THE DECISION PROBLEM FOR THE LOGIC OF PREDICATES AND OF OPERATIONS

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INTRODUCTION

Most work on the decision problem for the logic of predicates belongs to the class of preliminary formulas, posed in a way limited to prefixes and predicate variables. These investigations have recently, in many respects, been completed. Corresponding results are introduced at the beginning of the second chapter of this work in extensive detail.

It turns out that that part of these results can be anticipated from simple algebraic considerations and that the situation is far more general, at any rate for any theory of first order. The first chapter of our work is dedicated to this.

The content of the second chapter essentially generalizes the above-mentioned results on the decision problem for the logic of predicates to the case where there are assumed variables of operation. The second chapter can be considered, also, as an illustration of the first. The more difficult of the special results of the second chapter have been considered by us separately (see [3]).

In distinction to the case of absence of operations, adjoining the equality sign in the case of available operations materially alters the picture. Our work is comparable with [4] for instance. In the second chapter, we consider only formulas without the equality sign.

Clarification of the connections between the results of Chapter 2 and earlier results of other authors we postpone to §4 of Chapter 2.

CHAPTER 1. TIGHT SETS AND THE DECISION PROBLEM

§1. Tight Partially Ordered Sets

This section is algebraic in character. In particular, as is acceptable and convenient in contemporary general algebra, we shall use the same symbol to denote a partially ordered set, i.e., a definite model, and the fundamental set of this model. Analogously, the concept "subset of a partially ordered set" will have two meanings. One meaning is submodel, the other – subset of the basic set. Instead of "partially ordered set" we shall abbreviate "p.o. set." All information needed from the theory of partial ordering can be found in §4 of Chapter 1 of the book [6].

Definition 1. (Definition of tightness). A p.o. set M is called tight if each sequence $\{x_i\}_{i\in\omega}$ of elements of M contains an increasing (not necessarily strictly increasing) subsequence.

Here ω denotes, as usual, the natural order in the set of all natural numbers. Obviously, if M is a tight p.o. set, then:

THEOREM 1. M cannot contain an infinite subset of pairwise noncomparable elements and

THEOREM 2. M satisfies the descending chain condition.

Conversely, if a p.o. set satisfies the conditions of Theorems 1 and 2, then it is tight. We shall not use here, and hence will not show, this converse assertion.

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LEMMA 1. Suppose M is a tight p.o. set, $X \subseteq M$ and $\forall xy (x < y \,\&\, y \in X \longrightarrow x \in X)$ while M_0 is the collection of minimal elements from $M - X$. Then M_0 is finite and

$$M - X = \{y \in M : \exists x \in M_0 \quad (x \leq y)\}.$$

Proof. Finiteness of M_0 follows from Theorem 1. The second assertion of the Lemma is a consequence of Theorem 2.

This proves Lemma 1.

LEMMA 2. A subset and homeomorphic image of a tight p.o. set is tight. The direct product of a finite number of tight p.o. sets is tight.

Proof. The first assertion of the Lemma is obvious. For the second assertion there would suffice a demonstration for the case of two direct sets. Let A and B be tight p.o. sets and $\{(a_i, b_i)\}_{i \in \omega}$ be a sequence of elements of the p.o. set $A \times B$. On the strength of tightness of A, the sequence $\{a_i\}_{i \in \omega}$ contains an increasing subsequence $\{a_i\}_{i \in X}$. Here X is some infinite subset of ω. On the strength of the tightness of B, the sequence $\{b_i\}_{i \in X}$ contains an increasing subsequence $\{b_i\}_{i \in Y}$. Obviously, $\{(a_i, b_i)\}_{i \in Y}$ is an increasing subsequence of the sequence $\{(a_i, b_i)\}_{i \in \omega}$.

Lemma 2 is proved.

Definition 2. (Definition of the natural order of words). Let a and b run through the words of some alphabet. We put $a \leq b$ if a is a subword (not a splicing together of subwords) of b, i.e., if a arises from b by erasing some number (0, 1,...) of occurrences of letters. The resulting partial ordering will be called natural.

LEMMA 3. Let M be the set of all words of a finite alphabet Q, partially ordered in the natural manner. The p.o. set M is tight.

Proof. This proceeds by induction on $n = \bar{\bar{Q}}$. For $n = 0, 1$ the Lemma is obvious. Suppose $n > 1$ and that the Lemma is already proved for $n - 1$. We enumerate the elements of Q by the natural numbers from 0 to $n-1$:

$Q = \{q_0, q_1, \ldots, q_{n-1}\}$. Further, for each natural number m and for $r = 0, \ldots, n-1$ we put

$$q_{m \cdot n + r} = q_r, \quad d_0 = \Lambda, \quad d_{m+1} = d_m * q_{m+1}.$$

We use the symbol '$*$' here and farther on in this section to symbolize the operation of attaching to words. Λ-is the empty word.

LEMMA 4. If $a \in M$ and $\neg(d_m \leq a)$, then a can be written in the form $a = a_1 * \ldots * a_m$, where each a_k does not contain q_k (some a_k can be empty).

Proof by induction on m. The Lemma is trivial for $m = 0$. It is obvious for $m = 1$. Suppose $m > 1$ and that for $m - 1$ Lemma 4 is still demonstrated. Suppose b is the maximal beginning stretch of a which does not majorize d_{m-1}. Then $a = b * a_m$ for some a_m. On the strength of the induction assumption it is sufficient to show that a_m does not contain q_m. In this we may regard a_m as different from Λ. On the strength of maximality of b the word $a_m = q_{m-1} * c$ for some c and $d_{m-1} \leq b * q_{m-1}$. If c belongs to q_m, then, obviously, $d_m \leq a = b * q_{m-1} * c$. Hence c and along with it a_m do not contain q_m.

This proves Lemma 4. We proceed to prove Lemma 3.

We remark that if ℓ is the length of a word (i.e., in general the number of occurrences of letters in the word) $a \in M$, then $a < d_{\ell \cdot n}$. Suppose now that $\{a_i\}_{i \in \omega}$ is a sequence of elements of M. If each d_m is less than or equal to some a_i, then, obviously, the sequence $\{a_i\}_{i \in \omega}$ has an increasing subsequence. Hence we shall consider that for some m $\forall i \neg(d_m \leq a_i)$. On the strength of Lemma 4, each a_i can assume in this case the form $a_i = a_{i1} * \ldots * a_{im}$, where each a_{ik} does not contain q_k.

Consider now the direct product $M_1 \times \ldots \times M_m$, where M_k is the naturally ordered set of all words of the alphabet $Q - \{q_k\}$. This direct product is tight on the strength of the induction hypothesis and Lemma 2. From the sequence

$$\{(a_{i1}, \ldots, a_{im})\}_{i \in \omega}$$

whose elements are direct products we can choose an increasing subsequence $\{(a_{i1},\ldots,a_{im})\}_{i \in X}$. The corresponding subsequence $\{a_i\}_{i \in X}$ is an increasing subsequence of the sequence $\{a_i\}_{i \in \omega}$.

