\frown m): m < n\}$. The statements (i) and (ii) are proved.

We prove the statement (iii). Let $x = \lim x(s_n)$ and $N_i = \{s_n \mid (i+1): i < \mathrm{lh}(s_n)\}$ and $\varepsilon = \min(\{\omega\}\cup\{i: N_i \text{ is infinite}\})$. If ε is finite select a subsequence $n_0 < n_1 < \cdots$ such that $s_{n_0}\mid\varepsilon = s_{n_1}\mid\varepsilon = \cdots$ and $s_{n_0}(\varepsilon) < s_{n_1}(\varepsilon) < \cdots$, then $x = \lim x(s_{n_k}) = x(s_{n_0}\mid\varepsilon)$. Let $\varepsilon = \omega$. By Koenig's Lemma there exists a sequence $t_0\subset t_1\subset\cdots$ such that $t_i\in N_i$. Then $x = \lim x(t_n)\in\bigcap\bar G(t_n)$. Lemma 2 is proved.

We continue the proof of Theorem 1. Let $G(s)$ and $x(s)$ be as in Lemma 2 and B be the collection of points $x(s)$. By (ii), B is coherent. By (iii) and (i), $\bar B$ is disjoint from A.

Let $S = \{Y_g : g$ is a function from ω to $\{0,1\}\}$ where $Y_g = \{x(s): \alpha(s(k)) = g(k)$ for every $k < \mathrm{lh}\, s\}$. By (ii), every Y_g is coherent and every X_n is dense in every Y_g. Y_g is closed in B: if $x(s)$ does not belong to Y_g then $G(s)$ is disjoint from Y_g. Y_g is nwd in B: if $x(s) \in G$ then there exists n such that $G(s\hat{\ }n)$ is disjoint from Y_g and meets G.

Let $x \in \bar{Y}_g \cap \bar{Y}_h$ and $gm \neq hm$. If $x = x(s)$ then $\mathrm{lh}(s) \leqq m$. If $x \not\in B$ take strictly increasing sequence $t_0 \subset t_1 \subset \cdots$ such that $x \in \bigcap \bar{G}(t_n)$. But $\bar{G}(t_{m+1})$ is disjoint either from $\bar{Y}_g$ or from $\bar{Y}_h$ so $\bar{Y}_g \cap \bar{Y}_h$ is a part of $\{x(s) : \mathrm{lh}(s) \leqq m\}$ which is scattered. Theorem 1 is proved.

§2. Modesty and guard spaces

The notion of modest point sets in top. spaces was introduced in Part I. Its chain version was studied in [4]. Here we define modest point sets in vicinity spaces.

We work in a fixed vicinity space, p is a positive integer. A point set D is *perfunctorily p-modest* iff for every coherent and ewd sets $X_1, \cdots, X_p$ there exists a replete set Y such that $X_1, \cdots, X_p$ are dense in Y and $D \cap Y \subset X_1 \cup \cdots \cup X_p$. D is *p-modest* iff for each coherent X, $D \cap X$ is perfunctorily p-modest in the subspace X. D is *modest* iff it is p-modest for every p.

This definition is not exactly concordant with the definition of p-modesty in §2 of Part I, i.e. the definition in Part I is not the top. version of the above definitions. Let D be a point set in a T_1 space, D is p-modest in the sense of Part I iff it is p-modest in itself (i.e. in the subspace D) in the sense of the above definition. Unfortunately, the union of two point sets p-modest in themselves may be not p-modest in itself, see §4 in [4].

LEMMA 1. *If D and E are p-modest then $D \cup E$ is so.*

Proof is easy.

The above definition of modesty is compatible with the modesty definition in §4 of [4]. It fits our needs in this paper. We prove only those facts about modest sets in vicinity spaces which are used below. One may define ω-modest sets (as in Part I and [4]), to prove that modest sets form an ω_1-complete ideal (cf. theorem 4.3 in [4]) and generalize the following lemma according to lemma 2.2 in Part I.

LEMMA 2. *D is modest if $|D \cap X| < c$ for each separable replete X which is* nwd *in D.*

PROOF. It is enough to check that D is perfunctorily p-modest for arbitrary p. W.l.o.g., D is ewd. Let $X_1, \cdots, X_p$ be coherent and ewd. By Run-away Theorem there exists a countable, coherent and nwd A such that $D, X_1, \cdots, X_p$ are dense in A. By the condition of Lemma 2, $|D \cap \mathrm{rp}(A)| < c$. By Run-away Theorem there exist $B \subset A \cap (X_1 \cup \cdots \cup X_p)$ and a family S of replete subsets of $\mathrm{rp}(B)$ such that $|S| = c$, and $A \cap X_1, \cdots, A \cap X_p$ are dense in each $Y \in S$, and S is disjoint out of B. Clearly there exists $Y \in S$ such that $D \cap Y \subset X_1 \cup \cdots \cup X_p$. Q.E.D.

A family $\{X_1, \cdots, X_p\}$ is a *non-modesty witness* iff $X_1, \cdots, X_p$ are coherent and ewd and there exists no replete Y such that $X_1, \cdots, X_p$ are dense in Y and $Y \subset X_1 \cup \cdots \cup X_p$.

LEMMA 3. *Let $\{X_1, \cdots, X_p\}$ be a non-modesty witness, and $X = X_1 \cup \cdots \cup X_p$, and Y be a replete point set such that $X_1, \cdots, X_p$ are dense in Y. Then $Y - X$ is not meager in Y.*

PROOF. W.l.o.g., Y is the whole space. If $-X$ is meager then by Run-away Theorem there exists a replete Z such that $X_1, \cdots, X_p$ are dense in Z and $Z \subset X$. Q.E.D.

A *guard space* is a vicinity space together with a finite family (called the *guard*) of point sets (called the *guardians*) satisfying the following condition: every guardian is ewd and coherent, and for each point set X, if X is replete and every guardian is dense in X then $\{x \in X : x$ does not belong to any guardian$\}$ is not pseudo-meager in X. The guard of a guard space U is denoted by $\mathrm{Gd}(U)$ (or simply by Gd), a point set X in U is called *guarded* iff it is coherent and every guardian is dense in X. The (monadic) *guard language* is obtained from the vicinity language by adding the constant Gd. A *guard formula* is a formula in the guard language. The (monadic) *theory* of a guard space U is the theory of it in the guard language when the set variables range over all point sets in U.

We could allow countable guards in the above definition of guard spaces; some of our results about guard spaces are easily generalized for the case of countable guards. Finiteness of the guard allows us to interprete a guard space in the underlying vicinity space and the vicinity spaces are our main concern.

In the rest of this section we work in a fixed guard space with the guard Gd consisting of p guardians.

LEMMA 4. *Let X be replete and guarded. If $G \cap X \neq 0$ then $G \cap X - \bigcup \mathrm{Gd}$ is not pseudo-modest in $G \cap X$.*

Proof is clear.

LEMMA 5. *Suppose that D is $(p+1)$-modest and X is replete and guarded. Then $(X - \bigcup \mathrm{Gd}) - D$ is not pseudo-meager in X.*

PROOF. Let $X - \bigcup \mathrm{Gd} - D = \bigcup \{X_\alpha : \alpha < \kappa\}$ where $\kappa < c$ and every X_α is nwd in X, and $Y = (X - \bigcup \mathrm{Gd}) - \bigcup \{\bar{X}_\alpha : \alpha < \kappa\}$. Y is dense in X and for every $\alpha < \kappa$ and Z, if Y is dense in Z then X_α is nwd in Z.

Take a coherent countable A in Y with guarded $\mathrm{rp}(A) - \bigcup \mathrm{Gd}$. Since D is $(p+1)$-modest there exists $Z \subset X$ such that Z is replete and guarded, A is dense in Z and $Z - \bigcup \mathrm{Gd}$ is disjoint from $D - A$. Then $Z - \bigcup \mathrm{Gd} \subset A \cup \{X_\alpha : \alpha < \kappa\}$ hence $Z - \bigcup \mathrm{Gd}$ is pseudo-meager in Z which is impossible. Q.E.D.

