ELEMENTARY PROPERTIES OF ORDERED ABELIAN GROUPS

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In the paper a classification of ordered abelian groups by elementary properties is set up.

It will be shown that the elementary theory of ordered abelian groups is solvable if and only if the elementary theory of ordered sets is solvable. Now in [1] it is announced as a theorem that the elementary theory of ordered sets is solvable, and this then implies the solvability of the elementary theory of ordered abelian groups.

§1. Notation and definitions

1.1. Let σ be a signature; $\Phi(\sigma)$ is a set of formulas in LPC (the lower predicate calculus) of signature σ that do not contain free variables; $\Phi_n(\sigma)$ is the set of formulas in $\Phi(\sigma)$ having in their prenex form not more than n quantifiers. If $\mathfrak{M}$ is a model of σ, then $\Phi(\mathfrak{M})$ is the set of formulas in $\Phi(\sigma)$ that are true on $\mathfrak{M}$. By definition, $\Phi_n(\mathfrak{M}) = \Phi(\mathfrak{M}) \cap \Phi_n(\sigma)$. If K is a class of models of σ, then $\Phi(K) \overset{df}{=} \bigcap_{\mathfrak{M} \in K} \Phi(\mathfrak{M})$. $\Phi(K)$ is called the elementary theory of the class.

1.2. A predicate $P(x_1, \cdots, x_n)$ is called formular in the class K of models of signature σ if there exists a formula $\mathfrak{A}(x_1, \cdots, x_n)$ in LPC of signature σ such that $P(x_1, \cdots, x_n) \equiv \mathfrak{A}(x_1, \cdots, x_n)$ in all models of K. In the case $n = 0$ we obtain the definition of a formular proposition.

Let $\tau = (Q, R, \cdots)$ be a set of formular predicates in the class K of models of signature σ, and $\mathfrak{M} \in K$. $|\mathfrak{M}|$ denotes the set of elements of $\mathfrak{M}$. Since $Q, R, \cdots$ are formular predicates, their values in $|\mathfrak{M}|$ are defined. By $< |\mathfrak{M}|, \tau > = < |\mathfrak{M}|, Q, R, \cdots >$ we denote the corresponding model of signature τ. In particular, $\mathfrak{M} = < |\mathfrak{M}|, \sigma >$. Suppose, moreover, that P is a unary formular predicate in K. Then $< |\mathfrak{M}|, \tau >^P$ is by definition the submodel of $< |\mathfrak{M}|, \tau >$ containing precisely those elements of $|\mathfrak{M}|$ that satisfy P.

1.3. The following expressions will be treated as synonyms:

1. The elementary theory of the class K is solvable.
2. $\Phi(K)$ is solvable.
3. K is a solvable class.

1.4. p_i denotes the ith prime number in ascending order. If s is a positive integer, then $l(s)$ is the index of the greatest prime divisor of s, $(s)_i$ is the exponent of the greatest power of p_i dividing s, and $\pi(s)$ is the set of prime divisors of s.

1.5. We abbreviate "winning strategy," "ordered set," "ordered abelian group" by w.s., o.s., o.a.g., respectively.

1.6. Order is everywhere to be interpreted as linear order.

§2. $\Gamma_n(\mathfrak{M}_1, \mathfrak{M}_2)$

Following [2] we shall set forth below conditions for the equality $\Phi_n(\mathfrak{M}_1) = \Phi_n(\mathfrak{M}_2)$. Analogous conditions in the language of mappings were obtained independently by A. D. Taĭmanov [3].

Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be models of σ. $\Gamma_n(\mathfrak{M}_1, \mathfrak{M}_2)$ is a game with two players I and II. The players move in turn; I begins. At the kth move ($k = 1, \cdots, n$) he chooses first a number l_k equal to 1 or 2. Then he chooses in the model $\mathfrak{M}_{l_k}$ an element $a_k^{l_k}$. II at the kth move chooses the element $a_k^{3-l_k}$ in $\mathfrak{M}_{3-l_k}$. After n moves we have a correspondence:

$$\begin{array}{c} a_1^1 \longleftrightarrow a_1^2 \\ \cdots\cdots\cdots \\ a_n^1 \longleftrightarrow a_n^2. \end{array} \qquad (*)$$

If (*) is an isomorphism, then II wins; otherwise I wins.

Lemma 1. *If II has a w.s. in $\Gamma_n(\mathfrak{M}_1, \mathfrak{M}_2)$, then $\Phi_n(\mathfrak{M}_1) = \Phi_n(\mathfrak{M}_2)$.*

Now let σ be finite and $t = t(\sigma)$ be a positive integer such that $(t)_i$ is the number of i-ary predicates of σ.

Lemma 2. *There exists a primitive recursive function $N = N(t(\sigma), n)$ such that II has a w.s. in $\Gamma_n(\mathfrak{M}_1, \mathfrak{M}_2)$ if $\Phi_{N(t(\sigma),n)}(\mathfrak{M}_1) = \Phi_{N(t(\sigma),n)}(\mathfrak{M}_2)$.*

We shall also use the game $\Gamma_{(n_1,\cdots,n_k)}(\mathfrak{M}_1, \mathfrak{M}_2)$, which differs from $\Gamma_l(\mathfrak{M}_1, \mathfrak{M}_2)$ only in that at the kth move the players choose n_k elements each, not necessarily one by one. Obviously, we have

Lemma 3. *Let $n = n_1 + \cdots + n_l$. If II has a w.s. in $\Gamma_n(\mathfrak{M}_1, \mathfrak{M}_2)$, then II also has a w.s. in $\Gamma_{(n_1,\cdots,n_l)}(\mathfrak{M}_1, \mathfrak{M}_2)$.*

§3. m-chains

3.1. Let m be a positive integer or ω, and τ_m a signature containing a binary predicate $x < y$ and unary predicates $|x| = k$, $1 \leq k < m$.

$x \leq y$ is an abbreviation for $\neg(y < x)$,

$s \asymp y$ is an abbreviation for $\neg(x < y) \mathbin{\&} \neg(y < x)$,

$|x| \neq k$ is an abbreviation for $\neg(|x| = k)$.

If $x < y$, we say that x precedes y. If $|x| = k$, we say that the norm of x is k.

T_m is the class of models of signature τ_m satisfying the following axioms (universal quantifiers preceding all other formal symbols have been omitted here and later in suitable cases):

(3.1.1) $x \asymp x$

(3.1.2) $x \asymp y \longrightarrow y \asymp x$.

(3.1.3) $x \asymp y \,\&\, y \asymp z \longrightarrow x \asymp z$.

(3.1.4) $x_1 = x_2 \,\&\, y_1 = y_2 \,\&\, x_1 < y_1 \longrightarrow x_2 < y_2$.

(3.1.5) $x \asymp y \,\&\, |x| = k \longrightarrow |y| = k, \quad 1 \le k < m$.

(3.2.1) $x < y \longrightarrow \neg\, (y < x)$.

(3.2.2) $x < y \,\&\, y < z \longrightarrow x < z$.

(3.3.1) $|x| = i \longrightarrow |x| \ne j, \quad i \ne j, \quad 1 \le i, \quad j < m$.

Models of the class T_m are called m-chains. Thus, if we do not distinguish between elements connected by $\asymp$, an m-chain is an o.s. whose elements may have no norm at all or one and the same norm, the norm being a positive integer less than m; no connection is assumed between order and norm. T_1 is simply the class of o.s.

It is easy to see that

$$\Phi(T_m) = \Phi(T_\omega) \cap \Phi(\tau_m).$$

For the sake of brevity we shall also use the predicate $|x| = 0$, which means that x has no (positive) norm. For finite m, $|x| = 0$ is an abbreviation for $|x| \ne 1 \,\&\, \cdots \,\&\, |x| \ne m-1$. The predicate $|x| = 0$ does not occur in τ_m nor in the formulas of $\Phi(T_m)$.

3.2. **Lemma 4.** *Suppose that:*

1. *K is a solvable class of models of signature σ;*

2. *$\tau = (Q_1, Q_2, \cdots)$ is a set of formular predicates in K and $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ are corresponding formulas in LPC of signature σ;*

3. *P is a unary formular predicate in K;*

4. *L is a class of models of signature τ such that:* a) *for every $\mathfrak{M} \in K$, $< |\mathfrak{M}|, \tau >^P \in L$, and* b) *for every $\mathfrak{N} \in L$ there is an $\mathfrak{M} \in K$ for which $< |\mathfrak{M}|, \tau >^P \cong \mathfrak{N}$.*

Then $\Phi(L)$ is solvable.

Proof. Let $\mathfrak{A} \in \Phi(\tau)$. In $\mathfrak{A}$ we replace the predicates Q_i by the formulas $\mathfrak{A}_i$ and restrict the quantifiers to P. We obtain a formula $\mathfrak{L} \in \Phi(\sigma)$. Clearly, $\mathfrak{A} \in \Phi(L)$ is equivalent to $\mathfrak{L} \in \Phi(K)$.

3.3. **Lemma 5.** *The elementary theory of ω-chains is solvable if the elementary theory of o.s. is solvable.*

Proof. Let $S(x, y)$ be an abbreviation for $\neg\, \exists\, z (x < z < y) \,\&\, x < y$.

We set

$$|x| = k \overset{df}{\equiv} \exists x_1 \cdots \exists x_k \forall y [S(x, x_1) \,\&\, S(x_1, x_2) \,\&\, \cdots$$
$$\cdots \,\&\, S(x_{k-1}, x_k) \,\&\, \neg S(x_k, y)], \quad 1 \le k < \omega;$$
$$P(x) \overset{df}{\equiv} \neg \exists y S(y, x).$$

We use Lemma 4, with T_1, τ_1, T_ω, τ_ω in place of K, σ, L, τ, respectively. Obviously, if $\mathfrak{N} \in T_1$, then $<|\mathfrak{M}|, \tau_\omega>^P$ is in fact an ω-chain. Now let $\mathfrak{N}$ be an ω-chain. With an element $x \in \mathfrak{N}$ of norm k $(k = 0, 1, \cdots)$ we associate a set A_x of order type $1 + k + \omega^* + \omega$. We assume that $A_x = A_y$ for $x \asymp y$ and $A_x \cap A_y = \emptyset$ otherwise. We extend the partial order in $\bigcup A_x$ to a linear order by postulating that for $x < y$ every element of A_x shall precede every element of A_y. So we obtain an o.s. which we call associated with $\mathfrak{N}$. Clearly, this o.s. is what we were looking for.

Corollary. *Let m be finite. The class of models τ_m satisfying only the axioms* (3.1.1) – (3.2.2) *is solvable if the class of o.s. is solvable.*

3.4. Let $\mathfrak{M}$, $\mathfrak{N}$ be ω-chains and $\mathfrak{M}'$, $\mathfrak{N}'$ be the o.s. associated with them.

Lemma 6. *For $\Phi(\mathfrak{M}) = \Phi(\mathfrak{N})$ it is necessary and sufficient that $\Phi(\mathfrak{M}') = \Phi(\mathfrak{N}')$.*

The necessity follows from the fact that a w.s. for II in $\Gamma_n(\mathfrak{M}', \mathfrak{N}')$ can easily be constructed from a w.s. for II in $\Gamma_n(\mathfrak{M}, \mathfrak{N})$.