Lemma 3 is proved.

Definition 3. (Definition of W). By means of W we denote the collection of all words of the alphabet $\{\forall, \exists, \forall^\infty, \exists^\infty\}$, quasi-ordered in the following way: $A \leq B \pmod W$ means that A can be derived from B with the help of the operations:

σ_1 – cancellation of a single occurrence of $q = \forall$ or $\exists$ and

σ_2 – change of one occurrence of q^∞ into an occurrence of q^m, where $0 \leq m < \infty$, and

σ_3 – change of one occurrence of q^∞ into an occurrence of $q^\infty * q^\infty$.

Here, as usual, $q^0 = \Lambda$, $q^{m+1} = q^m * q$. In place of $A \leq B (\mathrm{mod}\, W)$ and $B \leq A \pmod W$ we shall write $A = B \pmod W$.

We recall that the quasi-ordering ρ is reflexive and transitive. The relation $x\rho y$ and $y \rho x$ is, as is natural, called quasi-equality. In a quasi-ordered set M, by identifying quasi-equal elements, we get some p.o. set, which we call the p.o. set corresponding to M.

Definition 1'. The quasi-ordered set M will be called tight if its corresponding p.o. set is tight.

LEMMA 5. The quasi-ordered set W is tight.

Proof. On the strength of Lemma 3, the natural form of the set of all words of the alphabet $\{\forall, \exists, \forall^\infty, \exists^\infty\}$ is tight. The mapping $A \to \{B : A = B \pmod W\}$ is a homomorphism of this tight p.o. set to the p.o. set corresponding to M. On the strength of Lemma 2 and Definition 1', W is tight.

LEMMA 6. If $A_1, \ldots, A_n, B \in W$ and $A = A_1 * \ldots * A_n$ and $A \leq B \pmod W$, then there can be found $B_i \in W$ such that $B = B_1 * \ldots * B_n \pmod W$ and $A_i \leq B_i \pmod W$.

Proof. It is sufficient to demonstrate this Lemma for $n = 2$. It is further sufficient to limit ourselves to the case where A arises from B by means of one application of one operation. The Lemma is obvious if this operation is σ_1. If the operation is σ_3, then simply $B = A \pmod W$. Suppose A arises from B by one application of the operation σ_2. The Lemma is obvious if the q^m arising as a result of the change wholly enters into A_1 or into A_2. We consider the remaining possibility:

$$A_1 = C * q^i, \quad A_2 = q^j * \mathfrak{D}, \quad B = C * q^\infty * \mathfrak{D}.$$

Obviously, the role of the desired B_1 and B_2 can be played by the words $C * q^\infty, q^\infty * \mathfrak{D}$.

Lemma 6 is proved.

Put $q_m = \forall$, if m is even and $q_m = \exists$ if m is odd, and as before $d_0 = \Lambda$, $d_{m+1} = d_m * q_{m+1}$.

LEMMA 7. Each $A \in W$ can be represented, and moreover uniquely, in the form

$$A = q_0^{\alpha_0} * q_1^{\alpha_1} * \ldots * q_m^{\alpha_m} \pmod W,$$

where α_i is a natural number of the symbol ∞ and $0 < \alpha_1, \ldots, \alpha_m$. Here $d_m \leq A \pmod W$ and $\neg (d_{m+1} \leq A \pmod W))$.

Proof. The existence of the representation referred to is demonstrated in an obvious manner by induction on the length of the word A. It is also obvious that here $d_m \leq A \pmod W$. Further the word

$$q_0^{\alpha_0} * q_1^{\alpha_1} * \ldots * q_m^{\alpha_m} = A_0 * A_1 * \ldots * A_m$$

for some $A_k \leq q_k^{\alpha_k} (mod W)$ preserve, for each of the operations $\sigma_1 - \sigma_3$, the property of the word's having such a resolution. Hence $\neg (d_{m+1} \leq A (mod W))$, and, accordingly, m is defined uniquely. Now let $A = B (mod W)$, where $B = q_0^{\beta_0} * q_1^{\beta_1} * \ldots * q_m^{\beta_m}$ and $0 < \beta_1, \ldots, \beta_m$.

On the strength of what has been outlined above, $B = B_0 * B_1 * \ldots * B_m$ for some $B_k \leq q_k^{\alpha_k} (mod W)$. It is easy to see that $B_k = q_k^{\beta_k}$. Hence $\beta_k \leq \alpha_k$ and, on the ground of symmetry, $\beta_k = \alpha_k$.

Lemma 7 is proved.

Definition 4. (Definition of standard sets of prefixes). Words of the alphabet $\{\forall, \exists\}$ are called prefixes. The collection of all prefixes we denote by $\mathcal{P}$. Suppose $A \in W$. Put

$$\Pi(A) = \{a \in \mathcal{P} : a \leq A (mod W)\}.$$

The set Π of prefixes we will call standard if $\Pi = \mathcal{P}$ or $\Pi = \Pi(A)$ for some $A \in W$.

LEMMA 8. Suppose $A, B \in W$. Then $A \leq B$ $(mod W)$ is equivalent with $\Pi(A) \subseteq \Pi(B)$.

The proof requires only the implication

$$\Pi(A) \subseteq \Pi(B) \rightarrow A \leq B (mod W).$$

Suppose $\Pi(A) \subseteq \Pi(B)$ and denote by means of ℓ the length of the word B. On the strength of Lemma 7, $A = q_0^{\alpha_0} * \ldots * q_m^{\alpha_m}$ for some m, α_0 and $\alpha_1, \ldots, \alpha_m > 0$. Denote $a = q_0^{\ell_0} * \ldots * q_m^{\ell_m}$, where $\ell_i = min\{\alpha_i, \ell+1\}$. Since $a \in \Pi(A) \subseteq \Pi(B)$, then $a \leq B (mod W)$. On the strength of Lemma 6, $B = B_0 * \ldots * B_m (mod W)$ for some $B_i \geq q_i^{\ell_i} (mod W)$. In fact $B_i \geq q_i^{\alpha_i} (mod W)$. This is obvious if $\ell_i = \alpha_i$. Suppose $\ell_i = \ell + 1$. Then the length of B_i is less than ℓ_i, and hence from $B_i \geq q_i^{\ell_i} (mod W)$ it follows that B_i even contains an occurrence of q_i^{∞}. Hence

$$A \leq B_0 * \ldots * B_m = B (mod W).$$

Lemma 8 is proved.

LEMMA 9. A collection of standard sets of prefixes, p.o. with respect to set theoretic relations, is tight.

The proof stems in an obvious way from Lemmas 5 and 8.

LEMMA 10. A union of increasing sequences of standard sets of prefixes is standard.