LEMMA 6. *Let D be $(p+1)$-modest. Then there exists a modest and everywhere big set disjoint from both $\bigcup \mathrm{Gd}$ and D.*

PROOF. Let S be an open basis for the whole space of cardinality $\leqq c$. Arrange S in a sequence $\langle G_\alpha : \alpha < c \rangle$ where every member of S appears c times. Let $\langle X_\alpha : \alpha < c \rangle$ be a sequence of all separable, replete and nwd subsets. By Lemma 5, $Y_\alpha = (G_\alpha - \bigcup \mathrm{Gd}) - D$ is not pseudo-meager. Pick up consecutively $y_\alpha \in Y_\alpha - \bigcup \{X_\beta : \beta < \alpha\} - \{y_\beta : \beta < \alpha\}$. $\{y_\alpha : \alpha < c\}$ is the desired set. Q.E.D.

§3. Family codes

A point set D in a guard space is $(p + \mathrm{Gd})$-*modest* (respectively $(p + \mathrm{Gd})$-*modest in itself*) iff for every point set $X_1, \cdots, X_p$ (respectively for every subset $X_1, \cdots, X_p$ of D) there exists a replete and guarded set V such that if $X = X_1 \cup \cdots \cup X_p$ is not empty, $X_1, \cdots, X_p$ are dense in X and the repletion of X is guarded then $X_1, \cdots, X_p$ are dense in V and $D \cap V \subset X$.

LEMMA 1. *If D is $(p + |\mathrm{Gd}(U)|)$-modest and disjoint from $\bigcup \mathrm{Gd}$ then D is $(p + \mathrm{Gd})$-modest. The union of two $(p + \mathrm{Gd})$-modest sets is $(p + \mathrm{Gd})$-modest. There exists a guard formula expressing $(p + \mathrm{Gd})$-modesty.*

Proof is clear.

THEOREM 2. *There exists a guard formula $\varphi(X, D, D^0, W)$ satisfying the following condition. Let U be a guard space, D and D^0 be point sets in U and P be* *a family of subsets of D. Suppose that D is* ewd, *coherent and* $(2+\mathrm{Gd})$-*modest in itself, D^0 is an* ewd *subset of D, $|P| \geqq 2$, every member of P is dense in D^0 and P is disjoint on D^0. Then there exists a point set W such that an arbitrary point set X satisfies $\varphi(X, D, D^0, W)$ in U iff $X \subset D$ and for each G there exist $A \in P$ and $H \subset G$ such that $A \cap H = X \cap H$.*

Here "P is disjoint on D^0" means that $\{A \cap D^0 : A \in P\}$ is disjoint. The condition $\forall G \exists AH(A \in P$ and $H \subset G$ and $A \cap H = X \cap H)$ may be expressed by the phrase "X is locally a member of P".

PROOF. Let U, D, D^0 and P be as in Theorem 2. We work in U and adapt the following terminology. Members of P are *colors.* A point set X is *monochromatic* iff $D \cap X \subset A$ for some color A, and X is *motley* iff $X - A$ is dense in $D \cap X$ for every color A. X is *auxiliary* iff it is guarded, replete, nwd and D^0 is dense in X. If $X \subset D$ then $X \cap D^0$ is denoted by X^0.

LEMMA 3. *There exists a point set W disjoint from D and all guardians and such that for each auxiliary set X:*
(i) *$|W \cap X| < c$ if X is monochromatic and separable, and*
(ii) *$W \cap X \neq 0$ if X is motley.*

PROOF OF LEMMA 3. There are exactly c separable auxiliary sets since U is of cardinality c. Let $\langle X_\alpha : \alpha < c\rangle$ and $\langle Y_\alpha : \alpha < c\rangle$ be sequences of all monochromatic and all motley separable auxiliary sets respectively. Any X_α is nwd in any Y_β. By Lemma 2.5, $Y'_\beta = (Y_\beta - \bigcup \mathrm{Gd}) - D$ is not pseudo-meager in Y_β hence there exists $y_\beta \in Y'_\beta - \bigcup\{X_\alpha : \alpha < \beta\}$. It is easy to see that $W = \{y_\beta : \beta < c\}$ is the desired set. Lemma 3 is proved (cf. lemmas 7.4 in [7] and 2.4 in Part I).

LEMMA 4. *Let W be as in Lemma* 3 *and X be a subset of D with* ewd *X^0. X is motley iff for each G there exist coherent $Y, Z \subset G \cap X$ such that Y^0, Z and all guardians are dense in the repletion of $Y \cup Z$ and W meets each auxiliary set C such that Y^0, Z are dense in C.*

PROOF OF LEMMA 4. Firstly suppose that X is not motley. There exist a color A and a non-empty open set G such that $A \cap \bar{G} = \bar{G} \cap X$. Let Y, Z be non-empty subsets of $G \cap X$ such that Y^0, Z and all guardians are dense in $\mathrm{rp}(Y \cup Z)$. Since D is $(2+\mathrm{Gd})$-modest in itself, there exists $V \subset \mathrm{rp}(Y \cup Z)$ such that V is replete, guarded, and Y^0, Z are dense in V, and V is disjoint from

$D-(Y^0 \cup Z)$. W.l.o.g., V is separable and nwd. Hence V is auxiliary and monochromatic. By Lemma 3, $|V \cap W| < c$. By Run-away Theorem, there exists a family S of replete subsets of V such that $|S| = c$, and Y^0, Z and all guardians are dense in each $C \in S$, and different elements of S are disjoint on W. Hence there exists $C \in S$ which is disjoint from W.

Now suppose that X is motley. Take arbitrary G. First case: there exist $H \subset G$ and a color A dense in $H \cap X^0$. Set up $Y = A \cap H \cap X^0$ and $Z = H \cap X - A$.

Second case: $G \cap X^0$ is motley. Select points $u_0, u_1, \cdots$ in G such that $\{u_n : n < \omega\}$ is coherent, X^0 and all guardians are dense in $\{u_n : n < \omega\}$ and for any color A, $|\{n : u_n \in A\}| \leqq 1$. Take $Y = Z = \{u_n : n < \omega\} \cap X^0$. Lemma 4 is proved.

Lemma 4 gives a guard formula $\forall G \psi(G, X, W)$ expressing the predicate "X is motley" in the case when $X \subset D$, X^0 is ewd and W is as in Lemma 3. It is easy to see that the formula $\forall H(H \subset G \rightarrow \psi(H, X, W))$ expresses the predicate "$G \cap X$ is motley" in the same case.

We finish the proof of Theorem 2. The desired formula $\varphi(X, W)$ says that $X \subset D$, and X^0 is ewd, and $G \cap X$ is not motley for any G, and for every G and Y, if $Y \subset D - X$ and Y is dense in G then $G \cap (X \cup Y)$ is motley. Q.E.D.

In the proof of Theorem 2 we build a specific formula $\varphi(X, D, D^0, W)$. It is denoted below $\mathrm{st}(X, D, D^0, W)$ and is called the formula of §3. We say that a point set X is a *storey* of a sequence $t = (D, D^0, \cdots, D^l, W)$ of point sets iff $\mathrm{st}(X, t)$ (i.e. $\mathrm{st}(X, D, D^0, W)$) holds.

COROLLARY 5. *Let U, D, D^0 be as in Theorem 2, $W \subset U$, $t = (D, D^0, W)$ and $X \subset D$. Then X is a storey of t iff for each G there exist $H \subset G$ and a t-storey Y such that $H \cap X = H \cap Y$.*

NOTE 6. W.l.o.g., all bound set variables in the formula $\mathrm{st}(X, D, D^0, W)$ *range over the subsets of D.*

Proof is easy.

