Sequential Composition of Walking Motions for a 5-Link Planar Biped Walker

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Abstract— **Work by the authors published elsewhere addressed the problem of designing controllers that induce exponentially stable, periodic walking motions at a given fixed speed for a 5-link, planar biped robot with one degree of underactuation in single support. The key technical tool was the hybrid zero dynamics, a 1-DOF invariant subdynamics of the full robot model. Further exploiting the features of the hybrid zero dynamics, this paper provides an additional control feature: the ability to compose such controllers in order to obtain walking at several discrete speeds with guaranteed stability during the transitions. This feature affords the construction of a feedback controller that takes the robot from one exponentially stable walking motion to another while providing local stabilization and disturbance rejection.**

I. Introduction

This work builds on the results in [9,10], which developed the notion of the hybrid zero dynamics for the walking motion of an N-link, planar, biped robot with one less degree of actuation than degree of motion freedom (DOF) during the single support phase. This two dimensional, invariant sub-dynamics of the complete hybrid model of the biped robot was shown to be key to designing exponentially stabilizing controllers for walking motions. In particular, exponentially stable orbits of the hybrid zero dynamics can be rendered exponentially stable in the complete hybrid model. The Poincaré map of the hybrid zero dynamics was proven to be diffeomorphic to a scalar, LTI system, rendering transparent the existence and stability properties of periodic orbits of the hybrid zero dynamics. A special class of output functions based on Bézier polynomials was used to simplify the computation of the hybrid zero dynamics, while at the same time inducing a convenient, finite parameterization of these dynamics. Parameter optimization was then applied to the hybrid zero dynamics to directly design a provably stable, closed-loop system which satisfied design constraints, such as walking at a given average speed and the forces on the support leg lying in the allowed friction cone. All of the results were illustrated on a 5-link walker (see Figure 1).

This work provides the ability to compose the above controllers in order to obtain walking at several discrete speeds with guaranteed stability during the transitions. For simplicity, this development will be specialized to the 5-link walker studied in [7,8], though the results may be directly extend to the class of N-link bipeds studied in [3,9].

Section II reviews the 5-link robot model studied. Section III summarizes the work of [9,10] on the hybrid zero dynamics. Section IV presents a class of finitely parameterized, almost linear output functions. Section V presents a method for serially composing two controllers so as to

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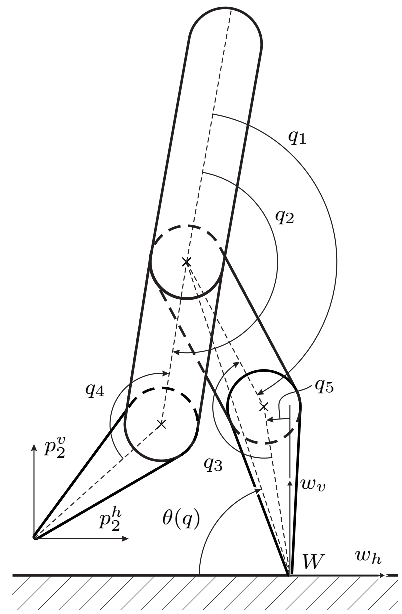

Fig. 1. Schematic of the 5-link robot considered with measurement conventions.

transition the robot from walking at a given fixed speed to another, without loss of stability. The controller design is motivated by a switching idea presented in [2]: controllers were first designed to accomplish the individual tasks of juggling, catching, and palming a ping-pong ball by a robot arm; these controllers were then sequentially composed via switching to accomplish the complex task of maneuvering the ping-pong ball in a 3-D workspace with an obstacle. The regions of attraction of each controller were first empirically estimated within the full state space of the robot. Switching from one controller to another without loss of stability was then accomplished by comparing the current state of the robot to the region of attraction of the controller for the next desired task. The problem faced in this work is more challenging in that the domains of attraction of any two of the individual controllers may have empty intersection, and hence a transition controller will be required in order to steer the robot from the region of attraction of one controller into the region of attraction of a second, "nearby", controller.