Proof. Let $\Pi_0 \subseteq \Pi_1 \subseteq \ldots$ be an increasing sequence of standard sets and $\Pi = \bigcup \Pi_i$. The Lemma is obvious if $\Pi = \mathcal{P}$. Suppose $\Pi \neq \mathcal{P}$. Then each $\Pi_i = \Pi(A_i)$ for some $A_i \in W$. On the strength of Lemma 7

$$A_i = q_0^{\alpha_{i0}} * \ldots * q_{m(i)}^{\alpha_{i m(i)}} \quad (mod W)$$

for suitable $m(i)$, α_{i0} and $\alpha_{i1}, \ldots, \alpha_{i m(i)} > 0$

Since $\Pi \neq \mathcal{P}$, it follows that $m = sup\{m(i) : i \in \omega\} < \infty$. Without affecting generality we can suppose that each $m(i) = m$. Put

$$\alpha_j = sup\{\alpha_{ij} : i \in \omega\} \text{ and } A = q_0^{\alpha_0} * \ldots * q_m^{\alpha_m}.$$

We show that $\Pi = \Pi(A)$. If $a \in \Pi$, then for some i, $a \leq A_i (mod W)$ and hence $a \leq A (mod W)$. Conversely, suppose the prefix $a \leq A (mod W)$. The word $A = B_0 * \ldots * B_m$ for some $B_j \leq q_j^{\alpha_j} (mod W)$. The operations $\mathcal{O}_1 - \mathcal{O}_3$ preserve the property of the word's having this resolution. Then $a = a_0 * \ldots * a_m$ for some $a_j \leq q_j^{\alpha_j} (mod\, W)$. Here $a_j = q_j^{e_j}$ for the entering natural number ℓ_j. It is easy to see that, for sufficiently large i, $\ell_j \leq \alpha_{ij}$ for every i. Here $a \leq A_i (mod\, W)$, and hence $a \in \Pi_i \subseteq \Pi$.

Lemma 10 is proved.

Definition 5. (Definition of closure of a set of prefixes). The closure of the set $\Pi \subseteq \mathcal{P}$ we denote $\bar{\Pi} = \{a \in \mathcal{P} : \exists b \in \Pi\, (a \leq b)\}$. (Natural order). In case $\Pi = \bar{\Pi}$ we call the set Π closed as usual.

THEOREM 1. Each closed set of prefixes is the union of a finite number of standard ones.

Proof. Suppose Π is a closed subset of $\mathcal{P}$. Each $a \in \Pi$ lies in a standard $\{b : b \leq a\} \subseteq \Pi$. On the strength of Lemma 10 and Zorn's Lemma, each standard subset from Π lies in some maximal one among the standard subsets of Π. Accordingly, Π is the union of its maximal standard subsets. Among themselves these maximal standard subsets are pairwise not comparable. On the strength of Lemma 9 their number is finite.

Theorem 1 is proved.

LEMMA 11. Suppose R is a lattice (structure), $M \subseteq R$ and M is tight and each element of R is the union of a finite number of elements of M. Then R is tight.

Proof. Suppose that R is not tight and $\{x_i\}_{i \in \omega}$ is a sequence of elements of R not having an increasing subsequence. Then there exists an $i_0 \in \omega$, such that

$$i_0 < j \longrightarrow \neg (x_{i_0} \leq x_j),$$

and also the set X of such i_0 is infinite. The sequence $\{x_i\}_{i \in X}$ satisfies the condition $i < j \longrightarrow \neg (x_i \leq x_j)$. Represent each x_i, for $i \in X$ in the form of a union of elements of M: $x_i = x_{i0} \vee \ldots \vee x_{im(i)}$. With the help of induction, we construct a decreasing sequence $X = X_0 \supset X_1 \supset X_2 \supset \ldots$ and a function $n = n(i)$ defined for each i from $Y = \{min\, X_k : k \in \omega\}$. Suppose X_k was constructed as well, and $i = min\, X_k$. If $i > j \in X_k$, then, for at least one $m \leq m(i)$ there holds $\neg (x_{im} \leq x_j)$. Since $X_k - \{i\}$ is a union of sets $X_{km} = \{j \in X_k : \neg (x_{im} \leq x_j)\}$. Put $n(i) = min\{m : X_{km}$ is infinite$\}$ and $X_{k+1} = X_{kn(i)}$. It is easy to see that the sequence $\{x_{i\,n(i)}\}_{i \in Y}$ of elements of M satisfies the condition $i < j \longrightarrow \neg (x_{in(i)} \leq x_{jn(j)})$, which contradicts the tightness of M.

Lemma 11 is proved.

THEOREM 2. The closed sets form a tight sublattice of the lattice of all sets of prefixes.

Proof. The union and intersection of two closed sets of prefixes is obviously closed. Accordingly the closure of a set really constitutes a sublattice. Its tightness, too, follows from Lemmas 9 and 11 and Theorem 1.

Theorem 2 is proved.

Definition 6. (Definition of Ξ). Ξ is a quasiordered set. The elements of Ξ are all possible sequences $\xi = (\xi_1, \xi_2, \ldots)$, where each ξ_i is a natural number (0, 1,...) or the symbol ∞, and $\xi \leq \eta\, (mod\, \Xi)$ means the same as

$$\forall i \left(\sum_{i \leq j} \xi_j \leq \sum_{i \leq j} \eta_j \right).$$

Definition 7. (Definition of Ξ^*). Ξ^* is a partially ordered set. The elements of Ξ^* are exactly those elements $\xi \in \Xi$, such that $\xi_1 \geq \xi_2 \geq \xi_3 \geq \ldots$ and either each $\xi_i = \infty$, or ξ ends in zeros. Order in Ξ^* is defined so: $\xi \leq \eta\, (mod\, \Xi^*)$ means the same as $\forall i\, (\xi_i \leq \eta_i)$.

LEMMA 12. For each $\xi \in \Xi$ we put $\xi^* = (\xi_1^*, \xi_2^*, \ldots)$, where $\xi_i^* = \sum_{i \leq j} \xi_j$. The mapping $\xi \to \xi^*$ is a homomorphism of Ξ to Ξ^*. This homomorphism induces an isomorphism of the quotient-sets of Ξ with respect to $\xi = \eta (mod\, \Xi)$ to Ξ^*.

The proof is obvious.

THEOREM 3. The p.o. set Ξ^* is tight.

COROLLARY. The quasi-ordered set Ξ is tight.

Proof of Theorem 3. If $\xi, \eta \in \Xi^*$, then both

$$\{max\{\xi_i, \eta_i\}\}_{i=1,2,\ldots} \text{ and } \{min\{\xi_i, \eta_i\}\}_{i=1,2,\ldots}$$

lie in Ξ^*. This easily follows. Hence Ξ^* is a lattice. We denote by Ξ_1 the collection of all those $\xi \in \Xi^*$ at which all different from zero elements are equal among themselves. Each element of the lattice Ξ^* is the union of a finite number of elements of Ξ. On the strength of Lemma 11, this suffices to prove the tightness of Ξ_1.