Sufficiency. Suppose that $\Phi(\mathfrak{M}') = \Phi(\mathfrak{N}')$. Then $\Phi(<|\mathfrak{M}'|, \tau_\omega>^P) = \Phi(<|\mathfrak{N}'|, \tau_\omega>^P)$. But $<|\mathfrak{M}'|, \tau_\omega>^P \cong \mathfrak{M}$, $<|\mathfrak{N}'|, \tau_\omega>^P \cong \mathfrak{N}$.

§4. Linear order in direct sums of cyclic groups

4.1. By σ_0 we denote the signature containing a single ternary predicate $S(x, y, z)$. For $S(x, y, z)$ we often write $x + y = z$.

$x = 0$ is an abbreviation for $\forall y S(x, y, y)$,

$x = y$ is an abbreviation for $\forall z (z = 0 \longrightarrow S(z, x, y))$,

$S(x, y, 0)$ is an abbreviation for $\forall z (z = 0 \longrightarrow S(x, y, z))$.

K^0 is the class of models of signature σ_0 satisfying the following axioms:

(4.1.1) $x = x$.

(4.1.2) $x = y \longrightarrow y = x$.

(4.1.3) $x = y \,\&\, y = z \longrightarrow x = z$.

(4.1.4) $x_1 = x_2 \,\&\, y_1 = y_2 \,\&\, z_1 = z_2 \,\&\, S(x_1, y_1, z_1) \longrightarrow S(x_2, y_2, z_2)$.

(4.2.1) $\exists z S(x, y, z)$.

(4.2.2) $S(x, y, z_1) \,\&\, S(x, y, z_2) \longrightarrow z_1 = z_2$.

(4.2.3) $S(x, y, u) \,\&\, S(u, z, w_1) \,\&\, S(y, z, v) \,\&\, S(x, v, w_2) \longrightarrow w_1 = w_2$.

(4.2.4) $S(x, y, z) \longrightarrow S(y, x, z)$.

(4.2.5) $\exists x(x = 0)$.

(4.2.6) $\exists x S(x, y, 0)$.

Thus, if we do not distinguish between elements connected by the sign $=$, then K^0 is an abelian group.

4.2. Let K_{ij} be the class of abelian groups consisting of direct sums of cyclic groups of order p_i^j and the null group.

Lemma 7. *Every nonzero element g in $G \in K_{ij}$ lies in a cyclic direct summand.*

Corollary. *Let $G \in K_{ij}$, $g \in G$ and $0 \le k \le j$. Then*

$$p_i^k g = 0 \longrightarrow p_i^{j-k} | g.$$

Definition. A correspondence

$$g^\nu \longleftrightarrow h^\nu, \qquad (1)$$

where $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$; G, $H \in K_{ij}$ is called regular (between elements of the groups G, $H \in K_{ij}$) if there exist elements $g_1, \cdots, g_r \in G$, $h_1, \cdots, h_r \in H$, and integers ξ_μ^ν such that $\{g_1\} + \cdots + \{g_r\}$ is a direct summand of G, $\{h_1\} + \cdots + \{h_r\}$ is a direct summand of H, and $g^\nu = \Sigma \xi_\mu^\nu g_\mu$, $h^\nu = \Sigma \xi_\mu^\nu h_\mu$.

Here $\{g\}$ denotes the cyclic subgroup generated by g and $+$ is the sign of a direct sum.

The smallest possible r is called the rank of the regular correspondence (1).

Let $\alpha(G)$, where $G \in K_{ij}$, be the number of cyclic direct summands in a decomposition of G. Then $\alpha_n(G) \overset{df}{=} \min(\alpha(G), n)$.

Lemma 8. *Let (1) be a regular correspondence of rank $r < n$ and $\alpha_n(G) = \alpha_n(H)$. Then for every $g \in G$ there is an $h \in H$ such that the extension of (1) by means of $g \longleftrightarrow h$ is a regular correspondence of rank $\le r + 1$.*

4.3. Let K_s, where s is a positive integer, be the class of abelian groups satisfying the axiom $\forall x(sx = 0)$.

By Prüfer's first theorem [4] a group $G \in K_s$ can be represented as a direct sum $G = \Sigma G_{ij}$, where $G_{ij} \in K_{ij}$ and $G_i \overset{df}{=} \Sigma_j G_{ij}$ is a Sylow subgroup.

By g_i we denote the component of $g \in G$ in G_i.

Remark. There exists an integer ζ_i, depending on s and i only, such that $g_i = \zeta_i g$. For future reference we fix a definite ζ_i for all s and i.

Definition. A correspondence

$$g^\nu \longleftrightarrow h^\nu, \qquad (2)$$

where $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$; G, $H \in K_s$ is called regular (between elements of G, $H \in K_s$) if there are direct decompositions $G = \Sigma G_{ij}$, $H = \Sigma H_{ij}$ with G_{ij}, $H_{ij} \in K_{ij}$ such that the correspondence

$$g^{\nu}_{ij} \longleftrightarrow h^{\nu}_{ij} \qquad (2_{ij})$$

is regular, where g^{ν}_{ij}, h^{ν}_{ij} are the components of g^{ν}, h^{ν} in G_{ij}, H_{ij}, respectively, $\nu = 1, \cdots, k$.

The maximal rank of the correspondences (2_{ij}) is called the rank of (2).

Let $G \in K_s$ and $G = \Sigma G_{ij}$, where $G_{ij} \in K_{ij}$. We set

$$\beta_n(G) \overset{df}{=} \Sigma\, \alpha_n(G_{ij}) \cdot (n+1)^{(s)_1 + \cdots + (s)_{i-1} + j - 1}.$$

Lemma 9. *Suppose that (2) is a regular correspondence of rank $r < n$ and $\beta_n(G) = \beta_n(H)$. Then for every $g \in G$ there is an $h \in H$ such that the extension of (2) by means of $g \longleftrightarrow h$ is a regular correspondence of rank $\leq (r+1)$.*

4.4. Let K^*_s be the class of abelian groups consisting of direct sums of cyclic groups of the orders $p_1^{(s)_1}, \cdots, p_{l(s)}^{(s)_{l(s)}}$ and the null group.

Lemma 10. *Let $G \in K^*_s$, $g \in G$ and $m \mid s$. Then $mg = 0 \longrightarrow (s/m) \mid g$.*

Proof. The proof is based on the Corollary to Lemma 7.

4.5. Let ρ_1 be the signature consisting of the predicates $S(x, y, z)$ and $x < y$.

L_s is the class of models $\mathfrak{M}$ of signature ρ_1 such that

a) $< |\mathfrak{M}|, S(x, y, z) > \in K^*_s$;

b) L_s satisfies the axioms (3.1.1)–(3.1.4), (3.2.1), (3.2.2) (with the same conventions about abbreviations);

c) $\mathfrak{M}$ satisfies the axioms

(4.3.1) $x \leq 0 \longrightarrow x = 0$,

(4.3.2) $x + y \leq \max(x, y)$,

(4.3.3) $px < y \longrightarrow \exists z(pz = px \;\&\; z < y)$,

where $p \in \pi(s)$.

The following simple properties of models of class L_s are easily checked, where m is an arbitrary integer, n divides s, and $o(x)$ denotes the order of the element x in the sense of group theory.

1) $x = y \longrightarrow x \asymp y$.
2) $mx \leq x$, $x_i \leq x$.
3) $(m, s) = 1 \longrightarrow mx \asymp x$, $(m, o(x)) = 1 \longrightarrow mx \asymp x$, $-x \asymp x$.
4) $x \not\asymp y \longrightarrow x + y \asymp \max(x, y)$.
5) $x = \max_i x_i$
6) $mx < y \longrightarrow \exists\, z(mz = mx \;\&\; z < y)$.
7) $D(x) \overset{df}{=} < \hat{y}\,(y < x),\ S(x, y, z) > \in K^*_s$.
8) $E(x) \overset{df}{=} < \hat{y}(y \leq x), S(x, y, z) > \in K_s$.
9) $F(x) \overset{df}{=} E(x) / D(x) \in K_s$.

In 7) and 9) it is assumed that $x \neq 0$. In addition, we set $D(0) = \emptyset$, $F(0) = 0$.

Let $G \in L_s$. A subgroup A is called convex if

$$x,\ y \in A \ \&\ x < z < y \longrightarrow z \in A.$$

The empty set is also counted among the convex subgroups. $D(x)$ and $E(x)$ are convex subgroups. $D(x)$ is serving. $E(x) \setminus D(x)$ is called a jump of G.

4.6. Let μ range over a set M of natural numbers. Let $E^1_\mu \setminus D^1_\mu$ be jumps of $G \in L_s$, $E^2_\mu \setminus D^2_\mu$ be jumps of $H \in L_s$, $F^1_\mu = E^1_\mu / D^1_\mu$, $F^2_\mu = E^2_\mu / D^2_\mu$.

Definition. A correspondence

$$D^1_\mu \longleftrightarrow D^2_\mu \tag{3}$$

is called n-regular if

1) $D^1_{\mu 1} \subsetneq D^1_{\mu 2} \equiv D^2_{\mu 1} \subsetneq D^2_{\mu 2}$,

2) $\beta_n(F^1_\mu) = \beta_n(F^2_\mu)$.

Definition. A correspondence

$$g^\nu \longleftrightarrow h^\nu, \tag{4}$$

where $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$, G, $H \in L_s$, is called n-regular if the following conditions hold:

1) $D(\Sigma \xi_\nu g^\nu) \longleftrightarrow D(\Sigma \xi_\nu h^\nu)$,

is an n-regular correspondence, where $\xi_1, \cdots, \xi_k$ range over all the integers.

2) Let $\Sigma \xi^1_\nu g^\nu, \cdots, \Sigma \xi^l_\nu g^\nu$ be all the elements of the form $\Sigma \xi_\nu g^\nu$ belonging to $E(\Sigma \xi^1_\nu g^\nu)$. Then $\Sigma \xi^1_\nu h^\nu, \cdots, \Sigma \xi^l_\nu h^\nu$ are all the elements of the form $\Sigma \xi_\nu h^\nu$ belonging to $E(\Sigma \xi^1_\nu h^\nu)$, and

$$\left.\begin{array}{c} \Sigma \xi^1_\nu g^\nu + D(\Sigma \xi^1_\nu g^\nu) \longleftrightarrow \Sigma \xi^1_\nu h^\nu + D(\Sigma \xi^1_\nu h^\nu), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Sigma \xi^l_\nu g^\nu + D(\Sigma \xi^1_\nu g^\nu) \longleftrightarrow \Sigma \xi^l_\nu g^\nu + D(\Sigma \xi^1_\nu h^\nu) \end{array}\right\} \tag{4'}$$

is a regular correspondence between elements of the groups

$$F(\Sigma \xi^1_\nu g^\nu),\ F(\Sigma \xi^1_\nu h^\nu) \in K_s.$$

Here (4′) is called a subcorrespondence related to $D(\Sigma \xi^1_\nu g^\nu)$. The rank of (4) is the maximum of the ranks of the subcorrespondences.

Corollary. *Let* (4) *be an n-regular correspondence and suppose* $m \mid s$. *Then* $m \mid \Sigma \xi_\nu g^\nu \longrightarrow m \mid \Sigma \xi_\nu h^\nu$.

For if $m \mid \Sigma \xi_\nu g^\nu$, then $(s/m) \Sigma \xi_\nu g^\nu = 0 \in E(\Sigma 0 \cdot g^\nu)$. Hence $(s/m) \Sigma \xi_\nu h^\nu = 0$ and, by Lemma 10, $m \mid \Sigma \xi_\nu h^\nu$.