§4. Towers

We work in a guard space U. Each non-empty open set G forms a subspace of U with the guard $\{X \cap G : X \text{ is a guardian in } U\}$. Let $\mathrm{P}(V_1, \cdots, V_n)$ be a predicate defined in every guard space and $X_1, \cdots, X_n$ be point sets in U. We say that $\mathrm{P}(X_1, \cdots, X_n)$ holds in G iff $\mathrm{P}(G \cap X_1, \cdots, G \cap X_n)$ holds in G. We define the *domain* of $\mathrm{P}(X_1, \cdots, X_n)$ in U as follows: $\mathrm{do}\,\mathrm{P}(X_1, \cdots, X_n) = \bigcup\{G : \mathrm{P}(X_1, \cdots, X_n) \text{ holds in } G\}$.

Let t be a sequence $(D, D^0, \cdots, D^l, W)$ of point sets where $0 \leqq l < \omega$. We say that t is a *tower* iff it satisfies the following conditions. D is $(2 + \mathrm{Gd})$-modest and ewd; $l \geqq 1$ and $D^0, \cdots, D^l$ are ewd, disjoint subsets of D; if A, B are storeys of t (see the end of §3) then $\mathrm{do}(A \cap D^0 = B \cap D^0) + \mathrm{do}(A \cap B \cap D^0 = 0)$ is ewd and $\mathrm{do}(A \cap D^1 \subset B \cap D^1) \cup \mathrm{do}(B \cap D^1 \subset A \cap D^1)$ is ewd and

$$\mathrm{do}(A \cap D^0 = B \cap D^0) \approx \mathrm{do}(A \cap D^1 = B \cap D^1) \approx \mathrm{do}(A = B);$$

and there exist no G and $W' \subset G$ such that the following holds in the subspace G: every storey of $t' = (D \cap G, D^0 \cap G, W')$ is a storey of

$$t \,|\, G = (D \cap G, D^0 \cap G, \cdots, D^l \cap G, W \cap G)$$

and there exists a storey of t' and for each storey X of t' there exists a storey Y of t' such that $Y \cap D^1 \subset X$ and $X - Y$ is dense in G.

LEMMA 1. *Let D be coherent and $(2 + \mathrm{Gd}(U))$-modest, $l \geqq 1$ and $D^0, \cdots, D^l$ be disjoint and* ewd *subsets of D. Let $\tau > 1$ be an ordinal and $\langle A_\alpha : \alpha < \tau\rangle$ be a sequence of subsets of D such that $\langle A_\alpha \cap D^0 : \alpha < \tau\rangle$ is disjoint and $\langle A_\alpha \cap D^1 : \alpha < \tau\rangle$ is strictly increasing in every non-empty open set. Then there exists W such that $t = (D, D^0, \cdots, D^l, W)$ is a tower and for each $X \subset D$, X is a storey of t iff $\Sigma_{\alpha<\tau}\mathrm{do}(X = A_\alpha)$ is* ewd.

PROOF. Use Theorem 3.2.

In the rest of this section t is a tower and A, B are t-storeys. We say that $A \leqq B$ iff $\mathrm{do}(A \cap D^1 \subset B)$ is ewd, and $A < B$ iff $A \leqq B$ and $B \cap D^1 - A$ is ewd. By induction on ordinal α we define relations "$A \approx \alpha$ modulo t" and "$A \geqq \alpha$ modulo t". $A \geqq \alpha$ modulo t iff $\mathrm{do}(A \approx \beta)$ is empty for each $\beta < \alpha$. $A \approx \alpha$ modulo t iff for each $\beta < \alpha$ there exists $B \approx \beta$ modulo t, and $A \geqq \alpha$ modulo t, and $A \leqq B$ for each $B \geqq \alpha$ modulo t. It is easy to check that, if $A \approx \alpha$ and $B \approx \alpha$ modulo t then $A \approx B$. Say $A < \alpha$ iff $\Sigma\{\mathrm{do}(A \approx \beta) : \beta < \alpha\}$ is ewd. The ordinal $\tau = \min\{\alpha : \text{there is no } A \approx \alpha \text{ modulo } t\}$ is called the *height* of t.

LEMMA 2. *Let $\mathrm{P}(A, G)$ be a predicate satisfying the following two conditions:* (i) *if $\mathrm{P}(A, G)$ holds and $H \subset G$ then $\mathrm{P}(A, H)$ holds, and* (ii) *if $\Sigma\{G_i : i \in I\}$ is dense in G and $A \cap G = \Sigma\{A_i \cap G_i : i \in I\}$ and each $\mathrm{P}(A_i, G_i)$ holds then $\mathrm{P}(A, G)$ holds. Suppose that for each $G \subset G^*$ there exist A and $H \subset G$ such that $\mathrm{P}(A, H)$ holds. Then there exists A such that $\mathrm{P}(A, C^*)$ holds.*

PROOF. Let $\{G_i : i \in I\}$ be an open basis for G^*. For each i there exist A_i and $G'_i \subset G_i$ such that $\mathrm{P}(A_i, G'_i)$ holds. By Lemma 1.4 in Part 1, there exist a set $J \subset I$ and a disjoint family $\{H_i : i \in J$ and $H_i \subset G'_i\}$ such that $\Sigma_{i\in J} H_i$ is dense in G^*. Let $j \in J$ and $X = (\Sigma_{i\in J} A_i \cap H_i) + (A_j - G^*)$. By Corollary 3.5, X is a storey of t. It is easy to see that $\mathrm{P}(X, G^*)$ holds. Q.E.D.

THEOREM 3. *There exists G such that for each t-storey A, $\mathrm{do}(A < \tau)$ is dense in G.*

PROOF. Suppose the contrary: for each G there exist A and H such that H is disjoint from $\mathrm{do}(A < \tau)$ i.e. $H \subset \mathrm{do}(A \geqq \tau)$. By Lemma 2 there exists $A \geqq \tau$.

Let $V = \bigcup\{G :$ there exists $A \geqq \tau$ such that for each $B \geqq \tau$, $A \leqq B$ in $G\}$. By Lemma 2 there exists $A \geqq \tau$ such that for each $B \geqq \tau$, $A \leqq B$ in V. If V is ewd then $A \approx \tau$ modulo t which is impossible.

Let G^* be the complement of $\bar{V}$. In the rest of the proof of Theorem 1 we work in the subspace G^*. Let B range over $\{A \cap G^* : A \geqq \tau\}$. For every B and $G \subset G^*$ there exist B' and $H \subset G$ such that $B' < B$ in H. By Lemma 2, for each B there exists B' such that $B' < B$ in G^*. Select a decreasing sequence $B_0 > B_1 > \cdots$. Let $C_0 = B_0$ and $C_{n+1} = B_{n+1} - \bigcup\{B_m \cap D^0 : m \leqq n\}$. By Theorem 3.2, there exists $W' \subset G^*$ such that for each $X \subset D \cap G^*$, X is a storey of $(D \cap G^*, D^0 \cap G^*, W')$ iff $\Sigma\{\mathrm{do}(X = C_n) : n < \omega\}$ is dense in G^* which contradicts the definition of tower t. Q.E.D.

We define the *arena* of t to be the union of those G that for each t-storey A, $\mathrm{do}(A < \tau)$ is dense in G. A *skeleton* of t is a sequence $\langle A_\alpha : \alpha < \tau\rangle$ of t-stories such that $A_\alpha \approx \alpha$ modulo τ.

§5. Constructive towers

We work in a fixed guard space. Let $l \geqq 2$ and $t = (D, D^0, \cdots, D^l, W)$ be a tower of height $\tau > 0$. Let $X^k = D^k \cap X$ for every $X \subset D$ and $0 \leqq k \leqq l$. The tower t is constructive iff it satisfies conditions (CT1)–(CT7) below. We alternate stating these conditions with definitions and explanations. In the course of exposition it is supposed that t satisfies conditions (CTi) stated beforehand.