Each $\xi \in \Xi_1$ we place in the pair relation (ξ_1, μ), where $\mu = \infty$, if $\xi = (\infty, \infty, \ldots)$, and $\mu = min\{i: \xi_i = 0\}$ otherwise. Obviously, the mapping $\xi \longrightarrow (\xi_1, \mu)$ is an isomorphic imbedding of Ξ_1 in the direct product of the set $\omega \cup \{\infty\}$ onto itself. Application of Lemma 2 completes the proof.

Theorem 3 is proved.

§ 2. Tight Sets and the Decision Problem

Let L be some language of the logic of predicates and first degree operations. The formulas of the language L are constructed by the usual rules with the help of connectives and of quantifiers from individual, predicate, and operational variables, and predicate and operational constants. In connection with the fact that we intend to consider the questions algorithmically, we assume that the variables and constants of the language are words of some fixed finite alphabet and that the following sets are recursive: the set of all formulas in the language L, the set of all individual variables, the set of all predicate variables, the set of all i-place predicate variables for each natural number i, the set of all operational variables and the set of all i-place operational variables for each natural number i.

Denote by $\xi_i(L)$ the number of different i-place predicate variables and by $\eta_i(L)$ the number of different i-place operational variables of the language L. For the sake of convenience of presentation we assume that $\xi_0(L) = \eta_0(L) = 0$. We also assume that if $1 \leqslant i \leqslant j$, then $\xi_j(L) > 0$ implies $\xi_i(L) = \infty$ and $\eta_j(L) > 0$ implies $\eta_i(L) = \infty$. Relative to the number of predicate and operational constants of the language L, no suppositions will be made. The number of individual variables is infinite.

A model of the language L is a nonempty set together with the predicates defined on it and the operational constants of the language L.

Definition 8. (Definition of $\Phi(\Pi, \xi, \eta)$). Let Π be some collection of prefixes and $\xi = (\xi_1, \xi_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots)$ be sequences from Ξ. By $\Phi(\Pi, \xi, \eta)$ we denote the collection of all prenex forms α of the language L which do not contain free individual variables and are such that

1. Prefixes from α lie in Π and

2. For each positive integer i the number of different i-place predicate (resp. operational) variables in α does not exceed ξ (resp. does not exceed η_i).

Let $\Phi = \Phi(\Pi, \xi, \eta)$, $\Phi' = \Phi(\Pi', \xi', \eta')$ and $\Pi' \subseteq \Pi$ and $\xi' \leq \xi \,(mod\, \Xi)$ and $\eta' \leq \eta \,(mod\, \Xi)$. The class Φ' formally may not be contained in the class Φ. But for each $\alpha' \in \Phi'$ there is, in the class Φ some α not essentially different from α'.

For example, if $\forall \exists \forall \in \Pi$ and $\xi_1 = 1$ and $\xi_2 = 2$ and

$$\alpha' = \forall xy\, [P_1(x) \,\&\, P_2(x) \longrightarrow Q_1(x,y)],$$

then, in the role of α we can take the formula

$$\forall x \exists u \forall y\, [\,[P_1(x) \,\&\, Q_2(x,x) \longrightarrow Q_1(x,y)] \,\&\, [P_1(u) \vee \neg P_1(u)]\,].$$

In other words, α arises from α' by means of the operations

A1. Introducing fictitious input quantifiers,

A2. replacing some medicate variables by others with more places, and

A3. proceeding analogously for the operational variables.

For each model $\mathfrak{M}$ of the language L the formulas α and α' are simultaneously well-formed or not well-formed in $\mathfrak{M}$ and simultaneously valid or not valid in $\mathfrak{M}$.

If a problem entering in Π is algorithmically decidable in each of the predicates $A(m,i) \equiv m \leq \xi_i$ and $B(m;i) \equiv m \leq \eta_i$ is decidable (recursive), then there exists an algorithm formed on α' corresponding to α.

There holds the following.

LEMMA 13. Suppose that $\mathfrak{A}$ is an algorithm which discerns well-formedness of any other property of formulas which is preserved by the operations A1-A3. And for each formula from Φ let the algorithm $\mathfrak{A}$ stop work after a finite number of steps. Then the analogous algorithm exists for Φ'.

Proof. Considering the sequence

$$\Phi(\Pi',\xi',\eta') \longrightarrow \Phi(\Pi',\xi',\eta) \longrightarrow \Phi(\Pi',\xi,\eta) \longrightarrow \Phi(\Pi,\xi,\eta),$$

we remark that it is sufficient to analyze three special cases of the lemma;

$$\begin{array}{ll} \xi' = \xi & \text{and } \eta' = \eta \ , \\ \Pi' = \Pi & \text{and } \eta' = \eta \ ., \\ \Pi' = \Pi & \text{and } \xi' = \xi \ . \end{array}$$

We go into the first of them only.

Suppose π_1 is a prefix of the formula α' and $\pi_1, \pi_2, \ldots$ is a sequence containing all prefixes majorizing π_1 in the sense of natural order. Suppose that α_i arises from α' by adding some number (0, 1,...) of occurrences of fictitious quantifiers and that the prefix of α_i is π_i. We force the algorithm $\mathfrak{A}$ to take one step $\alpha' = \alpha_1$, then two steps for each of the formulas α_1, α_2, then three steps for each of the formulas $\alpha_1, \alpha_2, \alpha_3$, etc. It is clear that in a finite number of steps we clarify whether α' has the indicated property.

Lemma 13 is proved.

All that has been said makes for the following natural

Definition 9. (Definition of $\Psi(\Pi,\xi,\eta)$). Let Π be a closed set of prefixes and let $\xi = (\xi_1, \xi_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots)$ be sequences from Ξ^*. The sequence $\Psi(\Pi,\xi,\eta)$ denotes the collection – not containing free individual variables – of all preceding variables in the formulas α of the language L such that

1. The prefix of α lies in Π and

2. For each positive integer i the number of different, not less than i-place, predicate (resp., operational) variables in α is $\leq \xi_i$ (resp. is $\leq \eta_i$).

If Π is standard, the class $\Psi(\Pi,\xi,\eta)$ will also be called standard.

LEMMA 14. Suppose for an arbitrary $\Phi(\Pi,\xi,\eta)$, $\dot{\xi}^*$ and η^* have the same meanings as in Lemma 12. Then $\Psi(\bar{\Pi},\xi^*,\eta^*)$ is the union of all those $\Phi(\Pi',\xi',\eta')$, such that $\Pi' \subseteq \bar{\Pi}$ and $\xi' \leq \xi \pmod{\Xi}$ and $\eta' \leq \eta \pmod{\Xi}$.