Section VI illustrates how the results of Section V affords the construction of a feedback controller that steers the robot from one walking rate to another to another while providing stabilization and a modest amount of robustness to disturbances and parameter mismatch between the design model and the actual robot. The results are illustrated via simulation; an animation of the robot's walking motion is available on [1].

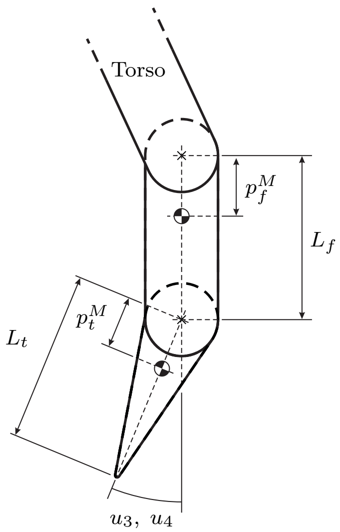

Fig. 2. Schematic of leg with measurement conventions.

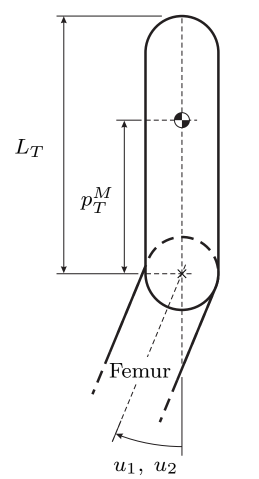

Fig. 3. Schematic of torso with measurement conventions.

Model parameters	Torso (T)	Femurs (f)	Tibias (t)
Mass (kg)	20	6.8	3.2
L_* (m)	0.625	0.4	0.4
Inertia (m^2kg)	2.22	1.08	0.93
p_*^M (m)	0.2	0.163	0.128

TABLE I
Model parameters

II. Robot model and modeling assumptions

The robot, depicted in Figure 1, is assumed to be planar and consist of a torso and two identical legs with knees; furthermore, all links have mass, are rigid, and are connected in revolute joints. All walking cycles will be assumed to take place in the sagittal plane and consist of successive phases of *single support* (meaning the stance leg is touching the walking surface and the swing leg is not) and *double support* (the swing leg and the stance leg are both in contact with the walking surface). During the single support phase, it is assumed that the stance leg acts as a pivot. It is further supposed that the walking gaits of interest are such that successive phases of single support are symmetric, and progress from left to right.

The two phases of the walking cycle naturally lead to a mathematical model of the biped consisting of two parts: the differential equations describing the dynamics during the single support phase, and a model of the contact event. The rigid contact model of [5] is assumed, which collapses the double support phase to an instant in time, and allows a discontinuity in the velocity component of the state, with the position remaining continuous. The biped model is thus *hybrid* in nature, consisting of a continuous dynamics and a re-initialization rule at the contact event.

Swing phase model: With 5-links, the dynamic model of the robot during the swing phase has 5-DOF. Let $q = (q_1, \cdots, q_5)'$ be the set of coordinates depicted in Figure 1, which describe the configuration of the robot with respect to the world reference frame W. Since only symmetric gaits are of interest, the same model can be used irrespective of which leg is the stance leg if the coordinates are relabeled after each phase of double support. Using the method of Lagrange, the model is written in the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu. \tag{1}$$

Table I lists the associated model parameters (see Figures 2 and 3 for details on the parameter definitions used here; the inertias listed here include the rotor inertia reflected through the gear reducer). Torques $u_i,\ i = 1$ to 4, are applied between each connection of two links, *but not between the stance leg and ground*. The model is written in state space form by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} \dot{q} \\ D^{-1}(q)\left[-C(q,\dot{q})\dot{q} - G(q) + Bu\right] \end{bmatrix} \\ &=: f(x) + g(x)u. \end{aligned} \tag{2}$$

where $x := (q', \dot{q}')'$. The state space of the model is taken as $T\mathcal{Q} := \{x := (q', \dot{q}')' \mid q \in \mathcal{Q},\ \dot{q} \in I\!R^5\}$, where $\mathcal{Q}$ is a simply-connected, open subset of $[0, 2\pi)^5$ corresponding to physically reasonable configurations of the robot, as done in [7].