DEFINITION. Let X, Y, Z range over t-storeys. $Y \approx X + 1$ iff $X < Y$ and there are no G and Z such that $X < Z < Y$ in G. $Y \approx X + (n+2)$ iff there exist Z such that $Z = X + (n+1)$ and $Y \approx Z + 1$. X is *limit* iff for every G and Y there exists Z such that if $Y < X$ in G then $Y < G < X$ in G. $Y \approx X + \omega$ iff $X < Y$, Y is limit and there is no G and limit Z with $X < Z < Y$ in G.

Note that a t-storey X is limit if $X \approx 0$ modulo t. In other respects the above definition is co-ordinated with terminology of §4. In the rest of the definition of constructivity A, B, C range over limit storeys of t.

(CT1) $\exists B(B \approx A + \omega)$.

Expressions $A + n$ and $A + \omega$ are used below as variables over $\{X : X \approx A + n\}$ and $\{B : B \approx A + \omega\}$ respectively.

Let $\alpha, \beta, \gamma, \delta$ be ordinals, $0 \leqq i, j \leqq 8$ and $(\alpha, \beta, i) < (\gamma, \delta, j)$ iff $\max(\alpha, \beta) \leqq \max(\gamma, \delta)$, and (α, β, i) precedes (γ, δ, j) lexicographically if $\max(\alpha, \beta) = \max(\gamma, \delta)$. Let $\gamma = nu_i(\alpha, \beta)$ iff (α, β, i) is the γth triple in the defined order. We write $nu(\alpha, \beta)$ instead of $nu_0(\alpha, \beta)$.

Definition. $C \approx nu_i(A, B)$ iff $A^2 \cup (B + 1)^2 \approx (C + 2 + i)^2$.

(CT2) $\exists C(C \approx nu_i(A, B))$.

(CT3) $\exists A, B, i(C \approx nu_i(A, B))$.

(CT4) Let $C_1 \approx nu_i(A_1, B_1)$ and $C_2 \approx nu_j(A_2, B_2)$. Then $C_1 \leqq C_2$ iff $A_1^1 \cup B_1^1 \subset A_2^1 \cup B_2^1$, $A_1 \leqq A_2$ in $\mathrm{do}(A_1^1 \cup B_1^1 = A_2^1 \cup B_2^1)$, $B_1 \leqq B_2$ in $\mathrm{do}(A_1 = A_2)$ and either $i \leqq j$ or $\mathrm{do}(A_1 = A_2$ and $B_1 = B_2)$ is empty.

Lemma 1. *If $C_1 \approx nu_i(A, B)$ and $C_2 \approx nu_i(A, B)$ then $C_1 \approx C_2$. If $C \approx nu_i(A_1, B_1)$ and $C \approx nu_j(A_2, B_2)$ then $A_1 \approx A_2$, $B_1 \approx B_2$ and $i = j$.*

Lemma 1 follows from (CT4). It allows us to use an expression $nu_i(A, B)$ as a variable over $\{C : C \approx nu_i(A, B)\}$.

Lemma 2. *Let $A \approx \omega\alpha$* modulo *t, $B \approx \omega\beta$* modulo *t and $\gamma = nu_i(\alpha, \beta)$. Then $\gamma < \tau$ and $nu_i(A, B) \approx \omega\gamma$* modulo *$t$.*

Proof by induction on γ.

We use Gödel's operations $\mathcal{F}_1, \cdots, \mathcal{F}_8$, see [5]. The constructible sets are defined as follows. Let $\gamma = nu_i(\alpha, \beta)$. If $i = 0$ then $F_\gamma = \{F_\delta : \delta < \gamma\}$. If $1 \leqq i \leqq 3$ then $F_\gamma = \mathcal{F}_i(F_\alpha, F_\beta)$. If $4 \leqq i \leqq 8$ then $F_\gamma = \mathcal{F}_i(F_\alpha)$.

DEFINITION.

$A \in B$ iff $A^2 \cup (B+1)^2 \approx (nu(A, B) + 11)^2$;

$A \approx \{A_1, A_2\}$ iff for each B, $\mathrm{do}(B \in A) \approx \mathrm{do}(B = A_1$ or $B = A_2)$;

$A \approx (A_1, A_2)$ iff $\exists BC[A \approx \{B, C\}$ and $B \approx \{A_1, A_1\}$ and $C \approx \{A_1, A_2\}]$;

$A \approx (A_1, A_2, A_3)$ iff $\exists B[B \approx (A_1, A_2)$ and $A \approx (B, A_3)]$;

$A \approx \mathcal{F}_1(A_1, A_2)$ iff $A \approx \{A_1, A_2\}$;

$A \approx \mathcal{F}_2(A_1, A_2)$ iff for each B, $\mathrm{do}(B \in A) \approx \mathrm{do}(B \in A_1) - \mathrm{do}(B \in A_2)$;

$A \approx \mathcal{F}_3(A_1, A_2)$ iff $\forall B, G[B \in A$ in G iff there exist B_1, B_2 such that $B_1 \in A_1$ in G, $B_2 \in A_2$ in G and $B \approx (B_1, B_2)$ in $G]$;

$B \approx \mathcal{F}_4(A)$ iff $\forall C_1, G[C_1 \in B$ in G iff there exist C, C_2 such that $C \approx (C_1, C_2)$ in G and $C \in A$ in $G]$;

$B \approx \mathcal{F}_5(A)$ iff $\forall C, G[C \in B$ in G iff there exist C_1, C_2 such that $C \approx (C_1, C_2)$ in G, $C_1 \in A$ in G, $C_2 \in A$ in G and $C_1 \in C_2$ in $G]$;

$B \approx \mathcal{F}_6(A)$ iff $\forall C, G[C \in B$ in G iff there exist C_0, C_1, C_2, C_3 such that $C_0 \approx (C_1, C_2, C_3)$ in G, $C_0 \in A$ in G and $C \approx (C_2, C_3, C_1)$ in $G]$;

$B \approx \mathcal{F}_7(A)$ iff $\forall C, G[C \in B$ in G iff there exist C_0, C_1, C_2, C_3 such that $C_0 \approx (C_1, C_2, C_3)$ in G, $C_0 \in A$ in G and $C \approx (C_3, C_2, C_1)$ in $G]$;

$B \approx \mathcal{F}_8(A)$ iff $\forall C, G[C \in B$ in G iff there exist C_0, C_1, C_2, C_3 such that $C_0 \approx (C_1, C_2, C_3)$ in G, $C_0 \in A$ in G and $C \approx (C_1, C_3, C_2)$ in $G]$.

(CT5) $A \in nu(A_1, A_2)$ iff $A < nu(A_1, A_2)$ or $A > nu(A_1, A_2)$ and there exists $B < nu(A_1, A_2)$ such that for each $C < A$, $\mathrm{do}(C \in A) \approx \mathrm{do}(C \in B)$.

(CT6) $nu_i(A, B) \approx \mathcal{F}_i(A, B)$ if $1 \leqq i \leqq 3$.

(CT7) $nu_i(A, B) \approx \mathcal{F}_i(A)$ if $4 \leqq i \leqq 8$.

The definition of constructivity is finished. Note that, if t is constructive then for each G the tower $t \mid G = (D \cap G, \cdots, W \cap G)$ in G is constructive.

THEOREM 3. *Let t be a constructive tower and $A \approx \omega\alpha$, $B \approx \omega\beta$* modulo *t. Then $F_\alpha \in F_\beta$ iff $A \in B$* modulo *t. Moreover, if $F_\alpha \notin F_\beta$ then* $\mathrm{do}(A \in B$ modulo $t)$ *is empty.*

PROOF simultaneously for t and every projection tower $t \mid G$. Hence the second statement of the theorem follows from the first one which is proved by induction on $nu(\alpha, \beta)$. Let $\beta = nu_i(\alpha_1, \alpha_2)$.

