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THE DECISION PROBLEM FOR STANDARD CLASSES

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§0. Introduction. The standard classes of a first-order theory T are certain classes of prenex T-sentences defined by restrictions on prefix, number of monadic, dyadic, etc. predicate variables, and number of monadic, dyadic, etc. operation variables. In [3] it is shown that, for any theory T, (1) the decision problem for any class of prenex T-sentences specified by such restrictions reduces to that for the standard classes, and (2) there are finitely many standard classes $K_1, \cdots, K_n$ such that any undecidable standard class contains one of $K_1, \cdots, K_n$. These results give direction to the study of the decision problem.

Below T is predicate logic with identity and operation variables. The Main Theorem solves the decision problem for the standard classes admitting at least one operation variable.

§1. Standard classes. From now on we restrict the terms "formula" and "sentence" to formulas and sentences in the language of first-order predicate logic including identity and operation variables. A *prefix* is a word in the alphabet $\{\forall, \exists\}$. $\forall^n$ and $\exists^n$ are the words consisting of n occurrences of $\forall$ and of n occurrences of $\exists$, respectively, where n is a natural number. ω denotes the countable cardinal and the set of natural numbers. Now let w be a word in the four-letter alphabet $\{\forall, \exists, \forall^\omega, \exists^\omega\}$. The set $P(w)$ of prefixes is defined thus:

$$P(\forall^n) = \{\forall^i : 0 \le i \le n\}, \qquad P(\exists^n) = \{\exists^i : 0 \le i \le n\},$$
$$P(\forall^\omega) = \{\forall^i : 0 \le i < \omega\}, \qquad P(\exists^\omega) = \{\exists^i : 0 \le i < \omega\},$$
$$P(w_1 w_2) = \{u_1 u_2 : u_1 \in P(w_1) \text{ and } u_2 \in P(w_2)\}.$$

For example, $P(\exists^\omega \forall^2 \exists^\omega) = \{\exists^i \forall^j \exists^k : i, j, k \in \omega \text{ and } j \le 2\}$.

A *place sequence* is a function $s : \{1, 2, \cdots\} \Rightarrow \{0, 1, \cdots, \omega\}$ such that $\{n : sn \ne 0\}$ is finite. The classes $R(s)$ and $F(s)$ are defined as follows. Let $r\alpha(n)$ and $f\alpha(n)$ be the number of n-place predicate variables and of n-place operation variables, respectively, in a formula α. Then α belongs to $R(s)$ iff

$$\sum \{r\alpha(i) : i \ge n\} \le \sum \{si : i \ge n\}$$

for each n, $1 \le n < \omega$, and α belongs to $F(s)$ iff

$$\sum \{f\alpha(i) : i \ge n\} \le \sum \{si : i \ge n\}$$

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for each n, $1 \leq n < \omega$. Also $P(\text{all})$ denotes the set of all prefixes; $R(\text{all})$ and $F(\text{all})$ denote the class of all formulas.

Let w be a word in the aforementioned four-letter alphabet or the word "all". Let s and t be place sequences or the word "all". The class $K(w, s, t)$ consists of all prenex sentences α such that the prefix of α belongs to $P(w)$ and $\alpha \in R(s) \cap F(t)$. *A standard class* is a class of the form $K(w, s, t)$.

Note. The elaborate definitions of $R(s)$ and $F(s)$ are needed because $(n + i)$-place variables can serve as n-place ones. If the inequalities of the definitions are replaced by $r\alpha(n) \leq sn$ and $f\alpha(n) \leq sn$ then the Main Theorem below will not hold.

Below 0, 1^n, 1^ω, and 2^1 denote the place sequences $(0, 0, \cdots)$, $(n, 0, 0 \cdots)$, $(\omega, 0, 0, \cdots)$, and $(0, 1, 0, 0, \cdots)$, respectively.