4.7. **Lemma 11.** *Given:*

0) G, $H \in L_{p^s}$, *where p is a prime number;* $E^1 \setminus D^1$ *and* $E^2 \setminus D^2$ *are jumps*

of G *and* H, *respectively;* $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$; $r \le n$; $g \in G$.

1) *The correspondence*

$$D(\Sigma \xi_\nu g^\nu) \longleftrightarrow D(\Sigma \xi_\nu h^\nu), \quad D^1 \longleftrightarrow D^2, \tag{5}$$

where $\xi_1, \cdots, \xi_k$ *range over all possible integers, is n-regular.*

2) *The correspondence*

$$g^\nu \longleftrightarrow h^\nu \tag{6}$$

where $\nu = 1, \cdots, k$, *is n-regular, and the rank* r *can be reached only in subcorrespondences relating to those* $D(\Sigma \xi_\nu g^\nu)$ *that are contained in* D^1.

3) $p^j g$ *is minimal among the elements of the form* $p^j g + \Sigma \xi_\nu g^\nu$ *for* $0 \le j < \eta$ *and* $p^j g \in E^1 \setminus D^1$.

4) $p^\eta g = \Sigma \xi^0_\nu g^\nu \in D^1$.

Then there is an $h \in H$ *such that the extension of* (6) *by means of* $g \longleftrightarrow h$ *is an n-regular correspondence; also, the rank* r *cannot be reached in subcorrespondences relating to those* $D(\Sigma \xi_\nu g^\nu)$ *that contain* D^1.

Proof. By the Corollary to the second Definition in 4.6, there is an $a \in H$ such that $p^\eta a = \Sigma \xi^0_\nu h^\nu \in D^2$. Since D^2 is serving, we may assume that $a \in D^2$.

By Lemma 9 there is a $b \in H$ such that the extension of the subcorrespondence relating to D^1 by means of $g + D^1 \longleftrightarrow b + D^2$ is a regular correspondence of rank $\le r$. Also $p^\eta b \in D^2$. Let $c \in D^2$ and $p^\eta c = p^\eta b$. We set $h = b - c + a$. Then h is the required element.

Lemma 12. *Given:*

0) $G, H \in L_{p^s}$; $E^1_\mu \setminus D^1_\mu$ *are jumps of* G, $E^2_\mu \setminus D^2_\mu$ *are jumps of* H, $\mu = 1, \cdots, s$; $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$; $g \in G$.

1) *The correspondence*

$$\begin{cases} D(\Sigma \xi_\nu g^\nu) \longleftrightarrow D(\Sigma \xi_\nu h^\nu), \\ D^1_\mu \longleftrightarrow D^2_\mu, \end{cases}$$

is n-regular.

2) *The correspondence*

$$g^\nu \longleftrightarrow h^\nu \tag{7}$$

is n-regular of rank $r < n$.

3) a_μ *is minimal among the elements of the form* $p^{s-\mu} g + \Sigma \xi_\nu g^\nu$ *and* $a_\mu \in E^1_\mu \setminus D^1_\mu$.

Then there is an $h \in H$ *such that the extension of* (7) *by means of* $g \longleftrightarrow h$ *is an n-regular correspondence of rank* $\le (r + 1)$.

Proof. Let $i < j$; then $p^{j-i} a_j$ is of the form $p^{s-i} g + \Sigma \xi_\nu g^\nu$, so that

$a_i \le p^{j-i} a_j \le a_j$. Let $p^j \mid g$ with $0 \le j \le \mu_0$. Then (assuming $g \ne 0$, i.e., $\mu_0 < s$; the case $g = 0$ is trivial)

$$\begin{array}{l} 0 = a_{\mu_0} < a_{\mu_0+1} \asymp \cdots \asymp a_{\mu_1} \\ \cdots\cdots\cdots\cdots \\ a_{\mu_{k-1}} < a_{\mu_{k-1}+1} \asymp \cdots \asymp a_{\mu_k} \\ \cdots\cdots\cdots\cdots \\ a_{\mu_{l-1}} < a_{\mu_{l-1}+1} \asymp \cdots \asymp a_{\mu_l} = a_s. \end{array}$$

Clearly, without violating the conditions of the lemma we may replace (cf. the kth row) $a_{\mu_{k-1}}$ by pa_{μ_k} and $a_{\mu_{k-1}+1}$ by $p^{\mu_k - \mu_{k-1} - 1} a_{\mu_k}$, $k = 1, \cdots, l$.

The conditions of Lemma 11 are satisfied with $D^1_{\mu_1}$, $D^2_{\mu_1}$, a_{μ_1} in place of D^1, D^2, g. Therefore there is a $b_1 \in H$ such that the extension of (7) by means of $a_{\mu_1} \longleftrightarrow b_1$ is an n-regular correspondence with the appropriate restriction on the ranks of subcorrespondences. In a similar way we extend the correspondence so obtained by means of $a_{\mu_2} \longleftrightarrow b_2$, etc. Suppose that we have found $b_1, \cdots, b_l$ such that the extension of (7) by means of $a_{\mu_1} \longleftrightarrow b_1, \cdots, a_{\mu_l} \longleftrightarrow b_l$ is n-regular of rank $\le (r+1)$, and let $a_{\mu_l} = a_s = g + \Sigma \xi^0_\nu g^\nu$. We set $h = b_l - \Sigma \xi^0_\nu h^\nu$, and then h is the required element.

4.8. Let $G \in L_s$. By G_i we denote the submodel of G that contains precisely the elements x for which $p_i^{(s)_i} = 0$. $G_i \in L_{p_i^{(s)_i}}$.

Lemma 13. *Given:*

0) $G, H \in L_s$; $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$; $g \in G$, $h \in H$.

1) *The correspondence* $g^\nu \longleftrightarrow h^\nu$ *is n-regular.*

2) *The correspondence* $g_i^\nu \longleftrightarrow h_i^\nu$, $g_i \longleftrightarrow h_i$ *is n-regular between elements of the models* G_i, H_i, $i = 1, \cdots, l(s)$.

3) *The correspondence*

$$D(\Sigma \xi_\nu g^\nu + \xi g) \longleftrightarrow D(\Sigma \xi_\nu h^\nu + \xi h) \tag{8}$$

is n-regular.

Then $g^\nu \longleftrightarrow h^\nu$, $g \longleftrightarrow h$ *is also an n-regular correspondence.*

Proof. Let $x = \Sigma \xi_\nu g^\nu + \xi g$, $y = \Sigma \xi_\nu h^\nu + \xi h$. $x_i = \zeta_i x$, $y_i = \zeta_i y$ (see 4.3). Since (8) is a regular correspondence, $x_i \asymp x$ is equivalent to $y_i \asymp y$. The rest follows from the Definition in 4.3.

Lemma 14. *Given:*

0) $G, H \in L_s$; $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$; $E^1_{ij} \setminus D^1_{ij}$ *and* $E^2_{ij} \setminus D^2_{ij}$, $p_i^j | s$, *are jumps of* G *and* H, *respectively;* $g \in G$.

1) *The correspondence*

$$g^\nu \longleftrightarrow h^\nu \tag{9}$$

is n-regular of rank $r > n$.

2) *The correspondence $D(\Sigma\,\xi_\nu g^\nu) \longleftrightarrow D(\Sigma\,\xi_\nu h^\nu)$, $D^1_{ij} \longleftrightarrow D^2_{ij}$ is n-regular.*

3) *a_{ij} is the minimal element of the form $p_i^{(s)i-j} g_i + \Sigma\,\xi_\nu g_i^\nu$ and $a_{ij} \in E^1_{ij} \setminus D^1_{ij}$.*

Then there is an $h \in H$ such that the extension of (9) by means of $g \longleftrightarrow h$ is an n-regular correspondence of rank $\le (r+1)$.

Proof. It is not hard to see that the conditions of Lemma 12 are satisfied with p_i in place of p. Therefore there is a b^i such that the correspondence $g_i^\nu \longleftrightarrow h_i^\nu$, $g_i \longleftrightarrow b^i$ between elements of the models G_i, H_i is n-regular. We set $h = \Sigma\, b^i$. Now the conditions of Lemma 13 can be verified.

4.9. Let $\rho_s = (S(x, y, z),\ x < y,\ P_j(x),\ 1 \le j < s)$.

L_s^* is the class of those models $\mathfrak{M}$ of signature ρ_s for which $< |\mathfrak{M}|,\ \rho_1 > \in L_s$ and $\mathfrak{M}$ satisfies, in addition, the following axioms, $P(x)$ being an abbreviation for $P_1(x) \vee \cdots \vee P_{s-1}(x)$:

(4.4.1) $P(x)\ \&\ x \asymp y \longrightarrow P(y)$.

(4.4.2) $P(x) \longrightarrow \exists y (P_1(y)\ \&\ x \asymp y)$.

(4.4.3) $P_1(x) \longrightarrow o(x) = s$.

(4.4.4) $P_1(x) \longrightarrow P_j(jx),\ \ 1 \le j < s$.

(4.4.5) $P_j(x)\ \&\ P_j(y)\ \&\ x \asymp y \longrightarrow x - y < x,\ \ 1 \le j < s$.

In other words, in $\mathfrak{M} \in L_s$ some of the jumps $E \setminus D$ have been chosen for which $F = E/D$ is cyclic of order s. The elements of the chosen jumps and only they satisfy P. Further, in every such cyclic F exactly one generator is chosen. Suppose, for example, that $g + D$ is the chosen generator of F. The elements of the coset $jg + D$ for $1 \le j < s$ satisfy the predicate P_j.

The results obtained above for L_s carry over to L_s^*. For this purpose it is sufficient:

a) to supplement the first Definition of 4.6 by the postulate that $E^1_\mu \setminus D^1_\mu$ is a chosen jump if and only if $E^2_\mu \setminus D^2_\mu$ is;

b) to supplement the second Definition of 4.6 by the postulate

$$P_j(\Sigma\,\xi_\nu g^\nu) \equiv P_j(\Sigma\,\xi_\nu h^\nu), \quad 1 \le j < s;$$

c) to attach asterisks in the statements of Lemmas 11–14 to the letters denoting the corresponding classes;

d) in the proof of Lemma 11, modified in this way, to consider also the case when g satisfies one of the predicates P_j.

Then b can be taken to be an arbitrary element of $E^2 \setminus D^2$ satisfying P_j, and the rest remains unaltered.

The new lemmas will be called $11^* - 14^*$, respectively.

§5. s-regularity

5.1. Let $\sigma^* = (S(x, y, z), x < y)$. K^* is the class of models of signature σ^* that satisfy the axioms (4.1.1) – (4.2.6) and, in addition, the following axioms:

(5.1.5) $x_1 = x_2 \& y_1 = y_2 \& x_1 < y_1 \longrightarrow x_2 < y_2$.

(5.3.1) $x < y \longrightarrow \neg (y < x)$.

(5.3.2) $x < y \& y < z \longrightarrow x < z$.

(5.3.3) $(x = y) \equiv [\neg (x < y) \& \neg (y < x)]$.

(5.4.1) $x < y \longrightarrow x + z < y + z$.

Thus, if we do not distinguish between elements connected by the sign $=$, then a model of the class K^* is an o.a.g.