Case $i = 0$. If $\alpha \leqq \beta$ the statement is clear. Let $\alpha > \beta$. Firstly suppose that $F_\alpha \in F_\beta$. Then there exists $\gamma < \beta$ such that $F_\alpha = F_\gamma$. Let $C \approx \omega\gamma$ modulo t and $C' < A$. There exist δ and G such that $C' \approx \omega\delta$ in G. W.l.o.g., G is the whole space for we may switch over to the tower $t \mid G$ in G. By the induction hypothesis $C' \in A$ iff $C' \in C$. By (CT5), $A \in B$ modulo t. Now suppose that $A \in B$ modulo t. By (CT5), there exists $C < B$ such that for each $C' < A$, $\mathrm{do}(C' \in A) \approx \mathrm{do}(C' \in C)$. W.l.o.g., $C \approx \omega\gamma$ for some γ. By the induction hypothesis $F_\delta \in F_\alpha$ iff $F_\delta \in F_\gamma$ for every $\delta < \alpha$. Hence F_α is equal to F_γ which belongs to F_β.

Other cases are not more difficult. Q.E.D.

Below, c^+ is the minimal ordinal of cardinality more than c.

LEMMA 4. *Let $0 < \varepsilon < c^+$ and ε is closed under the pairing function $nu(\alpha, \beta)$. Then there exists a constructive tower of height $\omega\varepsilon$ whose arena is the whole space.*

PROOF. By Lemma 2.6, there exist everywhere big, modest and disjoint point sets D^0, D^1, D^2 such that their union D is disjoint from every guardian. By Lemma 1.5 in Part I, each D^k can be partitioned into disjoint and ewd parts D^k_α where $\alpha < \omega\varepsilon$. Let $A_\alpha = D^0_\alpha + \Sigma_{\beta<\alpha} D^1_\beta + B_\alpha$ where B_α is defined as follows.

Let $\alpha = nu_i(\beta, \gamma) + n$. $B_\alpha = D^2_{\omega\beta} + D^2_{\omega\gamma+1}$ if either $n = i + 2$, or $n = 11$ and $F_\beta \in F_\gamma$. Otherwise $B_\alpha = D^2_\alpha$.

By Lemma 4.1, there exists a tower t such that $\langle A_\alpha : \alpha < \omega\varepsilon \rangle$ is a skeleton of t. t is the desired tower. Q.E.D.

§6. Raising towers

Let $t = (D, D^0, \cdots, D^l, W)$ be a tower in a fixed guard space, τ be the height of t and $A^k = A \cap D^k$ for every t-storey A and $0 \leqq k \leqq l$.

t is *stable* iff for each tower t' there exists $X \subset D^0$ such that for every t-storey A, $\mathrm{do}(A^0 \subset X) \approx \bigcup \{\mathrm{do}(A^0 = B \cap D^0) : B$ is a storey of $t'\}$.

LEMMA 1. *If* arena$(t) = 1$ *and there exists a skeleton $\langle A_\alpha : \alpha < \tau \rangle$ of t disjoint on D^0 then t is stable.*

PROOF. Take $X = \bigcup\{A^0_\alpha \cap G$: there exists a t'-storey B with $A^0_\alpha = B \cap D^0$ in $G\}$.

Note that the notion of stability is expressible by a guard formula for it is enough to speak in the above definition about towers $t' = (E, E^0, E^1, W')$. Similar notes are applied to some other definitions below.

LEMMA 2. *Suppose that t is stable. Then for each subset $I \subset \tau$ of cardinality $\leqq c$ there exists $X \subset D^0$ coding I in the following sense. If $\alpha < \tau$ and $A \approx \alpha$* modulo *t then either $\alpha \in I$ and* do$(A^0 \subset X)$ *is* ewd *or $\alpha \notin I$ and* do$(A^0 \subset X)$ *is empty.*

PROOF. By Lemma 2.6 there exist everywhere big and modest sets E^0 and E^1 disjoint among themselves and disjoint from D and every guardian. By Lemma 1.5 in Part I, each E^k can be partitioned into disjoint and ewd subsets E^k_α where $\alpha \in I$. Let $\langle B_\alpha : \alpha < \tau\rangle$ be a skeleton of t, $E = E^0 + E^1 + D^0$ and P be a sequence $\langle A_\alpha : \alpha \in I\rangle$ where $A_\alpha = E^0_\alpha + \Sigma\{E^1_\beta : \beta \leqq \alpha\} + B^0_\alpha$. By Lemma 4.1 there exists W' such that $t' = (E, E^0, E^1, W')$ is a tower with a skeleton P. Now use the stability of t to find the desired X. Q.E.D.

COROLLARY 3. *If t is stable then* $\exp|D^0|$ *is equal or greater than the cardinality of $\{I \subset \tau : |I| \leqq c\}$.*

A tower $t' = (E, E^0, \cdots, E^m, W')$ is an *extension* of t iff $D^1 \subset E^1$, and for each t-storey A there exists a t'-storey B (called a *t'-version* of A) such that $A^1 = B \cap E^1$, and for every t-storey A and t'-storey B, if $B \cap E^1 \subset A^1$ then B is a t'-version of some t-storey.

LEMMA 4. *If t' is an extension of t, $\alpha < \tau$, $A \approx \alpha$* modulo *t and B is a t'-version of A then $B \approx \alpha$* modulo *t'.*

PROOF by induction of α.

t is *exponential* on a point set E iff E is ewd and for each $X \subset E$ there exists a t-storey A with $A \cap E = X$. Recall that a top. space is almost separable iff every non-empty open point set in it has a non-empty open separable subset. For each cardinal κ, $\exp\kappa$ is the cardinality of the power set of κ.

LEMMA 5. *Suppose that t is exponential on some point set E. Then $\tau \geqq c$. Moreover $\tau \geqq \exp\aleph_1$ if the space is not almost separable.*

PROOF. See lemma 5.2 in Part I.

Let $\varphi_1(t)$ say that there exists an extension t' of t such that t' is exponential on some E and each t'-storey is a t'-version of a t-storey. Let $\varphi_2(t)$ say that t satisfies $\varphi_1(t)$ and for each tower t' satisfying $\varphi_1(t')$, if t is an extension of t' then each t-storey is a t-version of a t'-storey.

THEOREM 6. (i) *Suppose that the whole space is separable. Then $\varphi_1(t)$ holds iff $\tau \geqq c$, and $\varphi_2(t)$ holds iff $\tau = c$ and the arena of t is the whole space.* (ii) *If the whole space is not almost separable then each of $\varphi_1(t)$ and $\varphi_2(t)$ implies $\tau \geqq \exp \aleph_1$.*

PROOF. See Lemma 5 above and lemma 5.1 in Part I.

LEMMA 7. *Suppose that $\tau < c^+$ and the arena of t is the whole space. For each ordinal $\varepsilon < c^+$ there exists a constructive extension of t of height $\geqq \omega\varepsilon$.*

PROOF. W.l.o.g., $\omega\varepsilon \geqq \tau$ and $\omega\varepsilon$ is closed under the pairing function $nu(\alpha, \beta)$. By Lemma 2.6, there exist everywhere big and modest point sets E^0, E^1, E^2 which are disjoint among themselves and disjoint from D and all guardians. By Lemma 1.5 in Part I, each E^k can be partitioned into disjoint and ewd parts E^k_α where $\alpha < \omega\varepsilon$. Let $\langle D_\alpha : \alpha < \tau \rangle$ be a skeleton of t and $E = E^0 + (D^1 + E^1) + E^2$. We build a sequence $\langle A_\alpha : \alpha < \omega\varepsilon \rangle$ of subsets of E as follows. $A_\alpha \cap E^0 = E^0_\alpha$ for every α. $A_\alpha \cap (D^1 + E^1)$ is equal to D^1_α if $\alpha < \tau$ and it is equal to $D^1 + \Sigma_{\beta < \alpha} E^1_\beta$ otherwise. The construction of $\langle A_\alpha \cap E^2 : \alpha < \omega\varepsilon \rangle$ is similar to that of $\langle B_\alpha : \alpha < \omega\varepsilon \rangle$ in the proof of Lemma 5.4. By Lemma 4.1, there exists a tower $t' = (E, E^0, D^1 + E^1, E^2, W')$ such that $\langle A_\alpha : \alpha < \omega\varepsilon \rangle$ is a skeleton of t'. t' is the desired extension of t. Q.E.D.