We say that a class K of formulas is *decidable* if both satisfiability and finite satisfiability (that is, satisfiability in a finite model) are decidable for formulas in K. K is *conservative* [8] if there exists an algorithm $\alpha \Rightarrow \alpha'$ which associates a formula $\alpha' \in K$ with each formula α in such a way that α is satisfiable (finitely satisfiable) iff α' is so. We say that α is an *infinity axiom* if it has only infinite models.

Let L_0 be the class of all formulas without the identity sign. In accordance with the Main Theorems in [2] and [3] every $K = K(w, s, t) \cap L_0$ is either decidable (and contains no infinity axioms) or conservative, and there is a simple criterion for determining whether or not K is decidable. In [1] it is proved that $K(\exists^\omega\forall^2\exists^\omega, \text{all}, 0) \cap L_0$ is decidable (and contains no infinity axioms) and the same is claimed for $K(\exists^\omega\forall^2\exists^\omega, \text{all}, 0)$. This claim would settle the decision problem for all classes $K(w, s, 0)$, since it implies that $K(w, s, 0)$ and $K(w, s, 0) \cap L_0$ are either both decidable (and contain no infinity axioms) or both conservative (see [2]). As far as I know no proof of the claim has been published.

§2. Main Theorem. *Let K be a standard class $K(w, s, t)$, where t is a nonzero place sequence or the word "all". Then either K is decidable and included in at least one of the classes*

$$K(\exists^\omega, all, all), \quad K(all, 1^\omega, 1^1), \quad K(\exists^\omega\forall\exists^\omega, all, 1^1)$$

or else K is conservative and includes at least one of the classes

$$K(\forall, 0, 1^2), \quad K(\forall, 0, 2^1), \quad K(\forall^2, 2^1, 1^1).$$

Note. The infinity axiom $\forall x\exists y\exists z(x = fy \wedge x = fz \wedge y \neq z)$ belongs to $K(\text{all}, 1^\omega, 1^1) \cap K(\exists^\omega\forall\exists^\omega, \text{all } 1^1)$.

Proof of the Main Theorem. It is enough to check that the first three classes are decidable and the other three are conservative.

The decidability of $K(\exists^\omega, \text{all}, \text{all})$ is almost obvious. Let $\alpha = \exists x_1 \cdots \exists x_m B$ be a sentence in this class, where B is quantifier-free. Without loss of generality we may assume that each atom of B is in one of the following forms: $x_i = x_j, f(x_{i_1}, \cdots, x_{i_n}) = x_j$, or $p(x_{i_1}, \cdots, x_{i_n})$. For example, instead of $\exists x\exists y(fgx \neq y)$ we may consider the logically equivalent formula $\exists x\exists y\exists z(gx = z \wedge fz \neq y)$. But then if B is consistent in propositional logic with identity then α has a model of power $\leq m + 1$.

The decidability of $K(\text{all}, 1^\omega, 1^1)$ follows from [7]. Let α be a sentence in this class and let $p_1, \cdots, p_n$ be the predicate variables in α. The sentence α is not satisfiable iff $\forall p_1 \cdots \forall p_n \sim \alpha$ is a theorem of the second-order theory of a unary function with a countable domain. This theory is decidable according to Theorem 2.4 in [7]. Further, α is not finitely satisfiable iff

$$\exists p_0 \forall x(p_0(x)) \rightarrow \forall p_1 \cdots \forall p_n \sim \alpha$$

is a theorem of the weak second-order theory of a unary function. The latter theory is decidable according to Corollary 2.5 in [7].

The decidability of $K(\exists^\omega \forall \exists^\omega, \text{all}, 1^1)$ was proved recently by Saharon Shelah (unpublished).

We show that $K(\forall, 0, 1^2)$ is conservative in §3 below. In [6] it is announced that the dual class $K(\exists, 0, 1^2)$ is a reduction class for validity.

We also show in §3 that $K(\forall, 0, 2^1)$ is conserative; cf. [5].

