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A CONTRIBUTION TO THE ELEMENTARY THEORY OF LATTICE ORDERED ABELIAN GROUPS AND K-LINEALS

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According to [1], the class of all linearly ordered abelian groups (o-groups) is solvable, i.e. the elementary theory of this class is solvable. On the other hand, it is easy to see that the class of all partially ordered abelian groups is unsolvable. In connection with this (and clearly even without this connection) the question of the solvability of the class of all lattice ordered abelian groups (l-groups) is an interesting one. In particular, the question itself was formulated by A. I. Mal'cev at the International Congress of Mathematicians in Moscow in 1966 (see also [2]). In [3] the solvability of the universal theory of the class of all l-groups is proved, and the classification of l-groups by universal properties is undertaken.

Let us denote by L the class of all abelian l-groups satisfying the following requirements:

1. It is possible to define a multiplication of elements of G by real numbers making G into a K-lineal, i.e. into a vector space in which multiplication is compatible with the lattice operations, see [4]. In particular, G is a divisible group.
2. G is archimedean (see [5]).
3. The lattice $\hat{G}$ of filets (see [5]) of the l-group G is atomic.
4. $\hat{G}$ is a Boolean algebra.

Theorem 1. *Every class of l-groups containing L is unsolvable.*

In defining L atomicity may be replaced by nonatomicity. Requirement 4 may be replaced by the requirement that $\hat{G}$ be relatively complemented and with no units. Of course requirement 1 may be replaced by divisibility and countability. Requirement 1 may also be replaced by the requirement that G be countable and free. With all these variations Theorem 1 remains valid.

Thus what usual classes of l-groups are solvable? The solvability of certain classes of l-groups follows, according to general theorems of the theory of modules (see [6]), from the solvability of the class of all l-groups. Thus we have

Theorem 2. *The class of all l-groups with a finite number of filets is solvable.*

Proof. By *tree* we mean a finite partially ordered set satisfying the axiom $(x \leq y$ and $x \leq z) \longrightarrow (y \leq z$ or $z \leq y)$. From the classification of l-groups with a finite number of filets given in [2] it follows that up to elementary equivalence an l-group with a finite number of filets is none other than a generalized product of Feferman and Vaught (see [7] or [6]) of some o-groups with respect to the algebra of subsets of some tree. Thus from the theorem of Feferman and Vaught it follows that the solvability of the class of all l-groups with a finite number of filets comes out of the solvability of the class of all o-groups and from the solvability of the theory of trees with quantifiers as one-place predicates. The latter trivially follows from [8].

Note that many natural subclasses of the class of all l-groups with a finite number of filets are

* Editor's note. The present translation incorporates suggestions made by the author.

also solvable: the class of all divisible l-groups with a finite number of filets, the class of all l-groups of finite rank, the class of all free l-groups.

An l-group G is called *atomic* (respectively, *nonatomic*, *with unit*, *without unit*) if the lattice $\hat{G}$ is such. The *width* of an l-group G is the maximal number of strictly positive and pairwise orthogonal elements of G, if that number exists and is finite, and the symbol ∞ otherwise. An l-group is called a *K-group* if it admits a multiplication by reals making it into a K-space (see [4]). A K-group is the same thing as a complete (as in [5], i.e. conditionally complete as a lattice) and divisible l-group.

Theorem 3. *Every K-group G decomposes into a direct sum of atomic and nonatomic K-groups G_a and G_b. Two K-groups G and G′ are elementarily equivalent if and only if G_a and G'_a are elementarily equivalent and G_b and G'_b are elementarily equivalent. All nonatomic K-groups with unit (without unit) are elementarily equivalent, and such K-groups exist. All atomic K-groups without unit are elementarily equivalent, and such K-groups also exist. Finally, two atomic K-groups with unit are elementarily equivalent if and only if they have the same width, and there exist atomic K-groups with unit of arbitrary width.*

Theorem 4. *Let M be a class of K-groups and S the totality of all natural numbers n such that M contains a K-group of width n. The class M is solvable if and only if the set S is recursive. In particular, the class of all K-groups is solvable.*

Remark. A complete l-group G is the direct sum of a certain K-group G_0 and a certain complete l-group G_1 without divisible elements (see [5]). We do not investigate such G_1 here. Note only that in contradiction with the assertion on page 139 of [5], G_1 is not necessarily a subdirect sum of cyclic o-groups. In fact, let Q be extremal compact, separable and without isolated points. Such Q clearly exist. Let $C_\infty(Q)$ be the K-group of all continuous and almost everywhere finite (i.e. taking the values $\pm\infty$ on nowhere dense subsets) real functions on Q, see [4]. Denote by C^0 the l-subgroup of all functions in $C_\infty(Q)$ all finite values of which are integers. C^0 is a complete l-group without divisible elements and it does not embed into a direct sum of archimedean o-groups preserving $+$, $\bigwedge$, and $\bigvee$.

In conclusion, something about K-lineals. We consider a K-lineal X as a pair $\langle R, X'\rangle$ where R is the field of reals and X' is an l-group and a vector space over R, and for all x, y in X' and any positive ξ in R we have $x < y \longrightarrow \xi x < \xi y$. By *width* of the K-lineal X we mean the width of the l-group X'.

Theorem 5. *If M is a class of K-lineals and if the supremum of the widths of the K-lineals in M is ∞, then the class M is unsolvable. If n is a natural number and M is the class of all K-lineals of width $\leq n$ then the class M is solvable.*

If X is a K-lineal of width 1, denote by $J(X)$ the ordered set of jumps of the convex subgroups of X'.

Theorem 6. *Two K-lineals X and Y of width 1 are elementarily equivalent if and only if the ordered sets $J(X)$ and $J(Y)$ are elementarily equivalent. Let M be the class of K-lineals of width 1 and $J(M) = \{J(X): X \text{ in } M\}$. The class M is solvable if and only if $J(M)$ is. In particular, the class of all K-lineals of width 1 is solvable.*

The classification of K-lineals of finite width by elementary properties up to K-lineals of width 1 is analogous to the classification of l-groups with a finite number of filets up to o-groups.

Theorem 7. *Let n be a natural number and M the class of all K-lineals of width $\leq n$.*

Then M is solvable.

Theorem 8. *Let M be a class of K-spaces. M is solvable if and only if the width of the K-spaces in M is bounded by a finite number.*

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