A subset X of D^0 is *constructible* modulo t iff there exist a constructive extension t' of t realizing X in the following sense. There exists a limit t'-storey C such that for every limit t-storey A and its t'-version B, $\mathrm{do}(A^0 \subset X) \approx \mathrm{do}(B \in C \text{ modulo } t')$.

LEMMA 8. *Suppose that $\tau < c^+$ and the arena of t is the whole space. Then a subset X of D^0 is constructible* modulo *t iff for each G there exist $H \subset G$ and a constructible subset I of $\{\alpha : \omega\alpha < \tau\}$ satisfying the following conditions. Let $A \approx \omega\alpha$* modulo *t. If $\alpha \in I$ then* $\mathrm{do}(A^0 \subset X)$ *is dense in H. Otherwise* $\mathrm{do}(A^0 \subset X)$ *is disjoint from H.*

PROOF. Firstly suppose that X is constructive modulo t. Take a constructive extension $t' = (D', \cdots, W')$ of t and a t'-storey B such that for each t-storey A and its t'-version A', $\mathrm{do}(A^0 \subset X) \approx \mathrm{do}(A' \in B$ modulo $t')$. For an arbitrary G there exist β and $H \subset G$ such that $B \approx \omega\beta$ modulo t'. Let $I = F_\beta$, $A \approx \omega\alpha$ modulo t and A' be a t'-version of A. By Lemma 4, $A' \approx \omega\alpha$ modulo t'. By Theorem 5.3, $\mathrm{do}(A' \in B$ modulo $t)$ includes H if $\alpha \in I$, and it is disjoint from H otherwise.

Now suppose that for each G there exist appropriate H and I. We look for a constructive extension of t realizing X. Using Lemma 1.4 in Part I and a version of Lemma 5.3 in Part I, we may suppose that there exists a constructible subset I of $\{\alpha : \omega\alpha < \tau\}$ such that for every $A \approx \omega\alpha$ modulo t, either $\alpha \in I$ and $\mathrm{do}(A^0 \subset X)$ is ewd or $\alpha \notin I$ and $\mathrm{do}(A^0 \subset X)$ is empty. There exists $\beta < c^+$ such that $I = F_\beta$. By Lemma 7, there exists a constructive extension of t of height more than $\omega\beta$. By Theorem 5.3, t' realizes X. Q.E.D.

From Lemmas 2 and 8 follows

COROLLARY 9. *Suppose that t is stable, $\tau < c^+$ and the arena of t is the whole space. Then for each constructible subset I of $\{\alpha : \omega\alpha < \tau\}$ there exists $X \subset D^0$ such that X is constructive* modulo t *and for each $A \approx \omega\alpha$* modulo t, *either $\alpha \in I$ and* $\mathrm{do}(A^0 \subset X)$ *is* ewd *or $\alpha \notin I$ and* $\mathrm{do}(A^0 \subset X)$ *is empty.*

§7. Monadic theory of guarded spaces

Let M be the model $\langle c, <, \mathrm{P}\rangle$ where $<$ is the natural ordering of c and P is the 3-place predicate corresponding to the pairing function $nu(\alpha, \beta)$ (i.e. $\mathrm{P}(\alpha, \beta, \gamma)$ holds iff $\gamma = nu(\alpha, \beta)$). Let L be the monadic second-order language of M and T be the theory of M in L when the set variables range over constructible subsets of c. Each atomic formula of L is in one of the following forms: $v_i < v_j$, $\mathrm{P}(v_i, v_j, v_k)$ or $v_i \in V_j$. W.l.o.g., the logical operators of L are $\sim$, $\vee$ and $\exists$. Below, α ranges over c and I ranges over constructible subsets of c.

Let U be a separable guard space and $t = (D, D^0, D^1, D^2, W)$ be a constructive and stable tower in U of height c whose arena is the whole space. For every $X \subset D$ and $0 \leqq k \leqq 2$, $D^k \cap X$ is denoted by X^k. Let $\langle D_\alpha : \alpha < c\rangle$ be a skeleton of t, A range over the limit t-storeys and X range over the subsets of D^0 constructible modulo t. For each I select E_I such that for each $A \approx \omega\alpha$ modulo t, either $\alpha \in I$ and $\mathrm{do}(A^0 \subset E_I)$ is ewd or $\alpha \notin I$ and $\mathrm{do}(A^0 \subset E_I)$ is empty.

We translate formulas in L into guard formulas as follows:

$$(v_i < v_j)^* = (A_i < A_j), \quad (\mathrm{P}(v_i, v_j, v_k))^* = (A_k \approx nu(A_i, A_j)),$$

$$(v_i \in V_j)^* = (A_i^0 \subset X_j), \quad (\sim \varphi)^* = (\mathrm{do}\, \varphi^* = 0), \quad (\varphi \vee \psi)^* = (\varphi^* \vee \psi^*),$$

$$(\exists v_i \varphi)^* = (\exists A_i)\varphi^*, \quad (\exists V_i \varphi)^* = (\exists X_i)\varphi^*.$$

LEMMA 1. *$\varphi(\alpha_1, \cdots, \alpha_m, I_1, \cdots, I_n)$ holds (respectively fails) in M iff the domain of $\varphi^*(D_{\omega\alpha_1}, \cdots, D_{\omega\alpha_m}, E_{I_1}, \cdots, E_{I_n})$ is* ewd *(respectively empty) in U.*

PROOF by induction on φ. We prove Lemma 1 not only for U but also for every subspace of U open in U. The atomic case and the cases of negation and disjunction are straightforward. The two quantifier cases are similar; we adduce here only one of them.

Let $\varphi = (\exists v)\psi$. Fix the free variables of φ. If φ holds in M then some $\psi(\alpha)$ holds in M hence the domain of $\psi^*(D_{\omega\alpha})$ is ewd hence the domain of φ^* is ewd. Suppose that the domain of φ^* is not empty. Take G such that φ^* holds in G. There exists A such that $\psi^*(A)$ holds in G. There exist α and H such that $H \subset G \cap \mathrm{do}(A = D_\alpha)$. Hence $\psi^*(D_\alpha)$ holds in H. By the induction hypothesis $\psi(\alpha)$ holds in M hence φ holds in M. Q.E.D.

The constructive second-order theory of c is the full second-order theory of the ordinal c computed in the constructive universe.

THEOREM 2. *There exists an algorithm interpreting the constructive second-order theory of c in the monadic theory of any guard space.*

PROOF. It is clear that the constructive second-order theory of c is interpretable in T. Let φ be a sentence in L. Use Lemma 1 and Theorem 6.6 to build a guard sentence φ' such that φ holds (fails) in M iff φ' holds (fails) in every separable guard space. Let φ'' say that there exists a guarded and replete set X such that for each guarded and replete subset Y of X, φ' holds in the subspace Y. If φ holds (fails) in M then φ'' holds (fails) in every guard space. Q.E.D.

COROLLARY 3. *There exists an algorithm interpreting the constructive second-order theory of c in the monadic theory of any vicinity space U satisfying the following condition. If X is a separable, replete and* nwd *point set in U then X is not pseudo-meager in itself.*

PROOF. X is a guard space with empty guard.