The proof is obvious.

Definition 10. The sequence $\mathcal{F}$ denotes the collection, ordered by exclusion, of all possible $\Psi(\Pi,\xi,\eta)$.

THEOREM 4. The p.o. set $\mathcal{F}$ is tight.

Proof. Denote for the present by M the ordered by exclusion collection of all possible closed Π. On the strength of Theorem 2, M is tight. On the strength of Theorem 3, the p.o. set Ξ^* is tight. On the strength of Lemma 2, the direct product $M \times \Xi^* \times \Xi^*$ is tight. The mapping $(\Pi, \xi, \eta) \to \Psi(\Pi, \xi, \eta)$ is a homomorphism of this direct product to $\mathcal{F}$. Applying Lemma 2 completes the proof.

Theorem 4 is proved.

THEOREM 5. Suppose X is a subset of $\mathcal{F}$ such that

1. $\Psi_1 \in \mathcal{F}$ & $\Psi_2 \in X$ & $\Psi_1 \subseteq \Psi_2 \to \Psi_1 \in X$,

2. $\Psi(\Pi_1, \xi, \eta) \in X$ & $\Psi(\Pi_2, \xi, \eta) \in X \to \Psi(\Pi_1 \cup \Pi_2, \xi, \eta) \in X$

and M is the collection of minimal $\Psi \in \mathcal{F} - X$.

Then

1. M is finite;

2. $\Psi \in \mathcal{F} - X$ if and only if $\Psi \in \mathcal{F}$ and contains some $\Psi_0 \in M$;

3. Each $\Psi \in M$ is standard.

Proof. The first assertion of Theorem 5 stems, by Lemma 1, from the tightness of $\mathcal{F}$. The third we show by contradiction. Suppose $\Psi = \Psi(\Pi, \xi, \eta) \in M$ and Π are not standard. By Theorem 1, Π is the union of a finite number of standard Π_i. On the strength of minimality of Ψ the classes $\Psi(\Pi_i, \xi, \eta) \in X$. But then $\Psi = \Psi(\cup \Pi_i, \xi, \eta) \in X$. We have arrived at a contradiction.

Theorem 5 is proved.

As set X in Theorem 5 we can choose the collection of all those classes $\Psi \in \mathcal{F}$, in which a decidable problem is discernible for some property of the formulas. In place of decidability we could speak of primitive recursiveness, or, of degree of unsolvability $< d$ (or $\leq d$) in some reduction sense, d being a fixed degree.

Desiring to make the names of the elements of $\mathcal{F}$ constructive objects and for the sake of convenience of exposition in the following chapter, we give

Definition 11. Suppose $\Psi(\Pi, \xi, \eta) \in \mathcal{F}$. In the expression „$\Psi(\Pi, \xi, \eta)$"

1. We replace „Π"

1.1 by the word $\mathcal{A}ll$, if Π is the set of all prefixes,

1.2 by any $A \in W$, such that $\Pi = \Pi(A)$, or

1.3 by the expression $A_1 \cup \ldots \cup A_m$, if $\Pi = \Pi(A_1) \cup \ldots \cup \Pi(A_m)$;

2. We replace „ξ" (resp., „η")

2.1 by the word $\mathcal{A}ll$, if ξ (resp. η) is $(\infty, \infty, \ldots)$,

2.2 by the symbol $\emptyset$ if ξ (resp., η) is $(0, 0, \ldots)$, and

2.3 by the finite succession obtained from ξ (resp. from η) by omitting tails of zeros, in the remaining cases.

LEMMA 15. The following assertions hold:

1. Suppose A_1 and A_2 run through W. The predicate $A_1 \leq A_2 \ (\mathrm{mod}\ W)$ is primitively recursive.

2. Suppose Π_1 and Π_2 run through the names, indicated in Definition 11, of all possible closed sets of prefixes. The predicate $\Pi_1 \subseteq \Pi_2$ is primitively recursive.

3. Suppose ξ and η run through the names, indicated in Definition 11, of all possible sequences from Ξ^*. The predicate $\xi \leq \eta \ (\mathrm{mod}\ \Xi^*)$ is primitively recursive.

4. Suppose ψ_1 and ψ_2 run through the names, indicated in Definition 11, of all possible elements of $\mathcal{F}$. The predicate $\psi_1 \subseteq \psi_2$ is primitively recursive.

5. The problems entering in a closed Π and in $\psi(\Pi, \mathcal{S}, \eta)$ are primitively recursive.

Proof. We omit tiresome similarities. They are easily supplied. The needed information can be found in [7].

The first Assertion of Lemma 15 stems from the obvious equivalences:

$$
\begin{aligned}
A_1 * \forall \leqslant A_2 * \forall (\mathrm{mod}\, W) &\equiv A_1 \leqslant A_2 (\mathrm{mod}\, W), \\
A_1 * \forall^\infty \leqslant A_2 * \forall (\mathrm{mod}\, W) &\equiv A_1 * \forall^\infty \leqslant A_2 (\mathrm{mod}\, W), \\
A_1 * \exists \leqslant A_2 * \forall (\mathrm{mod}\, W) &\equiv A_1 * \exists \leqslant A_2 (\mathrm{mod}\, W), \\
A_1 * \exists^\infty \leqslant A_2 * \forall (\mathrm{mod}\, W) &\equiv A_1 * \exists^\infty \leqslant A_2 (\mathrm{mod}\, W), \\
A_1 * \forall \leqslant A_2 * \forall^\infty (\mathrm{mod}\, W) &\equiv A_1 \leqslant A_2 * \forall^\infty (\mathrm{mod}\, W), \\
A_1 * \forall^\infty \leqslant A_2 * \forall^\infty (\mathrm{mod}\, W) &\equiv A_1 \leqslant A_2 * \forall^\infty (\mathrm{mod}\, W), \\
A_1 * \exists \leqslant A_2 * \forall^\infty (\mathrm{mod}\, W) &\equiv A_1 * \exists \leqslant A_2 (\mathrm{mod}\, W), \\
A_1 * \exists^\infty \leqslant A_2 * \forall^\infty (\mathrm{mod}\, W) &\equiv A_1 * \exists^\infty \leqslant A_2 (\mathrm{mod}\, W)
\end{aligned}
$$

and analogously for $A_2 * \exists$ and $A_2 * \exists^\infty$.