Impact model: An impact occurs when the swing leg touches the walking surface, $S := \{(q,\dot{q}) \in T\mathcal{Q} \mid p_2^v = 0,\ p_2^h > 0\}$, also called the ground (see Figure 1 for definitions of p_2^h and p_2^v). The impact between the swing leg and the ground is modeled as a contact between two rigid bodies. In addition to modeling the change in state of the robot, the impact model accounts for the relabeling of the robot's coordinates that occurs after each phase of double support. Let R be the constant matrix such that $R\,q$ accounts for relabeling of the coordinates when the swing leg becomes the new stance leg. Then the impact model of [5] under standard hypotheses (see [4], for example), results in a smooth map $\Delta : S \to T\mathcal{Q}$,

$$x^+ = \Delta(x^-), \tag{3}$$

where $x^+ := (q^+, \dot{q}^+)$ (resp. $x^- := (q^-, \dot{q}^-)$) is the state value just after (resp. just before) impact. For later convenience, Δ is expressed as

$$\Delta(x^-) := \begin{bmatrix} \Delta_q\, q^- \\ \Delta_{\dot{q}}(q^-)\, \dot{q}^- \end{bmatrix} \tag{4}$$

where $\Delta_q := R$ and $\Delta_{\dot{q}}(q)$ is a 5×5 matrix of smooth functions of q.

Nonlinear system with impulse effects: The overall biped robot model can be expressed as a nonlinear system with impulse effects [11]

$$\begin{aligned} \dot{x} &= f(x) + g(x)u & x^- &\notin S \\ x^+ &= \Delta(x^-) & x^- &\in S, \end{aligned} \tag{5}$$

where, $x^-(t) := \lim_{\tau \nearrow t} x(\tau)$. Solutions are taken to be right continuous and must have finite left and right limits at each impact event (see [4] for details).

A *half-step* of the robot is defined to be a solution of (5) that starts with the robot in double support, ends in double support with the positions of the legs swapped, and contains no other impact event.

III. Summary of hybrid zero dynamics

In general, the *maximal internal dynamics of a system that are compatible with the output being identically zero* is called the *zero dynamics* [6]. In [10], this notion was extended to include the impact map common in many biped models. This section briefly summarizes the main results of [10], and due to space limitations, assumes familiarity with the zero dynamics of non-hybrid models.

Consider first the swing phase dynamics, (2), and note that if an output $y = h(q)$ depends only on the position variables, then, due to the second order nature of the robot model, the derivative of the output along solutions of (2) does not depend directly on the inputs. Hence its relative degree is at least two. Differentiating the output once again computes the accelerations, resulting in

$$\frac{d^2y}{dt^2} = L_f^2 h(q,\dot q) + L_g L_f h(q) u, \tag{6}$$

where the matrix $L_g L_f h(q)$ is called the decoupling matrix and depends only on the configuration variables. A consequence of the general results in [6] is that the invertibility of this matrix at a given point assures the existence and uniqueness of the zero dynamics in the neighborhood of that point. With a few extra hypotheses, these properties can be assured on a given open set.

Output function hypotheses: The output functions considered are assumed to be smooth functions satisfying the following hypotheses:

HH1) h is a function of only the position coordinates;

HH2) there exists an open set $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ such that for each point $q \in \tilde{\mathcal{Q}}$, the decoupling matrix $L_g L_f h(q)$ is square and invertible (i.e., h has vector relative degree $(2,\dots,2)'$);

HH3) there exists a smooth real valued function $\theta(q)$ such that $\Phi : \tilde{\mathcal{Q}} \to I\!R^5$ by $\Phi(q) := (h(q)', \theta(q))'$ is a diffeomorphism onto its image;

HH4) there exists a unique point $q^- \in \tilde{\mathcal{Q}}$ such that $(h(q^-), p_2^v(q^-)) = (0,0)$ and the rank of $[h',\ p_2^v]'$ at q^- equals 5.