5.2. Definition. An o.a.g. G is called s-regular, where s is a positive integer, if it satisfies the axiom

$$\forall x_1 \cdots \forall x_s \exists y (x_1 < \cdots < x_s \longrightarrow x_1 \leq y \leq x_s \& s \mid y). \tag{A}$$

From (A) it follows immediately that

$$\forall z \forall x_1 \cdots \forall x_s \exists y (x_1 < \cdots < x_s \longrightarrow x_1 \leq y \leq x_s \& y \equiv z (\text{mod } s)). \tag{B}$$

Lemma 15. *The definitions of s-regularity given above are equivalent.*

For let G be discrete and s-regular in the sense [6] (the remaining cases are trivial), and suppose that $x_1 < \cdots < x_s$. Among the elements of the form $x_1 + ke$, where e is the least positive element of G, there is $x_1 + re \equiv 0 \pmod s$. Without loss of generality we may assume that $0 \leq r < s$. But then $x_1 \leq x_1 + re \leq x_s$.

Lemma 16. *Let $s = m \cdot n$. An o.a.g. G is s-regular if and only if it is m-regular and n-regular.*

5.3. Definition. Let π be a set of prime numbers. An o.a.g. G is called π-regular if it is p-regular for every $p \in \pi$. If π is the set of all prime numbers, then G is called regular.

An archimedean group is regular.

From Lemma 16 it follows that an o.a.g. G is s-regular if and only if it is $\pi(s)$-regular.

Lemma 17. *An o.a.g. G is π-regular if and only if for every nonzero convex subgroup C the group G/C is π-complete (i.e., satisfies the axiom $\forall x (px = 0)$ for every $p \in \pi$).*

Proof. 1) Let G be p-regular, $C \neq 0$ a convex subgroup, and $g \in G$. In $g + C$ there is an element divisible by p, so that $p \mid g + C$. 2) Let G/C be π-complete for every nonzero convex subgroup C, and $x_1, \cdots, x_s \in G$ and

$x_1 < \cdots < x_s$. We set $y = \min(x_{i+1} - x_i)$. Let A be the union of all convex subgroups not containing y, B the intersection of all convex subgroups containing y, and $p \in \pi$. We examine separately two subcases.

2a) $A \neq 0$. Then $p \mid A + x_1$. Let $A + x_1 = A + pz$. Then $x_1 \leq pz + p \cdot |x_1 - pz| < x_1 + y \leq x_2$.

2b) $A = 0$. Since $p \mid b + x_1$, B contains $b \equiv - x_1 \pmod p$, i.e., $p \mid b + x_1$. But since B is archimedean, it is p-regular and satisfies axiom (B). Then we may assume that $0 \leq b \leq (p-1)$ and therefore $x_1 \leq x_1 + b \leq x_s$.

Corollary. *An extension of a π-regular group by a π-complete group is π-regular.*

5.4. Let π be a set of prime numbers, G an o.a.g., and $g \in G$, $g \neq 0$. Let $A(g)$ be the union of the convex subgroups not containing g, and $B(g)$ the intersection of the convex subgroups containing g. $B(g)/A(g)$ is archimedean and hence π-regular. Let $A_\pi(g)$ be the intersection of those convex subgroups C for which $B(g)/C$ is π-regular. Let $B_\pi(g)$ be the union of those convex subgroups C for which $C/A(g)$ is π-regular. By means of Lemma 17 and its Corollary it is easy to check:

Lemma 18. *Let C be an arbitrary convex subgroup of an o.a.g. G and g, $h \in G$ with g, $h \neq 0$.*

1) *$B_\pi(g)/A_\pi(g)$ is π-regular.*

2) *If $C \subset A_\pi(g)$, then $B_\pi(g)/C$ is not π-regular.*

3) *If $C \supset B_\pi(g)$, then $C/A_\pi(g)$ is not π-regular.*

4) $[A_\pi(g) \subset A_\pi(h)] \longrightarrow [B_\pi(g) \subseteq A_\pi(h)]$.

5) $A_\pi(g) = \bigcup_{p \in \pi} A_p(g)$.

6) $B_\pi(g) = \bigcap_{p \in \pi} B_p(g)$.

$B_\pi(g) \setminus A_\pi(g)$ will be called a *π-regular jump*, and $C_\pi(g) \overset{df}{=} B_\pi(g)/A_\pi(g)$ the *π-regular factor* corresponding to g.

Thus the set of nonzero elements of an o.a.g. splits into the set-theoretical union of disjoint π-regular jumps. In addition, we set $A_\pi(0) = \emptyset$, $B_\pi(0) = 0$, $C_\pi(0) = 0$. Instead of $A_{\pi(s)}(g)$, $B_{\pi(s)}(g)$, $C_{\pi(s)}(g)$ we shall write $A_s(g)$, $B_s(g)$, $C_s(g)$.

5.5. **Lemma 19.** *Let g, $h > 0$.*

$$[A_s(g) \subseteq A_s(h)]$$
$$\equiv [g \leq h \vee \forall x \exists y (g \geq x \geq h \longrightarrow |y| < sh \;\&\; y \equiv x \pmod s)].$$

Proof. 1) Let $A_s(g) \subseteq A_s(h)$ & $g \geq x \geq h$. Then $A_s(g) = A_s(h)$. By virtue of the s-regularity of $C_s(g)$ there is a $z \in B_s(g)$ such that

$$A_s(g) \leq A_s(g) + z \leq A_s(g) + (s-1)h$$

and

$$A_s(g) + z \equiv A_s(g) + x \pmod{s}.$$

Let $z - x + A_s(g) = su + A_s(g)$. We set $y = x + su$.

2) Let $A_s(g) \supset A_s(h)$. Then $A_s(h) \subset B_s(h) \subseteq A_s(g) \subset B_s(g)$ and $B_s(g)/B_s(h)$ is not s-complete. In $B_s(g) \setminus B_s(h)$ there is an x such that $x > 0$ and $s \nmid x + B_s(h)$. Then $h < x \leq g$, and if $|y| < sh$, then $y \in B_s(h)$ and therefore $y \not\equiv x \pmod{s}$.

Corollary. *The predicates $A_s(g) \subset A_s(h)$ and $A_s(g) = A_s(h)$ are formular in the class K^*.*

5.6. Let $\alpha_i(g)$ be the number of modulo p_i independent elements of $C_s(g)$ (in the usual sense), and let $\alpha_{in} = \min(\alpha_i(g), n)$. We define the (s, n)-norm of the convex subgroup $A_s(g)$, in symbols $|A_s(g)|_n$, to be the number

$$\sum_{i=1}^{l(s)} \alpha_{in}(g) \cdot (n+1)^{i-1}$$

if $C_s(g)$ is dense, and to be $\Sigma_{i=1}^{l(s)} n \cdot (n+1)^{i+1} + 1 = (n+1)^{l(s)}$ if $C(g)$ is discrete.

Lemma 20. *The predicate $|A_s(g)|_n = k$ is formular in the class K^*.*

§6. s-fundamentality

6.1. **Definitions.** The *s-fundament* $D_s(g)$ of an element g of an o.a.g. G is the union of those convex subgroups C for which $C \cap (g + sG) = \emptyset$. A convex subgroup is called *s-fundamental* if it is $D_s(g)$ for some g. A convex subgroup is called π-fundamental if it is p-fundamental for at least one $p \in \pi$.

Corollaries. *Let C be an arbitrary convex subgroup of an o.a.g. G and g, $h \in G$.*

1) $s \mid g \equiv D_s(g) = \emptyset$.

2) $D_s(g) \cap (g + sG) = \emptyset$.

3) *$C = D_s(g)$ is equivalent to $C = D_s(g + C)$, where $D_s(g + c)$ is the s-fundament of the element $g + C$ of the o.a.g. G/C.*

4) *If $D_s(g) \leq D_s(h)$, then $D_s(g + h) \leq D_s(h)$. If $D_s(g) < D_s(h)$, then $D_s(g + h) = D_s(h)$.*

5) *$D_n(g) \subseteq D_{mn}(g)$, where m and n are positive integers. It is easy to construct an example with $m > 1$ in which $D_n(g) \subset D_{mn}(g)$.*

6) *$D_{mn}(mg) = D_n(g)$.*

7) *Let $C = D_{p^s}(g) \supset D_p(g)$. Then there is an integer j such that $D_{p^j}(g) \subset D_{p^{j+1}}(g) = C$. Also, in C there is an element $h \equiv g \pmod{p^j}$. Let $g - h = p^j g'$. Then $D_p(g') = D_{p^{j+1}}(g - h) = D_p(g) = C$.*

Hence and from 6) it follows that C is p^s-fundamental if and only if it is *p-fundamental.*

8) $\forall g \forall h [D_s(g) \subseteq A_s(h) \vee B_s(h) \subseteq D_s(g)]$.

In other words, an s-fundamental subgroup either does not intersect an s-regular jump at all, or it contains it completely.

9) $D_s(g) \subseteq A_s(g)$.

10) $D_s(g) = \bigcap_{h \in g + sG} A_s(h)$.

11) $D_s(g) = \max(D_{p_i^{(s)_i}}, \ 1 \le i \le l(s))$.

Hence and from the preceding it follows that C is s-fundamental if and only if it is $\pi(s)$-fundamental.

12) *Let* $p \mid s$ *and* $D_s(pg) \subset D_s(h)$. *Then in* $D_s(h)$ *there is a* $g^1 \equiv ph \pmod s$. *Hence* $p | g^i$. *Let* $g^1 = pg^2$. *Then* $g^2 \in D_s(h)$.

13) *G is π-regular if and only if* $\forall g (D_\pi(g) \subseteq 0)$.

6.2. The set of elements h such that $D_s(h) \subset D_s(g)$ forms a subgroup which we denote by $D_s^*(g)$. $[D_s(g) \subset D_s(h)] \equiv [D_s^*(g) \subset D_s^*(h)]$; $E_s(g) \overset{df}{=} < \hat{h}(D_s(h) \subseteq D_s(g)), s >$; $F_s(g) \overset{df}{=} E_s(g)/D_s^*(g)$. $E_s(g) \setminus D_s^*(g)$ will be called an s-jump of the o.a.g. G.

If $\exists h (D_s(g) = A_s(h))$, then the convex subgroup $D_s(g)$ already has an (s, n)-norm. (See 5.6.)

Let $\neg \exists h (D_s(g) = A_s(h))$. Then the number

$$(n+1)^{l(s)} + \beta_n(F_s(g))$$

(see 4.3) will be called the (s, n)-norm of the s-fundamental subgroup $D_s(g)$ and will be denoted by $|D_s(g)|_n$.

Let

$$m = m(s, n) = (n+1)^{l(s)} + \sum_{p_i^j \mid s} n \cdot (n+1)^{(s)_1 + \cdots + (s)_{i-1} + j - 1} + 1$$
$$= (n+1)^{l(s)} + (n+1)^{(s)_1 + \cdots + (s)_{l(s)}}.$$

The convex subgroups $A_s(g)$ and $D_s(g)$ of the o.a.g. G with the (s, n)-norms assigned to them above and the inclusion relation form an m-chain (to within the notation of the predicates) which we shall denote by $T_{s,n}(G)$.

6.3. Let P_{sj} signify that there exists an h such that $D_s(g) = A_s(h)$ and that $G_s(h)$ is discrete and $A_s(h) + g \equiv A_s(h) + je \pmod s$ in $G/A_s(h)$, where $A_s(h) + e$ is the least positive element of $C_s(h)$, $1 \le j < s$.

Let

$$G_s = < G/sG, \ D_s(g) \subset D_s(h), \ P_{sj}(g), \quad 1 \le j < s >.$$

It is not difficult to verify that G_s is an element of L_s^* (see 4.9), but with the inessential provision that instead of $g < h$ and $P_j(g)$ we now write

$D_s(g) \subset D_s(h)$ and $P_{sj}(g)$, respectively.