From this first Assertion, and Lemma 8, follows the primitive recursiveness of the predicate $\Pi(A_1) \subseteq \Pi(A_2)$. Further

$$\Pi(A) \subseteq \bigcup_{i=1}^{m} \Pi_i \tag{1}$$

which is equivalent with the disjunction

$$\bigvee_{i=1}^{m} [\, \Pi(A) \subseteq \Pi_i \,]. \tag{2}$$

Indeed, (2) obviously implies (1). Conversely, suppose $A^{(n)}$ is obtained from A by replacing each occurrence of q^∞ by an occurrence of q^n. Each $A^{(n)}$ lies in some Π_i. By finiteness of m, some Π_i contains an infinite number of $A^{(n)}$, i.e., contains $\Pi(A)$. Hence the predicate

$$\Pi(A) \subseteq \bigcup_{i=1}^{m} \Pi(A_i)$$

is primitively recursive. Finally $\cup \Pi(B_j) \subseteq \cup \Pi(A_i)$ is equivalent with the conjunction $\wedge\, [\Pi(B_j) \subseteq \cup \Pi(A_i)]$. Assertion 2 is proved.

The remaining Assertions follow from the second and third.

Lemma 15 is proved.

THEOREM 5'. If the class $\psi(\Pi, \mathcal{S}, \eta)$ is assigned to correspond with Definition 11, then, under the conditions of Theorem 5, there also holds the following: the problem of occurrence of $\psi(\Pi, \mathcal{S}, \eta)$ in X is primitive recursive.

The proof is obvious.

§ 3. Remarks

1. Relative to the language L, we assumed that it did not contain propositional (i.e., zero place predicate) variables. This limitation is unessential. There is known a simple method of getting rid of

propositional variables sufficient for many purposes (see [1] for instance). Besides, we can consider possible the presence in L of propositional variables. In the definitions of $\Phi(\Pi, \xi, \eta)$ and $\Psi(\Pi, \xi, \eta)$ there could be on exhibit auxiliary parameters to characterize the amount of propositional variables. Thereupon the Theorems of §2 maintain their force.

2. Relative to the language L, we assumed also that it did not contain variables of zero place operations. This limitation, too, is unessential. Here too we can use additional parameters, characterizing the presence of zero place operational variables.

There is another consideration. Suppose that in the definion of $\Psi(\Pi, \xi, \eta)$ we consider that $\eta = (\eta_0, \eta_1, \dots)$, where η_i bounds the number of i-place operations. There holds

LEMMA 16. Let $\eta = (\eta_0, \eta_1, \dots)$ and $\eta^* = (0, \eta_1, \eta_2, \dots)$,

$$\Pi^* = \{\exists^n * \pi : n \in \omega \ \& \ n \leq \eta_0 \ \& \ \pi \in \Pi\}.$$

There exists a primitive recursive algorithm processing an arbitrary $\alpha \in \Psi(\Pi, \xi, \eta)$ and an $\alpha^* \in \Psi(\Pi^*, \xi, \eta^*)$, such that α and α^* are well-formed on the same models of the language L, and conversely.

Proof. For the translation from α to α^* we need to change the zero-place variables of operation by individual variables and to induce the existential quantifiers. For translation from α^* to α we need to proceed inversely. Lemma 16 is proved.

3. We remark further that the orderings with respect to inclusion of the collection of closed sets of prefixes and the p.o. set Ξ^* and $\mathcal{F}$ are distributive and complete lattices.

4. Theorems 5 and 5' do not depend on our assumptions that the formulas of the language L are constructive objects. These theorems maintain their force also for certain other logics.

CHAPTER 2. THE PROBLEM OF DECIDABILITY FOR THE SIMPLE LOGIC OF PREDICATES AND OPERATIONS

§ 1. Formulation of the Basic Result

For scanning this chapter, only the definitions from the previous chapter are needed. On the other hand, Chapter 2 can be considered a continuation of Chapter 1. In this chapter we make further assumptions relative to the language L. This will be the language of the simple logic of predicates and operations. For simplicity, we indicate as missing predicates and operations of constants. In particular, the quality sign is absent. At the beginning of §2 in Chapter 1 we supposed that certain sets have recurisve relations in the language L. In this chapter we suppose that these sets are primitive recursive. We shall also consider that for $i > 0$, $\xi_i(L) = \eta_i(L) = \infty$.

Definition 11 of Chapter 1 attributed standard names to the class $\Psi(\Pi, \xi, \eta)$. Analogously we award names to the classes $\Phi(\Pi, \xi, \eta)$ in those cases where Π is closed and each of the sequences ξ and η neither consists only of symbols ∞ nor has a tail of zeros. Thus, for example, $\Phi(\mathcal{A}ee, (0,1), \phi)$ is the collection of all prenex forms not containing free individual variables of the language L, in which to be sure there are not operational symbols and in which occurs exactly one predicate symbol and this a two-place one. Of course, we could skirt the question without the classes $\Phi(\Pi, \xi, \eta)$, but it is much more natural to limit these classes to predicate and operational variables.

We shall presume a certain Gödel numeration of the formulas of the language L. The Gödel number of the formula α will be denoted $[\alpha]$.

Definition 12. We shall say that the set Φ of the formulas of the language L is primitive recursive according to Loewenheim (and according to [9] it was the first formula discovered to be well-formed only in infinite domains), if there exists a primitive recursive function f such that if $\alpha \in \Phi$ is simply well-formed then it is well-formed also for a finite domain of power $\leq f[\alpha]$.

Let Φ_1 and Φ_2 be classes of formulas and g be a recursive function such that for $\alpha_1 \in \Phi_1$, $g([\alpha_1])$ is the Gödel number of some $\alpha_2 \in \Phi_2$. If, in this formula, α_1 is well-formed if and only if α_2 is well-formed, we say that the class Φ_1 reduces to Φ_2 with respect to well-formedness. If moreover Φ_1 is the class of all formulas of the language L then Φ_2 is called the class reduced with respect to well formedness. Analogous definitions are introduced for finite well-formedness, i.e., in respect of well-formedness in finite domains. If the function g reduces Φ_1 to Φ_2 both for simple well-formedness and for finite well-formedness, we say that Φ_1 conservatively reduces to Φ_2. If at the same time the function g is primitive recursive we can speak of primitive recursive and conservative reduction.

Definition 13. Suppose the function g affords a primitive recursive and conservative reduction of Φ_1 and Φ_2. We call this reduction simply strong if there exists primitive recursive function $h_1(m,n)$ and $h_2(m)$, such that if $g([\alpha_1]) = [\alpha_2]$, then well-formedness of α_1 on a set of power $\leq m$ implies well-formedness of α_2 on a set of power $\leq h_1(m, [\alpha_1])$ and well-formedness of α_2 on a set of power $\leq m$ implies well-formedness of α_1 on a set of power $\leq h_2(m)$.

Strong reducibility, conveniently for us, agrees with primitive recursiveness according to Loewenheim.

LEMMA 17. Suppose Φ_1 strongly reduces to Φ_2. Then if the class Φ_2 is primitive recursive according to Loewenheim, so is Φ_1, and if Φ_1 is a strongly reducing class, so is Φ_2.

The proof is obvious.

There holds (see [2])

THEOREM 6. Each class $\psi = \psi(\Pi, \xi, \Phi)$ satisfies at least one of the three following conditions.