COROLLARY 4. *There exists a uniform in p algorithm interpreting the constructive second-order theory of c in the monadic theory of any vicinity space U which is not p-modest and satisfies the following conditions: if X is a replete and separable subset of U, $Y \subset X$ and Y is pseudo-meager in X then Y is meager in X.*

PROOF. If U is not p-modest and satisfies the above condition then there exist a separable subspace V of U and subsets $X_1, \cdots, X_p$ of V such that $S = \{X_1, \cdots, X_p\}$ is a non-modesty witness in V. By Lemma 2.3, S is a guard in V. Now use Theorem 2 and its proof. Q.E.D.

A chain C is *p-modest* iff for each subchain C' of C, the vicinity space formed by C' is p-modest. This definition is consistent with the definition of p-modest chains in [4].

COROLLARY 5. *Assume that every pseudo-meager subset of the real line R is meager. Then there exists a uniform in p algorithm interpreting the constructive second-order theory of c in the monadic theory of any short chain which is not p-modest.*

§8. Quasi-separable guard-spaces

A top. space is *quasi-separable* iff each non-empty open point set in it includes a non-empty open separable subset. By theorem 8.1 in Part I, separability is not expressible in the guard language. We prove here that under some set-theoretic assumptions quasi-separability is expressible in the guard language. The Continuum Hypothesis is assumed in this section.

THEOREM 1. *There exist guard formulas $\varphi_1(X)$, $\varphi_2(X)$, $\varphi_3(X)$ such that for every guard space U and every point set X in U*

(i) *if U is separable then $\varphi_1(X)$ holds in U iff X is at most countable,*
(ii) *if U is quasi-separable then $\varphi_2(X)$ holds in U iff X is meager, and*
(iii) *if U is quasi-separable then $\varphi_3(X)$ holds in U iff X is everywhere big.*

PROOF. About formulas φ_1 and φ_2 see sections 6 and 7 in Part I. The desired formula $\varphi_3(X)$ says that for each G there exists $G' \subset G$ such that for each $H \subset G'$, $\varphi_1(H \cap X)$ fails in H. Q.E.D.

THEOREM 2. *Assume* $\exp \aleph_1 > \aleph_2$. *Then there exists a guard sentence expressing quasi-separability.*

PROOF. Let $\psi(t)$ say that t is a tower such that t satisfies formula φ_1 in §6 and for each limit storey A of t there exists a tower t' of height ω such that (i) if B is a t'-storey then B is a t-storey and $B < A$ modulo t, and (ii) if B is a t-storey and $B < A$ modulo t then there exists a t'-storey C such that $B < C$ modulo t. We prove that $\exists t\psi(t)$ is the desired sentence. Let U be a guard space.

Firstly suppose that U is quasi-separable. Build a tower $t = (D, D^0, D^1, W)$ such that the height of t is equal to c, the arena of t is the whole space and t has a skeleton $\langle A_\alpha : \alpha < c \rangle$ disjoint on D^0 (use Lemma 2.6, lemma 1.5 in Part I and Lemma 4.1). By lemma I.5.3 and theorem 6.6, t satisfies φ_1. Let A be a limit storey of t. We look for a tower t' satisfying the clauses (i) and (ii) above. By lemma 5.3 in Part I we may suppose that $A = A_\alpha$ for some α. Let $\langle \alpha_n : n < \omega \rangle$ be an increasing sequence converging to α. By Lemma 4.1 there exists a tower t' such that $\langle A_{\alpha_n} : n < \omega \rangle$ is a skeleton of t'. t' is appropriate.

Now suppose that U is not quasi-separable and $\psi(t)$ holds in U. By Theorem 6.6, there exists a storey A of t such that $A \approx \omega_2$ modulo t. Take t' satisfying the clauses (i) and (ii) above. Let $\langle C_n : n < \omega \rangle$ be a skeleton of t' and I_n be the collection of ordinals α such that $\mathrm{do}(C_n \approx \alpha$ modulo $t)$ is not empty. Clearly, $|I_n| \leqq \aleph_1$. Let $I = \bigcup\{I_n : n < \omega\}$, $\beta = \bigcup I$ and $B \approx \beta$ modulo t. Clearly, t' fails to satisfy the clause (ii). Q.E.D.

LEMMA 3. *The predicate "G is quasi-separable" and the predicate "D is everywhere big" are expressible each by means of the other in the guard language.*

PROOF. A guard space is quasi-separable iff there exists a point set D such that D is ewd and $D \cap G$ is not everywhere big for any G. A point set D is everywhere big iff D is ewd and for any G either G is not quasi-separable or G is quasi-separable and the formula φ_3 of Theorem 1 is satisfied by $D \cap G$ in G.
Q.E.D.

Let t be a sequence of four point sets.

LEMMA 4. *The predicate "D is everywhere big" is expressible by means of the predicate "t is a tower of height $\geqq c$" in the guard languages.*

PROOF. Let $\varphi(D)$ say that for each G, if $D \cap G$ is $(2 + \mathrm{Gd})$-modest then there exist D^0, D^1 and W such that $t = (D \cap G, D^0, D^1, W)$ is a tower of height $\geqq c$ in G and t is stable.

If D is everywhere big and $D \cap G$ is $(2 + \mathrm{Gd})$-modest use lemma 1.5 in Part I, Lemma 4.1 and Lemma 6.1. Suppose that D is not everywhere big. Take G with

$|D \cap G| < c$. By Lemma 2.2, $D \cap G$ is modest. Now use Corollary 6.3. Q.E.D.

Let ω_α^* be the cardinal ω_α in the constructive universe.

LEMMA 5. *The predicate "t is a tower of height $\geqq \omega_1^*$" is expressible in the guard language.*

PROOF. Let $\varphi(t)$ say that t is a tower and there exist towers t', t'' such that t is an extension of t', t' is an extension of t'', t'' is of height ω, t' is constructive and t' realizes every set which is constructive modulo t''. Q.E.D.

THEOREM 6. *Assume $c = \omega_1^*$. Then quasi-separability is expressible in the guard language.*

PROOF. See Lemmas 3–5.

It is easy to see that ω_1^* may be replaced in Lemma 5 and Theorem 6 by $\omega_2^*, \omega_3^*, \cdots$ and many others.

§9. Restricted monadic theories

In this section L is the guard language augmented by a point set constant W. Let U be a guard space and S be a family of point sets in U. For each point set W the pair (U, W) is an *augmented guard space*. The *S-theory* of an augmented guard space (U, W) is the theory of it in L when the set variables range over S. The *augmented* S-theory of U is the intersection of the S-theories of (U, W) when W range over the point sets in U. The *augmented modest theory* of U is the augmented S-theory of U when S is the family of modest subsets of U.

Let R be the real line and Q be the set of rational numbers. Assuming that R is not pseudo-meager Shelah interpreted the first-order logic in the augmented $PS(Q)$-theory of R, see theorem 7.11 in [7]. In [4] it is announced that the augmented modest theory of any complete short chain without jumps is undecidable if all pseudo-meager subsets of R are meager, see theorem 5.5 there. We generalize here these results.

THEOREM 1. *There exists an algorithm associating an L-sentence φ' with each first-order sentence φ in such a way that the following condition holds. Let U be a guard space and D be a* $(2 + \mathrm{Gd})$*-modest in itself and* ewd *point set in U disjoint from all guardians. Then φ is true iff φ' belongs to the augmented $PS(D)$-theory of U.*

PROOF. Here a *graph* is a model (M, P) where M is a non-empty set and P is a dyadic, reflexive and symmetric relation on M. A first-order formula φ is a *graph formula* iff P is the only non-logical constant in φ. The following lemma is well-known.