Under the (group) homomorphism $G \longrightarrow G/sG$ the subgroups $D_s^*(g)$ and $E_s(g)$ go over into $D(g+sG)$ and $E(g+sG)$. Conversely, the complete inverse images of $D(g+sG)$ and $E(g+sG)$ are $D_s^*(g)$ and $E_s(g)$. $F_s(g) \cong F(g+sG)$. The element $\zeta_i g$ (see the Remark in 4.3) goes over into $(g+sG)_i$.

For later reference we rephrase the necessary definition and Lemma 14*.

6.4. Definition. A correspondence

$$g^\nu \longleftrightarrow h^\nu, \tag{1}$$

where $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$, G and H o.a.g. is called (s, n)-*regular* if:

1) The correspondence $D_s(\Sigma \xi_\nu g^\nu) \longleftrightarrow D_s(\Sigma \xi_\nu h^\nu)$, where $\xi_1, \cdots, \xi_k$ range over all the integers, is an isomorphism of the submodels $T_{s,n}(G)$ and $T_{s,n}(H)$.

2) Let $\Sigma \xi_\nu^1 g^\nu, \cdots, \Sigma \xi_\nu^l g^\nu$ be all the elements of the form $\Sigma \xi_\nu g^\nu$ belonging to $E(\Sigma \xi_\nu^1 g^\nu)$. Then $\Sigma \xi_\nu^1 h^\nu, \cdots, \Sigma \xi_\nu^l h^\nu$ are all the elements of the form $\Sigma \xi_\nu h^\nu$ belonging to $E(\Sigma \xi_\nu^1 h^\nu)$, and the correspondence

$$\begin{array}{c} \Sigma \xi_\nu^1 g^\nu + D_s^*(\Sigma \xi_\nu^1 g^\nu) \longleftrightarrow \Sigma \xi_\nu^1 h^\nu + D_s^*(\Sigma \xi_\nu^1 h^\nu), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Sigma \xi_\nu^l g^\nu + D_s^*(\Sigma \xi_\nu^1 g^\nu) \longleftrightarrow \Sigma \xi_\nu^l h^\nu + D_s^*(\Sigma \xi_\nu^1 h^\nu) \end{array} \tag{1'}$$

between the elements of the groups $F_s(\Sigma \xi_\nu g^\nu)$, $F_s(\Sigma \xi_\nu h^\nu) \in K_s$ is n-regular.

3) $P_{sj}(\Sigma \xi_\nu g^\nu) \equiv P_{sj}(\Sigma \xi_\nu h^\nu)$, $1 \le j < s$.

The rank of the correspondence (1) is the maximal among the ranks of the correspondences of the form (1′).

Corollary. *Let* (1) *be an* (s, n)*-regular correspondence and* $m|s$. *Then*

$$m|\Sigma \xi_\nu g^\nu \longrightarrow m|\Sigma \xi_\nu h^\nu.$$

Lemma 21. *Given:*

0) G, H *are o.a.g;* $g^\nu \in G$, $h^\nu \in H$, $\nu = 1, \cdots, k$; $E_{ij}^1 \setminus D_{ij}^{1*}$ *and* $E_{ij}^2 \setminus D_{ij}^{2*}$, $(p_i^j|s)$, *are s-jumps of* G *and* H, *respectively;*

1) *The correspondence*

$$g^\nu \longleftrightarrow h^\nu \tag{2}$$

is (s, n)*-regular of rank* $r < n$.

2) *The correspondence* $D_s(\Sigma \xi_\nu g^\nu) \longleftrightarrow D_s(\Sigma \xi_\nu h^\nu)$, $D_{ij}^1 \longleftrightarrow D_{ij}^2$ *is an isomorphism of the submodels* $T_{s,n}(G)$ *and* $T_{s,n}(H)$.

3) a_{ij} *is an element with minimal s-fundament among the elements of the form* $p_i^{(s)i-j} \zeta_i g + \Sigma \xi_\nu \zeta_i g_i^\nu$ *and* $a_{ij} \in E_{ij}^1 \setminus D_{ij}^{1*}$.

Then there is an $h \in H$ *such that the extension of* (2) *by means of* $g \longleftrightarrow h$

is an (s, n)*-regular correspondence of rank* $\leq (r+1)$.

Remark. From the preceding it is clear and can also easily be verified directly that all possible $D_s(\xi_\nu g^\nu + \xi g)$ are exhausted by all the $D_s(\xi_\nu g^\nu)$ and D^1_{ij}.

6.5. **Lemma 22.** *Let* $g, h \in G$. *Then*

1) $D_s(g) \subset A_s(h)$ *is equivalent to*

$$\exists x[A_s(x) \subset A_s(h) \ \&\ x \equiv g (\mathrm{mod}\, s)].$$

2) $A_s(g) \subset D_s(h)$ *is equivalent to*

$$\forall y[y \equiv h(\mathrm{mod}\, s) \longrightarrow A_s(g) \subset A_s(y)].$$

3) $D_s(g) \subset D_s(h)$ *is equivalent to*

$$\exists x[x \equiv g(\mathrm{mod}\, s) \ \&\ A_s(x) \subset D_s(h)].$$

Lemma 23. *The predicate* $|D_s(g)|_n = k$ *is formular in* K^*.

Proof. The proposition $\beta(G) = k$ is formular in K_s. Let $\mathfrak{A}$ be a formula in $\Phi(\sigma_0)$ corresponding to the proposition $\beta(G) = k - (n+1)^{l(s)}$. Replacing in $\mathfrak{A}$ the predicate $S(x, y, z)$ by $D_s(x+y-z) \subset D_s(g)$ and restricting the quantifiers to $D_s(x) \subseteq D_s(g)$ we obtain the required formula.

§7. $T_{s,n}(G)$

7.1. **Lemma 24.** *For all* s *and* n *there exist formular in* K^* *(see 5.1) predicates* $g < h$ *and* $|g| = k$, $1 \leq k < m = m(s, n)$ *(see 6.2) such that if* $G \in K^*$, *then*

$$< |G|, \quad g < h, \quad |g| = k, \quad 1 \leq k < m >$$

is an m-chain isomorphic to $T_{s,n}(G)$.

Proof. We set

$$|g| = k^{df} \equiv [s|g \ \&\ |A_s(g)|_n = k] \vee [s \nmid g \ \&\ |D_s(g)|_n = k];$$

$$\begin{aligned} g < h^{df} \equiv\ & [s|g \ \&\ s|h \longrightarrow A_s(g) \subset A_s(h)] \\ & \&[s|g \ \&\ s\nmid h \longrightarrow A_s(g) \subset D_s(h)] \\ & \&[s\nmid g \ \&\ s|h \longrightarrow D_s(g) \subset A_s(h)] \\ & \&[s\nmid g \ \&\ s\nmid h \longrightarrow D_s(g) \subset D_s(h)]. \end{aligned}$$

In the isomorphism so constructed, to the subgroup $A_s(g)$ there correspond the elements of $B_s(g) \setminus A_s(g)$ that are divisible by s and all those elements h for which $D_s(h) = A_s(g)$. Thus to the empty set, which we count here among the convex subgroups and as an element of $T_{s,n}(G)$, there corresponds the set consisting of the single element zero.

If $\neg\, \exists h(D_s(g) = A_s(h))$, then to the convex subgroup $D_s(g)$ there corresponds $\hat{h}(D_s(h) = D_s(g))$.

Note that the aggregate of elements $T_{s,n}(G)$ depends only on $\pi(s)$. The

(s, n)-norm $|A_s(g)|_n$ also depends on n. The (s, n)-norm $|D_s(g)|_n$ also depends on $(s)_i$, $1 \le i \le l(s)$.

7.2. We introduce the following predicate $(m = m(s, n))$ which is formular in T_m (see 3.1):

$$\alpha_i(x),\ \beta_{ij}(x),\ \alpha_i(x),\ 1 \le i \le l(x),\ 1 \le j \le (s)_i.$$

We set $\alpha_i(x) = 1$, $\beta_{ij}(x) = 0$ if $|x| = (n+1)^{l(s)}$.

We set $\alpha_i(x) = 0$, $\beta_{ij}(x) = \beta_{ij}$ if

$$|x| = (n+1)^{l(s)} + \Sigma \beta_{ij}(n+1)^{(s)_1 + \cdots + (s)_{i-1} + j - 1}.$$

Finally, $\gamma_i(x) \overset{df}{=} \alpha_i(x) + \beta_{i1}(x) + \cdots + \beta_{i(s)_i}(x)$. Further, let

$$\begin{aligned}
Q_i^1(x, y) &\overset{df}{\equiv} \alpha_i(x) = 0 \ \&\ \gamma_i(x) \ne 0 \ \&\ s < y \\
&\qquad \&\ \forall z[x < z \le y \longrightarrow \gamma_i(z) = 0], \\
Q_i^1(x) &\overset{df}{\equiv} \alpha_i(x) \ne 0 \vee \exists y\, Q_i^1(x, y), \\
Q_i^2(x) &\overset{df}{\equiv} \alpha_i(x) = 0 \ \&\ \gamma_i(x) \ne 0 \ \&\ \neg\, \exists y\, Q_i^1(x, y) \\
&\equiv \exists j\,(\beta_{ij}(x) \ne 0) \ \&\ \forall y[x < y \longrightarrow \exists z(x < z \le y \ \&\ \gamma_i(z) \ne 0)], \\
Q_i^3(x) &\overset{df}{\equiv} Q_i^2(x) \ \&\ \forall y[x < y \longrightarrow \exists z(x < z \le y \ \&\ Q_i^1(z))], \\
Q_i^2(x, y) &\overset{df}{\equiv} Q_i^2(x) \ \&\ x < y \ \&\ \forall z(x < z \le y \longrightarrow \neg\, Q_i^1(z)), \\
Q_i^4(x) &\overset{df}{\equiv} \exists y\, Q_i^2(x, y).
\end{aligned}$$

Obviously, $\gamma_i(x) \equiv Q_i^1(x) \vee Q_i^2(x)$, $\neg\,(Q_i^1(x) \ \&\ Q_i^2(x))$, $Q_i^2(x) \equiv Q_i^3(x) \vee Q_i^4(x)$, $\neg\,(Q_i^3(x) \ \&\ \neg\, Q_i^4(x))$.

7.3. It is easy to check that $T_{s,n}(G)$ satisfies the following axioms:

(a) $\exists x(|x| = 0 \ \&\ \forall y(x \le y))$.

(b) $x < y \ \&\ |y| = 0 \longrightarrow \exists z(x < z < y \ \&\ |z| \ne 0)$.

(c_1) $|x| > (n+1)^{l(s)} \longrightarrow \exists y(s < y)$.

(c_2) $x < y \ \&\ |x| > (n+1)^{l(s)} \longrightarrow \exists z(x < z < y \ \&\ |z| \le (n+1)^{l(s)}$.

Suppose that $x, y \in T_{s,n}(G)$ and $Q_i^1(x, y)$ and also that $g \in G$ and $g \in y \setminus x$. This makes sense, because x and y are subgroups of G. Let $p_i \nmid g$. Then $\gamma_i(D_{p_i}(g)) \ne 0$ and $D_{p_i}(g) \subseteq A_{p_i}(g) \subset y$. But, by definition $Q_i^1(x, y)$, in which case $D_{p_i}(g) \subseteq x$. This means (by Corollaries 3) and 13) in 6.1) that y/x is p_i-regular. But then $\beta_{ij}(x) = 0$ for $1 \le j < (s)_i$.