Swing phase zero dynamics (cf. [10, Lemma 1]): Hypotheses HH1–HH4 ensure that $Z := \{x \in T\tilde{\mathcal{Q}} \mid h(x) = 0, L_f h(x) = 0\}$ is a smooth two dimensional submanifold of $T\mathcal{Q}$; moreover, the feedback control

$$u^*(x) = -(L_g L_f h(x))^{-1} L_f^2 h(x) \tag{7}$$

renders Z invariant under the swing phase dynamics in the sense that, every $z \in Z$, $f_{\rm zero}(z) := f(z) + g(z)u^*(z) \in T_z Z$. Z is called the *zero dynamics manifold* and $\dot z = f_{\rm zero}(z)$ is called the (swing phase) *zero dynamics*.

Hybrid zero dynamics (cf. [10, Theorem 2 and 5]): Requiring that the swing phase dynamics be invariant under the impact map, that is, $\Delta(S \cap Z) \subset Z$, results in the *hybrid zero dynamics*,

$$\begin{array}{rcll} \dot z &=& f_{\rm zero}(z) & z^- \notin S\cap Z \\ z^+ &=& \Delta(z^-) & z^- \in S\cap Z. \end{array} \tag{8}$$

It is shown in [10] that along all solutions of (8), the output h is identically zero, hence this is a valid zero dynamics for the hybrid model. Let θ be as in HH3 and let γ_0 be the last row of D and $\gamma(q,\dot q) := \gamma_0(q)\,\dot q$. Then in the local coordinates, $(\xi_1,\xi_2) := (\theta(q), \gamma(q,\dot q))$, the swing phase zero dynamics of (2) become

$$\begin{array}{rcl} \dot\xi_1 &=& \kappa_1(\xi_1)\,\xi_2 \\ \dot\xi_2 &=& \kappa_2(\xi_1) \end{array} \tag{9}$$

where κ_1 and κ_2 are smooth functions of ξ_1. Furthermore, $S \cap Z$ can be shown to be diffeomorphic to $I\!R$ per $\sigma : I\!R \to S\cap Z$, where $\sigma(\omega) := [\sigma_q',\ (\sigma_{\dot q}(q^-)\,\omega)']'$, $\sigma_q := q^-$,

$$\sigma_{\dot q}(q^-) := \begin{bmatrix} \frac{\partial h}{\partial q}(q^-) \\ \gamma_0(q^-) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{10}$$

and q^- is given by HH4. In addition, θ, when evaluated along any half-step of the zero dynamics, is a strictly monotonic function of time and thus achieves its maximum and minimum values at the end points. Thus, the extrema of $\theta(q)$ over a half-step are $\theta^- := \theta(q^-)$ and $\theta^+ := \theta \circ \Delta_q(q^-)$. Without loss of generality, it is assumed that $\theta^+ < \theta^-$; that is, that along any half-step of the hybrid zero dynamics, θ is *monotonically increasing.*

Poincaré analysis of the zero dynamics (cf. [10, Theorem 7]) Assume that a smooth output function h on (5) satisfies HH1–HH4 function, and take the Poincaré section to be $S \cap Z$ so that the Poincaré map is the partial map $\rho : S\cap Z \to S \cap Z$ defined as in [4]. In local coordinates $(\zeta_1,\zeta_2) := (\theta(q), \frac{1}{2}\gamma^2(q,\dot q))$, the Poincaré map can be explicitly computed. Indeed, $\Delta : (\zeta_1^-, \zeta_2^-) \to (\zeta_1^+, \zeta_2^+)$ is $\zeta_1^+ = \theta^+$ and $\zeta_2^+ = \delta_{\rm zero}^2\,\zeta_2^-$, where $\delta_{\rm zero} := \gamma_0(q^+)\,\Delta_{\dot q}(q^-)\,\sigma_{\dot q}(q^-)$, a constant that may be computed *a priori*. In these coordinates, the hybrid zero dynamics (9) may be integrated to obtain

$$\rho(\zeta_2^-) = \delta_{\rm zero}^2\,\zeta_2^- + V_{\rm zero}(\theta^-), \tag{11}$$

where for $\theta^+ \le \zeta_1 \le \theta^-$,

$$V_{\rm zero}(\zeta_1) := \int_{\theta^+}^{\zeta_1} \frac{\kappa_2(\zeta)}{\kappa_1(\zeta)}\, d\zeta. \tag{12}$$

The domain of definition of (11) is

$$\left\{ \zeta_2^- > 0 \mid \delta_{\rm zero}^2\,\zeta_2^- + K \ge 0 \right\}. \tag{13}$$

where $K := \min_{\theta^+ \le \xi_1 \le \theta^-} V_{\rm zero}(\zeta_1) \le 0$.