In accordance with [3], the subclass of $K(\forall^2, 2^1, 1^1)$ consisting of sentences without the identity sign is conservative. Hence $K(\forall^2, 2^1, 1^1)$ is conservative.

§3. $K(\forall, 0, 1^2)$ and $K(\forall, 0, 2^1)$. First we show $K(\forall, 0, 1^\omega)$ to be conservative by encoding the domino problem (see [8]). A *domino type* is a quadruple $D =$ (left(D), top(D), right(D), bottom(D)) of natural numbers. A *domino set* is a finite set of domino types. Let P be a domino set. A function $C: \omega \times \omega \Rightarrow P$ is called a *P-covering* if, for every x and y,

$$\text{right}(C(x, y)) = \text{left}(C(x + 1, y)) \quad \text{and} \quad \text{top}(C(x, y)) = \text{bottom}(C(x, y + 1)).$$

A P-covering is called *periodic with period m* if $C(x, y) = C(x + m, y) = C(x, y + m)$ for all x and y.

PROPOSITION 3.1 (SEE [4]). *There is an algorithm $\alpha \Rightarrow P_\alpha$ which associates a domino set P_α with each formula α in such a way that α is satisfiable (finitely satisfiable) iff there exists a P_α-covering (a periodic P_α-covering, respectively).*

If $P = \{D_1, \cdots, D_n\}$ is a domino set, let $P(x)$ be the conjunction of the following formulas, where $f, g, h_1, \cdots, h_n$ are monadic operation variables:

$$fgh = gfx, \qquad \bigvee \{h_i x = x : 1 \le i \le n\},$$
$$\bigwedge \{h_i x = x \rightarrow h_j x \ne x : 1 \le i < j \le n\},$$
$$\bigvee \{h_i x = x \wedge h_j f x = f x : \text{right}(D_i) = \text{left}(D_j)\},$$
$$\bigwedge \{h_i x = x \wedge h_j g x = g x : \text{top}(D_i) = \text{bottom}(D_j)\}.$$

LEMMA 3.1. *If there exists a P-covering (a periodic P-covering) then $\forall x P(x)$ is satisfiable (finitely satisfiable, respectively).*

PROOF. Here and below $|M|$ denotes the universe of a model M.

Let $C: \omega \times \omega \Rightarrow P$ be a P-covering (a periodic P-covering with period m). Let R be the ring of integers (the ring of residue classes mod m or, if $m = 1$, the ring of residue classes mod 2). A model (a finite model) M for $\forall x P(x)$ is constructed as follows:

$$|M| = |R| \times |R|, \qquad f(a, b) = (a + 1, b), \qquad g(a, b) = (a, b + 1),$$

$$h_i(a, b) = \begin{cases} (a, b) & \text{if } C(a, b) = D_i \\ \text{an arbitrary element of } |M| \text{ distinct from } (a, b) & \text{if } C(a, b) \ne D_i. \end{cases} \quad \text{Q.E.D.}$$

LEMMA 3.2. *If* $\forall x P(x)$ *is satisfiable (finitely satisfiable) then there exists a P-covering (a periodic P-covering, respectively).*

PROOF. Let M be a model for $\forall x P(x)$. Pick an arbitrary $a \in |M|$ and define C as follows: $C(k, l) = D_i$ iff $M \models h_i f^k g^l a = f^k g^l a$. Then C is a P-covering. Moreover, if M is finite then C is periodic. Q.E.D.

THEOREM 3.1. $K(\forall, 0, 1^\omega)$ *is conservative.*

PROOF. The algorithm $\alpha \Rightarrow \forall x P_\alpha(x)$ is a desired conservative reduction to $K(\forall, 0, 1^\omega)$. Q.E.D.