1. Π and ξ_1 are finite,
2. ψ is primitive recursive according to Loewenheim,
3. ψ is a strongly reducing class.

ψ is primitive recursive according to Loewenheim if and only if $\xi_2 = 0$ or

$$\Pi \subseteq \Pi(\exists^\infty \forall^\infty) \cup \Pi(\exists^\infty \forall^2 \exists^\infty).$$

ψ is a strongly reducing class if and only if it contains at least one of the following nine classes:

$$\begin{aligned}
\Phi 1 &= \Phi(\forall\exists\forall,\ (\infty, 1),\ \emptyset),\\
\Phi 2 &= \Phi(\forall^3\exists,\ (\infty, 1),\ \emptyset),\\
\Phi 3 &= \Phi(\forall^\infty\exists,\ (0, 1),\ \emptyset),\\
\Phi 4 &= \Phi(\forall\exists\forall^\infty,\ (0, 1),\ \emptyset),\\
\Phi 5 &= \Phi(\exists^\infty\forall^3\exists,\ (0, 1),\ \emptyset),\\
\Phi 6 &= \Phi(\forall^3\exists^\infty,\ (0, 1),\ \emptyset),\\
\Phi 7 &= \Phi(\exists^\infty\forall\exists\forall,\ (0, 1),\ \emptyset),\\
\Phi 8 &= \Phi(\forall\exists^\infty\forall,\ (0, 1),\ \emptyset),\\
\Phi 9 &= \Phi(\forall\exists\forall\exists^\infty,\ (0, 1),\ \emptyset).
\end{aligned}$$

It is true that in place of the primitive recursiveness according to Loewenheim in [2] the discussion is concerned with effective recognition of well-formedness and finite well-formedness, and in place of strongly reducing classes the discussion involves reducing classes according to well-formedness and finite well-formedness. We do not here have in mind new recognition and reduction algorithms. We simply remark the stronger properties of those algorithms which are mentioned in [2]. The verification of the admissibility itself, we do not go into here.

(As might have been done also in the definition of strong reducibility, we may consider that $h_2(m) \equiv m$.)

The present Chapter is dedicated to the proof of the following theorem.

THEOREM 7. Each class $\psi = \psi(\Pi, \xi, \eta)$ satisfies at least one of the following three conditions:

1. Π and ξ_1 are finite and $\eta_1 = 0$;

2. ψ is primitive recursive according to Loewenheim;

3. ψ is a strongly reducing class.

ψ is primitive recursive according to Loewenheim if and only if one, at least, of the following four possibilities holds:

1. $\xi_1 = 0$ (trivial possibility);
2. $\xi_2 = \eta_2 = 0$;
3. $\Pi \subseteq \Pi(\exists^{\infty}\forall^{\infty}) \cup \Pi(\exists^{\infty}\forall^{2}\exists^{\infty})$ and $\eta_1 = 0$;
4. $\Pi \subseteq \Pi(\exists^{\infty}\forall\exists^{\infty})$.

ψ is a strongly reducing class if and only if it contains at least one of the classes $\Phi 1 - \Phi 9$ of Theorem 6 or one of the following classes:

$$\Phi 10 = \Phi(\forall^2,\ (0,1),\ (1)),$$
$$\Phi 11 = \Phi(\forall^3,\ (1)\ ,\ (0,1)).$$

It is not hard to see that the first formulation of Theorem 7 follows from Theorem 6 and two assumptions of Theorem 8. The primitive recursiveness according to Loewenheim of the class $\Phi(\exists^{\infty}\forall\exists^{\infty}, \mathcal{All}, \mathcal{All})$ is proved in [3]. It need be shown only, then, that the class $\Phi(\mathcal{All}, (\infty), (\infty))$ is primitive recursive according to Loewenheim and that the sets $\Phi 10$ and $\Phi 11$ are strongly reducing classes. The proof of these facts provides the content of the remainder of this work.

§ 2. Strongly Reducing Classes

THEOREM 8. The class $\Phi 10$ of Theorem 7 is a strongly reducing class.

Proof. Let $\alpha = \forall x \exists u \forall y \mathfrak{A}(x,u,y)$ be a formula from $\Phi 1$ (see Theorem 6), let P be the symbol of a two place predicate from $\mathfrak{A}$, and let $P_1, \dots, P_{m-1}$ be all symbols of one place predicates from $\mathfrak{A}$. Let $\cdot\,'\,\cdot$ be the symbol of a one place function. Assume $x^0 = x,\ x^{i+1} = (x^i)'$. We denote by $\mathfrak{A}^*$ the formula obtained from $\mathfrak{A}$ by replacing, for each $\vartheta = x, u, y$ and each $i = 1, \dots, m-1$, the occurrence of each subformula $P_i(\vartheta)$ by an occurrence of subformula $P(\vartheta, \vartheta^i)$.. For brevity, we denote by $\mathfrak{B}(x)$ the formula $P(x', x)$ and by $\mathfrak{L}(x)$ the formula

$$\bigvee_{i=0}^{m-1} \mathfrak{B}(x) \quad \& \quad \bigwedge_{0 \leq i < j \leq m-1} \neg(\mathfrak{B}(x^i) \ \& \ \mathfrak{B}(x^j)).$$

Put

$$\alpha^* = \forall xy\, [\, \mathfrak{L}(x)$$
$$\&\ [\, \mathfrak{B}(x)\ \&\ \mathfrak{B}(y) \longrightarrow \mathfrak{A}^*(x, x^m, y)]\,]\,.$$

It is easy to see that the mapping $\alpha \Longrightarrow \alpha^*$ effects a strong reduction of $\Phi 1$ to $\Phi 10$.

Theorem 8 is proved.

In May, 1966, V. P. Orevkov communicated to the author his result that the class of contractions by derivability is the set of formulas of the form $\exists xy\,(\mathfrak{D}_1 \& \mathfrak{D}_2)$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are disjunctions of elementary formulas or their negations, containing 2 symbols of one place operations and a fixed number of symbols of two place and one place predicates. The proof of this result has not be published. If we abstract from the present fact only two conjunctions, the Orevkov result is an immediate consequence of Theorem 8. In this article we are interested in the structure of the non-quantified parts only and in the definitions of the present symbols of predicates and operations.

THEOREM 9. The class $\Phi 11$ of Theorem 7 is a strongly reducing class.

Proof. Let
$\alpha \in \Phi 10$,

P be the symbol of a two place predicate from α ,

f be the symbol of a one place operation from α ,

p be the symbol of a one place predicate and

F be the symbol of a two place operation.