Thus $T_{s,n}(G)$ satisfies the axiom

$$\exists y\, Q_i^1(x, y) \longrightarrow \gamma_i(x) = \beta_{i(s)_i}(x). \tag{d_i}$$

We denote the conjunction of all the axioms in this subsection by $\mathfrak{C}_m^*$. The class of m-chains satisfying $\mathfrak{C}_m^*$ will be denoted by T_m^*.

7.4. **Lemma 25.** *Let $T \in T_m^*$. Then there is an o.a.g. G such that $\Phi(T_{s,n}(G)) = \Phi(T)$.*

Proof. Without loss of generality we may assume that T is countable. We shall construct an o.a.g. G for which $T_{s,n}(G) \cong T$.

Let $k = \Sigma_{i=1}^{l(s)} \alpha_i (n+1)^{i-1}$ and $0 \le \alpha_i \le n$. We denote by G_k^1 the archimedean-ordered direct sum of α_1 groups of type $R_{p_1}, \cdots, \alpha_{l(s)}$ groups of type $R_{p_{l(s)}}$, where R_m is the naturally ordered group of rational numbers (under addition) whose denominators are prime to m. By G_k^2 we denote R_m, where $m = \Pi_{\alpha_i = 0} p_i$.

Now let $y \in T$ and $|y| = k$. With y we associate the o.a.g. $G_y = G_k^1 +\cdot\, G_k^2$, where the sign $+\cdot$ means the lexicographically ordered direct sum, and G_k^1 is a convex subgroup of G_y. Let $y \in T$ and $|y| = (n+1)^{l(s)}$. With y we associate the o.a.g. G_y isomorphic to the group of integers.

Let $A_0 = \Sigma^{\cdot}_{|y| \le (n+1)^{l(s)}} G_y$. Here $\Sigma^{\cdot}$ means the lexicographically ordered direct sum in which the elements having nonzero components only in G_{y_1} are smaller in absolute value than the elements having nonzero components only in G_{y_2} for $y_1 < y_2$.

By a_y we denote one of the elements having nonzero components only in G_y such that $p_i \nmid a_y$ when $p_i | s$.

Let B be the minimal complete o.a.g. containing A_0 as a subgroup.

A. Suppose that $\mathrm{E}y\, Q_i^1(x, y)$. It is easy to see that $\hat{y}\, Q_i^1(x, y)$ is convex in T and contains a sequence converging to x. We split $\hat{y}\, Q_i^1(x, y)$ into $\gamma_i(x) = \beta_{i(s)_i}(x)$ subsets each of which contains a sequence converging to x. And if y and z lie in one and the same subset, we select in B the elements

$$\frac{a_y - a_z}{p_i^{\xi}}, \qquad \xi = 1, 2, \cdots \tag{1}$$

One of the elements of the form a_y, where $Q_i^1(x, y)$, shall be denoted by a_x.

Let A_{0xi} be the least subgroup of B containing A_0 and the selected elements (1), and let D be the largest convex subgroup of A_{0xi} not containing the elements a_y with $Q_i^1(x, y)$. It is easy to see that $\alpha_i(D) = 0$ and $\beta_{ij}(D) = 0$ for $j < (s)_i$, while $\beta_{i(s)_i}(D) = \beta_{i(s)_i}(x)$.

Let A_1 be the least subgroup of B containing all A_{0xi}, where $i = 1, \cdots l(s)$ and x ranges over all elements for which $\exists y\, Q_i^1(x, y)$.

B. We enumerate all elements $z \in T$ for which $Q_i^3(z)$: $z_1, z_2, \cdots$.

Since $Q_i^3(z)$, there exists a sequence $\{y_{k\nu}\}$ such that

B.1. $Q_i^1(y_{k\nu})$,

B.2. $y_{k1} > y_{k2} > \cdots$,

B.3. $\lim y_{k\nu} = z_k$.

Since $\{y_{i\nu}\} \cap \{y_{j\mu}\}$ is finite for $i \neq j$, we may obviously assume that

B.4. $\{y_{i\nu}\} \cap \{y_{j\mu}\} = \emptyset$.

In B we select the elements

$$\frac{a_{k\nu} - a_{k\mu}}{p_i^j}, \tag{2}$$

where $a_{k\nu} = a_{y_{k\nu}}$,

$$\nu \equiv \mu \equiv \beta_{i1}(z_k) + \cdots + \beta_{i(j-1)}(z_k) + \xi \,(\operatorname{mod} \gamma_i(z_k)),$$

where $0 < \xi \leq \beta_{ij}(z_k)$, $k = 1, 2, \cdots$.

Let A_{1i} be the least subgroup of B containing A_1 and the selected elements (2), and let D be the largest convex subgroup of A_{1i} not containing the elements $a_{y_{k\nu}}$ with a fixed k and $\nu = 1, 2, \cdots$. It is easy to see that

$$\alpha_i(D) = 0, \quad \beta_{ij}(D) = \beta_{ij}(z_k), \quad 1 \leq j \leq (s)_i.$$

Let A_2 be the least subgroup of B containing all A_{1i}, $i = 1, \cdots, l(s)$.

C. Suppose that $x, y \in T$ and $Q_i^2(x, y)$. Every element of the segment $[x, y]$ satisfies the predicate $\gamma_i(x) \neq 0 \longrightarrow Q_i^4(x)$. Let Z be the largest convex submodel of T containing $[x, y]$ any element of which satisfies the predicate $\gamma_i(x) \neq 0 \longrightarrow Q_i^4(x)$. On the set of elements $z \in Z$ for which $\gamma_i(z) = 0$ we define an equivalence relation

$$P_i(x, y) \overset{df}{\equiv} \forall z (x \leq z \leq y \lor y \leq z \leq x \longrightarrow \gamma_i(z) = 0).$$

For all x, y with $P_i(x, y)$ we select in B the elements

$$\frac{a_x - a_y}{p_i^{\xi}}, \quad \xi = 1, 2, \cdots. \tag{3}$$

We fix an element in each equivalence class of $P_i(x, y)$. Let Z^* be the submodel of Z containing all the elements $x \in Z$ with $\gamma_i(x) \neq 0$ and among the elements $y \in Z$ with $\gamma_i(y) = 0$ those that are fixed.

We enumerate all elements $x \in Z^*$ with $\gamma_i(x) \neq 0 : x_1, x_2, \cdots$.

Just as under B, we associate with every x_k a sequence $\{y_{k\nu}\}$ of elements of Z^* such that

C.1.1. $\gamma_i(y_{k\nu}) = 0$,

C.1.2. $y_{k1} > y_{k2} > \cdots$,

C.1.3. $\lim y_{k\nu} = x_k$,

C.1.4. $\{y_{i\nu}\} \cap \{y_{j\mu}\} = \emptyset$.

We may assume, furthermore, that $\bigcup_k \{y_{k\nu}\}$ exhausts all those $y \in Z^*$ with

$\gamma_i(y) = 0$ for which there exists an $x \in Z^*$ such that $x < y$ & $\gamma_i(x) \neq 0$.

In B we select the elements

$$\frac{a_{k\nu} - a_{k\mu}}{p_i^j}, \tag{4}$$

where $a_{k\nu} = a_{y_{k\nu}}$,

$$\nu \equiv \mu \equiv \beta_{i1}(x_k) + \cdots + \beta_{i(j-1)}(x_k) + \xi \,(\mathrm{mod}\, \gamma_i(x_k)),$$

where $0 < \xi \leq \beta_{ij}(x_k)$, $k = 1, 2, \cdots$.

Now let $a_k = a_{y_{k1}}$, $\xi_k = \min(j, \beta_{ij}(x_k) \neq 0)$, and $a^1_{k\nu}$ be one of the elements (4) for which $\mu > \nu$.

With each $y_{k\nu}$ we associate a sequence $\{x_{k(l)}\}$ of elements of Z^* with the following properties:

C.1.1. $\gamma_i(x_{k(l)}) \neq 0$,

C.2.2. $x_{k(1)} = x_k$,

C.2.3. $x_{k(1)} < x_{k(2)} < \cdots$,

C.2.4. $\lim_{k(l)} = y_{k\nu}$ (in Z^*).

In B we select the elements

$$a^2_{k\nu} = \frac{a^1_{k\nu} - a_{k(2)}}{p_i^{\xi k(2)}}, \quad a^3_{k\nu} = \frac{a^2_{k\nu} - a_{k(3)}}{p_i^{\xi k(3)}}, \cdots \tag{5}$$

for all k, ν.

D. Suppose that $y \in T$, $\gamma_i(y) = 0$, $\neg \exists x Q^1_i(x, y)$ and $\neg \exists x Q^2_i(x, y)$. Let $y_1, y_2, \cdots$ be all such elements.

In B we select the elements

$$\frac{a_{yk}}{p_i^{\xi}}, \quad \xi = 1, 2, \cdots; \; k = 1, 2, \cdots. \tag{6}$$

Let G be the least subgroup of B containing A_2 and the elements selected in (3)–(6) for all possible Z and i.

Then G is the required o.a g.

Remark. If s is a power of the prime number p_i, then $Q^1_i(x) \longrightarrow \alpha_i(x) \neq 0$, $P_i(x, y) \longrightarrow x \asymp y$, i.e., A and the beginning of B correspond to the fact that we "fuse" the decomposable p_i-regular jumps. (For $\pi_1 < \pi_2$, a π_1-regular jump decomposes, in general, into a set-theoretical sum of π_2-regular jumps.)

The situation of a p_i-fundamental subgroup is completely determined by the section that it induces in the series of p_i-regular jumps.

Thus, let s be a power of p_i and G^1 the least subgroup of B containing A_0 and the elements (2), (4) for all possible Z. Then $T_{s,n}(G^1)$ differs from T

only in that in place of the elements y with $\gamma_i(y) = 0$ there stand the elements y^1 with $\alpha_i(y^1) = 1$. This fact makes it necessary to select the elements (5) and (6).

§8. u-closure

Definition. The 1-closure of a submodel M of an o.a.g. G is the submodel M_1 consisting of the elements of M, their opposites, and zero. The u-closure of M, where u is a positive integer, is the submodel M_u consisting of all elements x of the form

$$x = \sum_{i=1}^{u^2} a_i, \quad a_i \in M_1.$$

Lemma 26. *Given:*

0) $w \geq 1$, $v = (3w^2)!$, $u = v^2$; *G, H o.a.g.; M a submodel of G, N a submodel of H, ϕ an isomorphism of M onto N admitting an extension to an isomorphism*

$$\phi^* Mu = Nu; \quad g \in G, \quad h \in H.$$

1) $vg \in M_v$, $vh \in N_v$, $\phi^* vg = vh$

or

2) $vg \overline{\in} M_v$, $vh \overline{\in} N_v$ *and the submodels* $\langle M_v, vg \rangle$ *and* $\langle N_v, vh \rangle$ *are isomorphic as o.s., where this isomorphism coincides on* M_v *with* ϕ^*.

Then the extension of ϕ by means of $g \longleftrightarrow h$ is an isomorphism ψ admitting an extension to an isomorphism of $\psi^* \langle M, g \rangle_w = \langle N, h \rangle_w$.

Proof. 1) $vg \in M_v$. It is easy to verify that if $x \in \langle M, g \rangle_w$, then $vx \in M_v$ and $v\psi^* x = \phi^* vx$.