LEMMA 2. *There exists an algorithm associating a graph formula φ' with each first-order formula φ in such a way that φ is true iff φ' is true in the graph theory.*

Let U and D be as in Theorem 1, D^0 and D^1 be disjoint ewd subsets of D, and $t = (D, D^0, D^1, W)$ where W is a point set in U. Recall that a storey of t is an arbitrary subset X of D such that the statement $\mathrm{st}(X, t)$ holds in U where st is the formula of §3. By Note 3.6, all bound set variables in the statement $\mathrm{st}(X, t)$ range over the subsets of D. Let $X^k = D^k \cap X$ for every t-storey X and $k = 0, 1$. We say that *t defines a graph* iff it satisfies the following conditions: there exists a storey of t, and $\mathrm{do}(X = Y) + \mathrm{do}(X^0 \cap Y^0 = 0)$ is ewd for every t-storey X, Y, and X^1 is ewd for every t-storey X.

With every graph formula $\varphi(v_1, \cdots, v_n)$ and t-storeys $A_1, \cdots, A_n$ associate the *t-domain* of $\varphi(A_1, \cdots, A_n)$ as follows:

$$\mathrm{do}_t P(A, B) = \mathrm{int}\,\mathrm{cl}(A^1 \cap B^1),$$

$$\mathrm{do}_t(\varphi_1 \vee \varphi_2)(\cdots) = \mathrm{do}_t\varphi_1(\cdots) \cup \mathrm{do}_t\varphi_2(\cdots),$$

$$\mathrm{do}_t \sim \varphi(\cdots) = U - \mathrm{cl}\,\mathrm{do}_t\varphi(\cdots),$$

$$\mathrm{do}_t \exists v \varphi(v, \cdots) = \bigcup\{\mathrm{do}_t\varphi(A, \cdots) : A \text{ is a storey of } t\}.$$

LEMMA 3. *If $\varphi(v_1, \cdots, v_n)$ is true in the graph theory, t defines a graph and $A_1, \cdots, A_n$ are t-storeys then* $\mathrm{do}_t\varphi(A_1, \cdots, A_n)$ *is* ewd.

PROOF OF LEMMA 3. Easy induction on the length of a deduction of φ from the axioms Pvv and $Puv \to Pvu$. Q.E.D.

Let (ω, P) be a graph. Partition D^0 and D^1 into ewd disjoint parts X_n and Y_{mn} respectively: $D^0 = \Sigma\{X_n : n < \omega\}$, $D^1 = \Sigma\{Y_{mn} : m, n < \omega\}$. Let $E_n = X_n + \Sigma\{Y_{mn} : (m, n) \in P\}$. By Theorem 3.2, there exists W_p such that for each subset Z of D, Z is a storey of $t_P = (D, D^0, D^1, W_P)$ iff $\Sigma\,\mathrm{do}(Z = E_n)$ is ewd.

LEMMA 4. *For each graph formula $\varphi(v_1, \cdots, v_m)$ and every natural number $n_1, \cdots, n_m$, the t_P-domain of $\varphi(E_{n_1}, \cdots, E_{n_m})$ is* ewd *if $\varphi(n_1, \cdots, n_m)$ holds in (ω, P) and it is empty otherwise.*

PROOF OF LEMMA 4. Easy induction on φ.

Theorem 1 follows from Lemmas 2–4.

THEOREM 5. *There exists an algorithm associating an L-sentence φ^* with each first-order sentence φ in such a way that the following condition holds. Let U be a guard space and S be a family of point sets in U such that* (i) *if $X \in S$ then $PS(X) \subset S$ and* (ii) *S contains a guarded set. Then φ is true iff φ^* belongs to the augmented S-theory of U.*

PROOF. φ^* says that there exists $D \in S$ such that D is guarded and $(2 + \mathrm{Gd})$-modest in itself and φ' belongs to the $PS(D)$-theory of U. Q.E.D.

If X ranges over separable sets only in the definition of guard spaces Theorems 4 and 5 remain true.

§10. The real line

Recall that a chain is a linearly ordered set, and a chain is short iff it embeds neither ω_1 nor ω_1^*. It is easy to check (and known) that the interval topology of any chain is normal. Any short chain forms, in the way described in §1, a vicinity space of degree $\leqq 2$ which is first-countable and of cardinality at most c. The last follows from the partition theorem of Erdös and Rado, see [1]. The short, coherent (i.e. without jumps and end-points) and complete chains are called *lines* in this section. A line is isomorphic to the real line R iff it is separable.

Let L be the monadic second-order language of order. The property to be a line is expressible in L. We assume the Continuum Hypothesis. Hence each line forms a guard space with the empty guard.

LEMMA 1. *Let U be quasi-separable line. U is not separable iff there exist point sets D and V in U such that $D \subset V$, D is discrete, V forms a line and D is everywhere big in V.*

PROOF. Suppose that U is not separable. For every point a and b in U define aEb iff the interval between a and b is either empty or separable. E is an equivalence relation with convex equivalence classes. Form D selecting an inner point from each equivalence class of E having at least two points. Let $V = D \cup \{a : \{a\}$ is an equivalence class of $E\}$. D and V are the desired point sets.

The other implication is clear. Q.E.D.

THEOREM 2. *Assume that there exists an L-sentence such that for each line U, U satisfies φ iff U is quasi-separable. Then there exists an L-sentence ψ such that for each line U, U satisfies ψ iff U is isomorphic to the real line R.*

PROOF. See Lemma 1 and Lemma 8.3.
Let $\aleph_\alpha^*$ be the αth cardinal of the constructive universe.

COROLLARY 3. *Assume* $\exp \aleph_1 > \aleph_2$ *or* $\aleph_1 = \aleph_1^*$. *Then there exists an L-sentence defining R up to isomorphism.*

PROOF. Use Lemma 2 and §8.

THEOREM 4. *Let U be a non-separable line L-equivalent to R. Then there exists an* ewd $D \subset U$ *such that D does not embed any uncountable subset of R.*

PROOF. By corollary 6.3 in Part I, there exists an L-formula $\varphi(X)$ such that for each $X \subset R$, $\varphi(X)$ holds in R iff X is countable. There exist ewd $D \subset U$ such that $\varphi(D)$ holds in U. Moreover, if $X \subset D$, $X \subset V \subset U$ and V is coherent and complete then $\varphi(X)$ holds in V. If $X \subset D$ is separable take $V \subset U$ such that $X \subset V$ and V is separable, coherent and complete. Then V is isomorphic to R, $\varphi(X)$ holds in V hence X is countable. Q.E.D.

THEOREM 5. *There exists an L-sentence holding in the real line and failing in every (non-separable) Suslin line.*

PROOF. Let $\psi(t)$ be the formula built in the proof of Theorem 8.2. According to that proof the sentence $\exists t\psi(t)$ holds in the real line. Let S be a non-separable Suslin line and t be a tower in S satisfying $\psi(t)$. By Theorem 6.6, there exists a storey A of t such that $A \approx \omega_1$ modulo t. Take a tower t' satisfying the clauses (i) and (ii) in the definition of ψ. Let $\langle C_n : n < \omega \rangle$ be a skeleton of t' and I_n be the collection of ordinals α such that do($C_n \approx \alpha$ modulo t) is not empty. Every $|I_n| \leqq \aleph_0$ for S is Suslin. Let $I = \bigcup\{I_n : n < \omega\}$, $\beta = \bigcup I$ and $B \approx \beta$ modulo t. Clearly, t' fails to satisfy the clause (ii). Q.E.D.

THEOREM 6. *There are chains with the same monadic theories whose completions do not have the same monadic theories.*

PROOF. By theorem 6.3 in [7], there exists an uncountable subchain C of the real line R monadically equivalent to the rational chain Q. Let C' (respectively Q') be $C \times Q$ (respectively $Q \times Q$) ordered lexicographically. Use the Ehrenfeucht Game Criterion to check monadic equivalence of C' and Q'. Let C^* be the completion of C'. The completion of Q' is isomorphic to R. Let ψ be an L-sentence saying that the universe is linearly ordered and there exist D and V such that $D \subset V$, D is discrete, V forms a line and the formula φ_3 of Theorem 8.1 is satisfied by D in V. Clearly, ψ holds in C^* and fails in R. Q.E.D.

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