In α we replace the operation $f(x)$ by the operation $F(x,x)$ and the predicate $P(x,y)$ by the predicate $p(F(x,y))$. We get some α^* . The mapping $\alpha \Rightarrow \alpha^*$ affords a strong reduction of $\Phi 10$ to $\Phi 11$. We show this. Let $\mathfrak{M}=\langle M,P,f\rangle$ be a model for α , i.e., $\mathfrak{M}$ is an algebraic system for which α obtains true values. On the strength of universality of the formula α each subsystem of the system $\mathfrak{M}$, closed with respect to the operation f, is also a model of α . Hence without affecting generality, we can consider that $M=\{a, fa, ffa, \dots\}$ for some a . In case M is finite, let m be the least number for which $f^m(a)=f^k(a)$ for $k<m$. If $k>1$, we discard from M the element $a,\dots,f^{k-2}(a)$. We again get a model for α . If $k=0$ we add to M a new element b and put $fb=fa$,

$$P(b, f^i(a)) \text{ is equivalent to } P(a, f^i(a)),$$

$$P(f^i(a), b) \text{ is equivalent to } P(f^i(a), a).$$

Accordingly, without affecting generality, we can consider that either M is infinite or, for some $m>1$ the elements $a, fa, \dots, f^{m-1}(a)$ are distinct and $f^m(a)=f(a)$. Denote $a_i=f^i(a)$. We place

$$p(a_{i+1}) \text{ is equivalent to } P(a_i, a_i),$$

$$p(a_0) \text{ is equivalent to } \neg p(a_1),$$

$$F(a_i, a_i)=a_{i+1}.$$

Suppose $a_i \neq a_j$ and, for definiteness, that the assertion $p(a_0)$ showed up as true. Further, assume that:

$$P(a_i, a_j) \text{ implies } F(a_i, a_j)=a_0,$$

$$\neg P(a_i, a_j) \text{ implies } F(a_i, a_j)=a_1.$$

As a result we get a model for α^*. Then the construction of the model for α according to the model for α^* is obvious.

Theorem 9 is proved.

§ 3. The Case of Absence of Two Place Symbols

Theorem 10. The set $\Phi(\mathcal{All}, (\infty), (\infty))$ is primitive recursive according to Loewenheim.

Proof. Let

$$\alpha \in \Phi(\mathcal{All}, (\infty), (\infty))$$

let $p_1, \dots, p_m$ be all the different symbols of one place predicates from α and $f_1, \dots, f_m$ be different symbols of one place operations not encountered in α .

Substitute in α for each individual variable x in each sub-formula $p_i(x)$ the sub-formula $p_1(f_i(x))$. We get some α^* . Obviously the mapping $\alpha \Longrightarrow \alpha^*$ brings about strong reduction of

$$\Phi(\mathcal{All}, (\infty), (\infty)) \quad \kappa \quad \Phi(\mathcal{All}, (1), (\infty)).$$

On the strength of Lemma 17 it remains to show the primitive recursiveness according to Loewenheim of this last set. We remark that in the foregoing part of the proof we could have confined it to introduction only of a single new operation symbol.

Suppose $\alpha \in \Phi(\mathcal{All}, (1), (\infty))$ and that P is the symbol of a one place predicate from α . Each term t of α has the form $t=x$ or $t=f_j f_{j-1} \dots f_1(x)$, where f_i and f_j are not necessarily different. In the first case the height of the term t we call zero and in the second, the number j . We denote by m the term of the maximum height from α and by T the set of all possible terms of heights $\leq m$ with symbols of operations from α . Terms translating into each other by remaining individual variables will not be distinguished in T . Then T is finite.

Let $\mathfrak{M}$ be a model for α , i.e., some set M in which operations P and predicate symbols from α are such that α obtains true values. Define in M an equivalence relation E as follows:

$$E(a,b) \text{ is equivalent to } \bigwedge_{t\in T} [p(t(a)) \sim p(t(b))].$$

The set M/E we denote by $\mathcal{N}$. Obviously $\overline{\overline{\mathcal{N}}} \leq 2^{\overline{\overline{T}}}$. We assume that

$$p(a/E) \text{ is equivalent to } p(a).$$

Further, we fix, in each class of a/E some element which we shall call sa . Assume that for every symbol of operation f from α

$$f(a/E) = f(sa)/E.$$

The definition of P and of the operation symbols convert $\mathcal{N}$ into some algebraic system $\mathfrak{N}$. We show that $\mathfrak{N}$ is a model of α . With this, Theorem 10 will be proved.

Each atomic sub-formula of α has the form $p(t)$, where $t\in T$. Hence it suffices for us to prove that for all $t\in T$ and every

$$a\in M \qquad p(t(a)) \text{ is equivalent to } p(t(a/E)). \tag{1}$$

For $i\leq m$ we denote by T_i the collection of all terms from T of height $\leq i$ and by E_i the relations in M such that $E_i(a,b)$ is equivalent to

$$\bigwedge_{t\in T_i} [p(t(a)) \sim p(t(b))].$$

Thus $T=T_m$ and $E=E_m$. For $i<m$ we have that

$$E_{i+1}(a,b) \quad \text{implies} \quad E_i(a,b).$$

LEMMA 18. If $t\in T_i$, then

$$t(a/E) = c/E \qquad \text{implies} \quad E_{m-i}(t(a),c).$$

Proof. For $i=0$ the Lemma is obvious. Suppose $i=1$, $t(x)=f(x)$ for some f and $f(a/E) = c/E$. On the other hand

$$f(a/E) = f(sa)/E \qquad \text{and } E(a,sa).$$

Hence we have $E_{m-1}(fa,f(sa))$ and $E_m(f(sa),c)$, and hence $E_{m-1}(fa,c)$. Further by induction. Suppose $t=f(\tau)\in T_{i+1}$ and

$$\tau(a/E) = b/E \text{ and } f(b/E) = c/E.$$

We have $E_{m-i}(\tau(a),b)$ and $E_{m-1}(f(b),c)$.
From which we obtain $E_{m-i-1}(t(a),f(b))$ and $E_{m-i-1}(t(a),c)$.

Lemma 18 is proved.

Substituting $i=m$ in Lemma 18 we get

$$t(a/E) = c/E \text{ implies } p(t(a)) \sim p(c).$$

Since $P(c/E)$ is equivalent with $p(c)$, there follows (1).

Theorem 10 is proved. And along with it, Theorem 7.

We shall not discuss the class $\Phi(\exists^\infty\forall\exists^\infty, \mathcal{A}ll, \mathcal{A}ll)$ here (see [3] for this). The class $\Phi(\mathcal{A}ll, (\infty),(\infty))$ in fact is easily led to the class $\Phi(\exists^\infty\forall\exists^\infty, \mathcal{A}ll, \mathcal{A}ll)$. According to [8] we solve as to deducibility the interesting class of formulas containing both $\Phi(\forall^\infty\exists\forall^\infty, \mathcal{A}ll, \mathcal{A}ll)$ and $\Phi(\mathcal{A}ll, (\infty),(\infty))$. For the latter see also [5]. For the class Φ_{10} see the remark following the proof of Theorem 8.

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