Now let $x, y, z \in \langle M, g \rangle_w$.

If $x + y = z$, then $\psi^* x + \psi^* y = (1/v)\ (\phi^* vx + \phi^* vy) = (1/v)\phi^* vz = \psi^* z$.

If $x < y$, then $\psi^* x - \psi^* y = (1/v)\ (\phi^* vx - \phi^* vy) < 0$.

2) $vg \overline{\in} M_v$. Let $x, y, z \in \langle M, g \rangle_w$ and $x = \alpha g + x^1$, $y = \beta g + y^1$, $z = \gamma g + z^1$, where $x^1, y^1, z^1 \in M_w$.

2a) $x + y = z$. If $\alpha + \beta = \gamma$, then obviously, $\psi^* x + \psi^* y = \psi^* z$.

Let $\alpha + \beta \neq \gamma$. Then $0 = x + y - z = (\alpha + \beta - \gamma)g - (z^1 - x^1 - y^1) = vg - \dfrac{v}{\alpha + \beta - \gamma}(z^1 - x^1 - y^1)$ and hence $vg \in M_v$.

2b) $x < y$. If $\alpha = \beta$, then obviously, $\psi^* x < \psi^* y$. Let $\alpha \neq \beta$, and for the sake of definiteness $\alpha > \beta$. Then $\dfrac{v}{\alpha - \beta}(x - y) = vg - \dfrac{v}{\alpha - \beta}(y^1 - x^1) < 0$, i.e., $vg < \dfrac{v}{\alpha - \beta}(y^1 - x^1) \in M_v$.

Then also $vh < \dfrac{v}{\alpha - \beta}(\phi^* y^1 - \phi^* x^1) \in N_v$, and hence $\dfrac{v}{\alpha - \beta}(\psi^* x - \psi^* y) < 0$, $\psi^* x < \psi^* y$.

§9. Classification

9.1. We introduce the following primitive-recursive functions:

$$\alpha(k) = (3k^2)!, \quad \beta(k) = (\alpha(k))^2.$$

$$\begin{cases} \gamma(0) = 1, \\ \gamma(k+1) = \beta(\gamma(k)). \end{cases}$$

$$\begin{cases} \delta(0) = 1, \\ \delta(k+1) = \delta(k) \cdot \alpha(\gamma(k)). \end{cases}$$

$$\begin{cases} \epsilon(0) = 0, \\ \epsilon(k+1) = 2[\epsilon(k) \cdot \alpha(\gamma(k)) + \delta(k)]. \end{cases}$$

$$l^1(k) = (k)_1 + \cdots + (k)_{l(k)}. \quad t(k) = l^1(\delta(k)) + 2.$$

For $T_{\delta(n),n}(G)$ we shall write $T_n(G)$.

Let s be a positive integer. We define on the o.a.g. G a function $\rho(g)$ as follows:

1) $\rho(0) = 0$.

2) If $C_s(g)$ is dense, then $\rho(g) = \infty$.

3) If $C_s(g)$ is discrete, $e + A_s(g)$ the least positive element of $C_s(g)$ and m an integer, then

$$\rho(g) = \begin{cases} |m| & \text{if } g + A_s(g) = me + A_s(g). \\ \infty & \text{if } \forall m[|g + A_s(g)| > me + A_s(g)]. \end{cases}$$

It is easy to see that the value of $\rho(g)$ does not depend on s. In the definition of $\rho(g)$ we could have used, instead of $A_s(g)$ and $C_s(g)$, also $A(g)$ and $C(g) = B(g)/A(g)$, where $B(g) \setminus A(g)$ is an archimedean jump containing g.

9.2. **Lemma 27.** *Suppose that G and H are o.a.g. and that k moves $(k = 0, 1, \cdots, n)$ have been made in $\Gamma_n(G, H)$:*

$$g^\nu \longleftrightarrow h^\nu, \quad \nu = 1, \cdots, k. \tag{A_k}$$

Let us also assume that:

(a_k) *The correspondence (A_k) can be extended to an isomorphism*

$$\phi_k < g^1, \cdots, g^k >_{\gamma(n-k)} = < h^1, \cdots, h^k >_{\gamma(n-k)}.$$

(b_k) *The correspondence*

$$\frac{\delta(n)}{\delta(n-k)} g^\nu \longleftrightarrow \frac{\delta(n)}{\delta(n-k)} h^\nu, \quad \nu = 1, \cdots, k \tag{B_k}$$

is $(\delta(n), n)$-regular of rank $\leq k$.

(c_k) *If $x \in < g^1, \cdots, g^k >_{\gamma(n-k)}$, then*

$$\rho(x) < \epsilon(n-k) \vee \rho(\phi_k x) < \epsilon(n-k) \longrightarrow \rho(x) = \rho(\phi_k x).$$

(d_k) *After* k *moves in the game*

$$\Gamma_{(t(n), \cdots, t(n))}(T_n(G),\ T_n(H))$$

with n *moves, we have reached the following position*:

$$\begin{cases} D_{\delta(n-k)}(\Sigma \xi_\nu g^\nu) \longleftrightarrow D_{\delta(n-k)}(\Sigma \xi_\nu h^\nu), \\ A_{\delta(n)}(x) \longleftrightarrow A_{\delta(n)}(\phi_k x), \end{cases}$$

where $\xi_1, \cdots, \xi_k$ *range over all possible integers,* $x \in \langle g^1, \cdots, g^k \rangle_{\gamma(n-k)}$, *and* II *has a w.s. in the game.*

Then II *also has a w.s. in* $\Gamma_n(G, H)$.

The proof is trivial for $k = n$. Suppose that $k < n$ and that the lemma is proved for $k+1$. Suppose also that I chooses at the $(k+1)$th move of $\Gamma_n(G, H)$ the element $g^{k+1} \in G$. By the symmetry of the conditions it is sufficient to restrict ourselves to the discussion of this case.

For the sake of brevity we introduce the following notation:

$$\delta(n) = s,\ \ \gamma(n-k-1) = w,\ \ \alpha(\gamma(n-k-1)) = v,$$
$$\gamma(n-k) = u,\ \ \langle g^1, \cdots, g^k \rangle = M,\ \ \langle h^1, \cdots, h^k \rangle = N.$$

A. Suppose it happens that $vg \in M_v$. As a consequence of the definition in 6.4, $v | \phi_k(vg)$. In this case II at the $(k+1)$th move of $\Gamma_n(G, H)$ chooses the element $h^{k+1} = (1/v)\phi_k(vg)$. The condition (a_{k+1}) is satisfied by Lemma 26. We recall that if $x \in \langle M, g \rangle_w$, then $vx \in M_v$. This implies (c_{k+1}), because $\epsilon(n-k) < v\epsilon(n-k-1)$. Moreover, $\delta(n-k) = v\,\delta(n-k-1)$ so that $\delta(n)/\delta(n-k-1) = (\delta(n)/\delta(n-k))\,v$. This implies (d_{k+1}) and (b_{k+1}).

B. Now we come to the discussion of the case $vg \in M_v$.

B.1. Let c_1 be closest to the left, and c_2 closest to the right of vg^{k+1} in M_v and $d = c_2 - c_1$ and suppose, for the sake of definiteness, that $vg^{k+1} - c_1 \leq c_2 - vg^{k+1}$. Then we set $g = vg^{k+1} - c_1$.

Let a_{ij} be the element with the least s-fundament among the elements of the form

$$p_i^{(s)i-j}\zeta_i g + \sum_\nu \xi_\nu \zeta_i\ \frac{\delta(n)}{\delta(n-k)}\ g^\nu$$

(see the Remark in 4.3).

We shall assume that at the $(k+1)$th move of the game

$$\Gamma_{(t(n), \cdots, t(n))}(T_n(G),\ T_n(H))$$

player I has chosen elements of $D_s(a_{ij})$ for all i and j such that $p_i^j \mid s$, and also an element of $A_s(g)$ in $T_n(g)$ (if $k = 0$, then $\emptyset \subset G$). Let D_{ij} and A be the corresponding elements which II has chosen in $T_n(H)$, using his w.s. in the game (if $k = 0$, then apart from these elements II also chooses $\emptyset \subset H$).

By Lemma 21 there is a $b \in H$ such that the extension of (B_k) by means of $g \longleftrightarrow b$, and hence also the extension of (B_k) by means of

$$\frac{\delta(n)}{\delta(n-k)}\, g \longleftrightarrow \frac{\delta(n)}{\delta(n-k)}\, b$$

is (s, n)-regular of rank $\leq (k+1)$.

Note that (see 6.3) for $1 \leq j \leq s$ and $(\delta(n)/\delta(n-k)) \mid j$

$$P_{\delta(n)j}\left[\frac{\delta(n)}{\delta(n-k)} \cdot x\right] \equiv P_{\delta(n-k)\frac{j\,\delta(n-k)}{\delta(n)}}(x).$$

Hence $P_{\delta(n-k)j}(g) \equiv P_{\delta(n-k)j}(b)$, $1 \leq j < \delta(n-k)$.

B.2. Suppose that we have succeeded in finding an $h \in H$ such that

a) $h \equiv b \pmod{\delta(n-k)}$,

b) $A_s(h) = A$,

c) $A < 2h + A \leq \phi_k d + A$ in H/A,

d) $\rho(g) < v \cdot \epsilon(n-k-1) \vee \rho(h) < v \cdot \epsilon(n-k-1) \longrightarrow \rho(g) = \rho(h)$.

Without loss of generality we may assume here that

e) $0 < 2h \leq \phi_k d$.

For if $2h > \phi_k d$, then $a = 2h - \phi_k d \in A$, and instead of h we can take $h^* = h - \delta(n-k) \cdot a$.

Since $vg^{k+1} = c_1 + g$, by the Corollary to the Definition in 6.4 we have $v \mid \phi_k c_1 + h$. In that case II chooses at the $(k+1)$th move of $\Gamma_n(G, H)$ the element $h^{k+1} = (1/v)(\phi_k c_1 + h)$.

B.2.1. Then the condition (a_{k+1}) is satisfied by Lemma 26.

B.2.2. By the choice of b condition (b_{k+1}) is satisfied.

B.2.3. Let $x = \xi g - \Sigma \xi_\nu g^\nu$ be an arbitrary element of $\langle M, g\rangle_w$. To verify condition (d_{k+1}) we only have to show now, obviously, that if $A_s(x) = A_s(g)$, then $A_s(\phi_{k+1}x) = A$, and if $A_s(x) \neq A_s(g)$, then there is an $a \in M_v$ such that $A_s(x) = A_s(a)$ and $A_s(\phi_{k+1}x) = A_s(\phi_k a)$. Here we may confine the discussion to the case $\xi \neq 0$. Observe that

$$\left|\frac{v}{\xi}x\right| = \left|vh^{k+1} - \frac{v}{\xi}\Sigma_\nu g^\nu\right| \geq g,$$

and hence $A_s(x) \supseteq A_s(g)$.

Let $y = (v/\xi)\Sigma \xi_\nu g^\nu \leq c_1$, then $|(v/\xi)x| = g + a$, where $a \in M_u$ and $a > 0$. In that case $|(v/\xi)\phi_{k+1}x| = h + \phi_k a$. If $A_s(a) \subseteq A_s(g)$, then $A_s(x) = A_s(g)$ and $A_s(\phi_{k+1}x) = A$, and if $A_s(a) \supset A_s(g)$, then $A_s(x) = A_s(a)$ and $A_s(\phi_{k+1}x) = A_s(\phi_k a)$.