THEOREM 3.2. $K(\forall, 0, 1^2)$ *is conservative.*

PROOF. Let $\alpha = \forall x B(x)$ be a sentence in $K(\forall, 0, 1^\omega)$ containing the operation variables $f_1, \cdots, f_m$. Then let α' be the sentence

$$\forall x[f_1 x = hgx \wedge f_2 x = hg^2 x \wedge \cdots \wedge f_m x = hg^m x \wedge B(hx)].$$

It is enough to prove that α' is satisfiable (finitely satisfiable) iff α is so. For clearly there is an α'' in $K(\forall, 0, 1^2)$ such that (i) α' logically implies α'', and (ii) any model for α'' becomes a model for α' by the addition of appropriate assignments to the operation variables $f_1, \cdots, f_m$.

If M' is a model for α' then α holds in the submodel M of M' such that $|M| = \{h\alpha : a \in |M'|\}$. Now let M be a model for α. A model M' for α' is constructed as follows:

$|M'| = |M| \times \{0, 1, \cdots, m\}$,

$g(a, i) = (a, i + 1)$ for $i < m$ and $g(a, m) = (a, 0)$,

$h(a, 0) = (a, 0)$ and $h(a, i) = (f_i a, 0)$ for $i > 0$,

$f_i(a, j) = hg^i(a, j)$ and in particular $f_i(a, 0) = (f_i a, 0)$. Q.E.D.

THEOREM 3.3. $K(\forall, 0, 2^1)$ *is conservative.*

PROOF. Let $\alpha = \forall x B(x)$ be a sentence in $K(\forall, 0, 1^2)$ containing the operation variables g, h. Then let α' be

$$\forall x[fx = F(x, x) \wedge gx = F(x, fx) \wedge hx = F(fx, x) \wedge B(gx) \wedge B(hx)].$$

As in the preceding proof it suffices to prove that α' is satisfiable (finitely satisfiable) iff α is so.

If M' is a model for α' then α holds in the submodel M of M' such that $|M| = \{ga : a \in |M'|\} \cup \{ha : a \in |M'|\}$.

Let M be a model for α and let R be the ring of residue classes mod 3. α' holds in any algebra M' such that:

$|M'| = |M| \times |R|$, $F[(a, n), (a, n)] = (a, n + 1)$,

$F[(a, n), (a, n + 1)] = (ga, 0)$, $F[(a, n + 1), (a, n)] = (ha, 0)$,

$f(a, n) = (a, n + 1)$, $g(a, n) = (ga, 0)$, $h(a, n) = (ha, 0)$. Q.E.D.

REFERENCES

[1] K. GÖDEL, *Zum Entscheidungsproblem des logischen Funktionenkalküls*, ***Monatshefte für Mathematik und Physik***, vol. 40 (1933), pp. 433–443.

[2] Y. GUREVICH, *On the effective recognizing of satisfiability of predicate formulas*, ***Algebra and Logic***, vol. 5 (1966), pp. 25–55.

[3] ———, *The decision problem for the logic of predicates and operations*, ***Algebra and Logic***, vol. 8 (1969), pp. 284–308.

[4] Y. Gurevich and I. Koriakov, *Remark on the R. Berger's paper on the domino problem*, ***Siberian Mathematical Journal***, vol. 13 (1972), pp. 459–463.

[5] V. Kostirko, *The reduction class* $\forall x \forall y \exists z F(x, y, z) \wedge \forall^n A(F)$, ***Cybernetics (Kiev, USSR)***, 1971, no. 5, pp. 1–3.

[6] V. Lifshitz, *Some reduction classes and undecidable theories*, ***Proceedings of the Leningrad Branch of Steklov Math Institute***, vol. 4 (1967), pp. 65–68.

[7] M. O. Rabin, *Decidability of second order theories and automata on infinite trees*, ***Transactions of the American Mathematical Society***, vol. 141 (1969), pp. 1–35.

[8] H. Wang, *Dominoes and the AEA case of the decision problem*, ***Proceedings of the Symposium on Mathematical Theory of Automata***, Polytechnic Institute of Brooklyn, New York, 1962.

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