Let $y \geq c_2$, then $|(v/\xi)x| = a^1 - g$, where $a^1 \in M_u$ and $a^1 \geq 2g$, and the verification is similar.

But since $y \in M_v$, either $y \leq c_1$ or $y \geq c_2$.

B.2.4. Let us verify (c_{k+1}). We keep the notation of B.2.3. Letting $\rho(x) < \epsilon(n-k-1)$, the case $\rho(\phi_{k+1}x) < \epsilon(n-k-1)$ can be analyzed similarly. We confine the discussion to an analysis of the case $|(v/\xi)x| = a^1 - g$, where $a^1 \in M_u$ and $a^1 \geq 2g$. The case $|(v/\xi)x| = g + a$, where $a \in M_u$ and $a > 0$, is simpler. Here $|(v/\xi)\phi_{k+1}x| = \phi_k a^1 - h$. If $A_s(a^1) \subset A_s(g)$, then

$$\left|\frac{v}{\xi}\right| \cdot \rho(x) = \rho\left(\left|\frac{v}{\xi}x\right|\right) = \rho(g) < v \cdot \epsilon(n-k-1).$$

So, $|v/\xi| \cdot \rho(\phi_{k+1}x) = \rho(h) = \rho(g)$. Hence, $\rho(x) = \rho(\phi_{k+1}x)$. If $A_s(a^1) \supset A_s(g)$, then $|v/\xi| \cdot \rho(x) = \rho(a^1) < \epsilon(n-k)$ and therefore $|v/\xi| \cdot \rho(\phi_{k+1}x) = \rho(\phi_k a^1) = \rho(a^1)$ and $\rho(x) = \rho(\phi_{k+1}x)$. If $A_s(a^1) = A_s(g)$, then $\rho(g) \leq \rho(a^1 - g) < v \cdot \epsilon(n-k-1)$ and therefore $\rho(g) = \rho(h)$. Moreover, $2(a^1 - g) = (a^1 - 2g) + a^1 \geq a^1$ and therefore $\rho(a^1) \leq 2\rho(a^1 - g) < \epsilon(n-k)$. Hence $\rho(a^1) = \rho(\phi_k a^1)$ and $\rho(a^1 - g) = \rho(\phi_k a^1 - h)$ and $\rho(x) = \rho(\phi_{k+1}x)$.

B.3. We now come to the choice of h. We introduce additional notation: $\delta(n-k) = t$; B is the upper subgroup of the s-regular jump of the o.a.g. H whose lower subgroup is A; $C = B/A$. Note that since $D_t(g) = D_s(\frac{s}{t}g) \subseteq A_s(\frac{s}{t}g) = A_s(g) \subseteq A_s(d)$ we have by the choice of D_{ij} and A that $D_t(b) \subseteq A \subseteq A_s(\phi_k d)$.

Without loss of generality we may assume that $b \in B \setminus A$ and $b > 0$.

1) $C_s(g)$ is dense. Then C is also dense. If $A \subset A_s(\phi_k d)$, then we set $h = b$. If $A = A_s(\phi_k d)$, then by the fact that $C_s(g)$ is t-regular and dense there is a $c \in H$ such that $A < 2c + A \leq \phi_k d + A$ and $c + A \equiv b + A \pmod t$. Let $c - b + A = ta + A$. We set $h = b + ta$.

2) $C_s(g)$ is discrete. Then C is also discrete. Suppose that $e \in G$ and $e + A_s(g)$ is the smallest positive element of $C_s(g)$ Suppose that $f \in H$ and $f + A$ is the least positive element of C. Since $C_s(g)$ is discrete and t-regular, among the numbers $1, \cdots, t$ there is a j such that $g + A_s(g) \equiv je + A_s(g) \pmod t$. By (1) (see B.1), in this case $b + A \equiv jf + A \pmod t$.

2a) $\rho(g) < v\epsilon(n-k-1)$. Then $\rho(g) \equiv j \pmod t$. Therefore there is a c such that $\rho(g)f - b + A = tc + A$. We set $h = b + tc$.

2b) $\rho(g) \geq v \cdot \epsilon(n-k-1)$.

Among the elements $v \cdot \epsilon(n-k-1)f + A, \cdots, (v \cdot \epsilon(n-k-1) + t - 1)f + A$ there is $lf + A \equiv jf + A \pmod t$. Let $lf - b + A = tc + A$. We set $h = b + tc$.

Thus, in all possible cases we have picked an element h satisfying the conditions a) – d).

This proves Lemma 27.

9.3. Now we set $k = 0$ in Lemma 27. Then we see that if H has a w.s. in $\Gamma_{(t(n), \dots, t(n))}(T_n(G), T_n(H))$, then also $\Gamma_n(G, H)$. By Lemmas 1–3 this leads to

Theorem 1 (the classification theorem). *Let G and H be o.a.g. There exists a primitive-recursive function $N = N(n)$ such that if $\Phi_{N(n)}(T_n(G)) = \Phi_{N(n)}(T_n(H))$, then $\Phi_n(G) = \Phi_n(H)$.*

Remarks. 1) The equation giving $N = N(n)$ can be written down explicitly. 2) There exists a primitive-recursive function $M = M(n, k)$ such that if $\Phi_{M(n,k)}(G) = \Phi_{M(n,k)}(H)$, then $\Phi_k(T_n(G)) = \Phi_k(T_n(H))$. This follows immediately from Lemma 24.

Corollary 1. *$\Phi(G) = \Phi(H)$ if and only if for every n*

$$\Phi(T_n(G)) = \Phi(T_n(H)).$$

Let G be an o.a.g. We construct the following ω-chain which we denote by $T_\omega(G)$. We take $T_1(G)$. To the right of it we write down an element without norm such that every element of $T_1(G)$ precedes x_1. Next, we write down $T_2(G)$ in its natural order so that x_1 precedes every element of $T_2(G)$. Then we write down an x_2 without norm to the right, etc.

Corollary 2. *$\Phi(G) = \Phi(H)$ is equivalent to $\Phi(T_\omega(G)) = \Phi(T_\omega(H))$.*

Let $T'_\omega(G)$ and $T'_\omega(H)$ be o.s. associated with $T_\omega(G)$ and $T_\omega(H)$, respectively (see 3.4).

Corollary 3. *$\Phi(G) = \Phi(H)$ is equivalent to $\Phi(T'_\omega(G)) = \Phi(T'_\omega(H))$.*

9.4. **Corollary 4.** *Let G and H be $\delta(n)$-regular and discrete o.a.g. Then $\Phi_n(G) = \Phi_n(H)$.*

Corollary 5. *Let G and H be $\delta(n)$-regular and dense o.a.g., $\alpha_i(G)$ (or $\alpha_i(H)$, respectively) the number of modulo p_i independent elements of G (or H), and* $\min(\alpha_i(G), n) = \min(\alpha_i(H), n)$ *for all i with $p_i \mid \delta(n)$. Then $\Phi_n(G) = \Phi_n(H)$.*

From Corollaries 4 and 5 there follows a criterion for elementary equivalence of regular groups that can be found in [5].

Theorem 1 implies the validity of the conditions for elementary equivalence to be found in [7] for the class of o.a.g. discussed there. When G lies in that class, then $\Phi_n(G)$ is finite for every n. The condition that the convex subgroups can be totally ordered may be omitted.

Corollary 6. *Let C be a convex subgroups of an o.a.g. G. Then $\Phi(G) = \Phi(C + \cdot G/C)$.*

We remark that Corollaries 4–6 can be proved directly by means of Lemmas 1 and 26.

§10. Solvability of the elementary theory of o.a.g.

10.1. **Lemma 28.** *Suppose that K is a class of models of signature σ and that $\Phi(K)$ is recursively axiomatized. Suppose further that for every n we can effectively construct formulas $\mathfrak{A}_{n1}, \cdots, \mathfrak{A}_{n\phi(n)}$ such that*

a) $\Phi(K) \vdash \mathfrak{A}_{n1} \vee \cdots \vee \mathfrak{A}_{n\phi(n)}$;

b) $\mathfrak{A}_{ni}$ *holds in* K;

c) *if* $\mathfrak{M}, \mathfrak{N} \in K$ *and* $\mathfrak{A}_{ni} \in \Phi(\mathfrak{M}) \cap \Phi(\mathfrak{N})$, *then* $\Phi_n(\mathfrak{M}) = \Phi_n(\mathfrak{N})$.

Then $\Phi(K)$ *is solvable.*

Theorem 2 (**on solvability**). *The elementary theory of o.a.g. is solvable if and only if the elementary theory of o.s. is solvable.*

Proof. A. Suppose the class of o.s. to be solvable. Then by Lemma 5 $\Phi(T_\omega)$ is solvable.

Let $m = m(\delta(n), n)$. Since the signature τ_m in $\Phi_{N(n)}(\tau_m)$ is finite, where $N = N(n)$ is the function in Theorem 1, there is only a finite number of formulas that have prenex form and whose quantifier-free form has disjunctive normal form. Let these be $\mathfrak{L}_1^1, \mathfrak{L}_2^1, \cdots$. We form all possible formulas $\mathfrak{L}_i^2 = \mathfrak{L}_1 \,\&\, \mathfrak{L}_2 \,\&\, \cdots$, where $\mathfrak{L}_j$ is $\mathfrak{L}_j^1$ or $\neg\, \mathfrak{L}_j^1$. The set of these formulas satisfies the conditions a) and b) in 10.1 for $K = T_m^*$ and $N(n)$ in place of the previous n. Observe that (see §7) $(\mathfrak{L}_m^* \longrightarrow \mathfrak{L}_i^2) \in \Phi(T_m)$ is equivalent to $\mathfrak{L}_i^2 \in \Phi(T_m^*)$. But $\Phi(T_m) = \Phi(T_\omega) \cap \Phi(\tau_m)$.

Using the solvability of $\Phi(T_\omega)$ we select among the formulas $\mathfrak{L}_i^2$ those for which $\neg\, \mathfrak{L}_i^2 \,\overline{\in}\, \Phi(T_m^*)$, i.e., that hold in T_m^*. Let these be $\mathfrak{L}_1, \cdots, \mathfrak{L}_{\phi(n)}$. By Lemma 24 we are in a position to write down formulas $\mathfrak{A}_{n1}, \cdots, \mathfrak{A}_{n\phi(n)}$ such that for every o.a.g. G and $i = 1, \cdots, \phi(n)$, $\mathfrak{A}_{ni} \in \Phi(G)$ is equivalent to $\mathfrak{L}_i \in \Phi(T_n(G))$.

Using Lemmas 25 and 27 we find that a)–c) for $\mathfrak{A}_{n1}, \cdots, \mathfrak{A}_{n\phi(n)}$ and $K = K^*$ hold.

B. Let $\Phi(K^*)$ be solvable. We choose an integer $s > 1$. The predicate $x < y \overset{df}{=} A_s(x) \subset A_s(y)$ is formular in K^*. The predicate $P(x)$, which signifies that $C_s(x)$ is discrete, is formular in K^*. To complete the proof of Theorem 2 it now remains to apply Lemma 4.

Corollary. *The class of semigroups of positive elements of o.a.g., i.e., of commutative semigroups with cancellation satisfying the axiom*

$$\exists z(z = x - y \vee z = y - x), \quad x \neq 0, \quad x \neq -y,$$

is solvable.

From Theorem 1 and Lemma 28 it also follows that the class of regular (and hence archimedean; see [5]) o.a.g. is solvable.

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