If $\delta_{\rm zero}^2 \ne 1$ and $\zeta_2^* := V_{\rm zero}(\theta^-)/(1-\delta_{\rm zero}^2)$ is in the domain of definition of ρ, then it is the unique fixed point of ρ. Moreover, if ζ_2^* is a fixed point, then ζ_2^* is an asymptotically stable equilibrium point of

$$\zeta_2(k+1) = \rho(\zeta_2(k)) \tag{14}$$

if, and only if, $\delta_{\rm zero}^2 < 1$, and in its domain of attraction is (13), that is, the entire domain of definition of ρ.

While the domain of definition of the Poincaré map is (13), not all solutions of the zero dynamics satisfy the modeling assumptions. In particular, the assumption that the stance leg acts as a pivot during the swing phase limits the ratio and sign of the ground reaction forces on the stance leg end. This limit can be reflected as an upper bound on the mathematical domain of definition of ρ; see [9].

IV. Finitely parameterized zero dynamics

The outputs $y \in I\!R^4$ are chosen as

$$y = h(q,\alpha) = H_0 q - h_d(\theta(q),\alpha), \tag{15}$$

with terms defined as follows.
1. H_0 is an 4×5 matrix of real coefficients specifying what is to be controlled.
2. $\theta(q) := cq$, where c is a 1×5 row vector of real coefficients, is a scalar function of the configuration variables and should be chosen so that it is monotonically increasing along a half-step of the robot ($\theta(q)$ is playing the role of time). Define θ^+ and θ^- to be the initial and final values of θ, respectively, along a half-step.
3. Normalization of θ to take values between zero and one,

$$s(q) := \frac{\theta(q) - \theta^+}{\theta^- - \theta^+}. \tag{16}$$

4. Bézier polynomials of order $M \geq 3$

$$b_i(s) := \sum_{k=0}^{M} \alpha_k^i \frac{M!}{k!(M-k)!} s^k (1-s)^{M-k}. \tag{17}$$

5. For α_k^i as above, define the 4×1 column vector $\alpha_k := (\alpha_k^1, \cdots, \alpha_k^4)'$ and the $4\times(M+1)$ matrix $\alpha := [\alpha_0, \cdots, \alpha_M]$.
6.

$$h_d(\theta(q),\alpha) := \begin{bmatrix} b_1 \circ s(q) \\ \vdots \\ b_4 \circ s(q) \end{bmatrix}. \tag{18}$$

The matrix of parameters α is said to be regular if the output satisfies hypotheses HH1-HH4 of Section 3 and hypothesis HH5 of Section 4 of [10], which together imply the invertibility of the decoupling matrix and the existence of a two dimensional, smooth, zero dynamics associated with the single support phase of the robot. Let Z_α be the (swing phase) zero dynamics manifold. Let Γ_α be any feedback satisfying assumptions CH2-CH5 of Section III.C of [9] so that Z_α is invariant under the swing phase dynamics in closed loop with Γ_α and is locally finite-time attractive otherwise. Note that standard results imply that $\Gamma_\alpha|_{Z_\alpha} = -(L_g L_f h)^{-1} L_f^2 h$ [6], and thus (i) $\Gamma_\alpha|_{Z_\alpha}$ is uniquely determined by the choice of parameters used in the output and is completely independent of the choice of feedback used to drive the constraints to zero in finite time; and (ii) even though Γ_α is necessarily not smooth, $\Gamma_\alpha|_{Z_\alpha}$ is as smooth as the robot model.

For a regular parameter value, α, the definition of the outputs and basic properties of Bézier polynomials yield a very simple characterization of $S \cap Z_\alpha$, the configuration and velocity of the robot at the end of a phase of single support. Define

$$q_\alpha^- = H^{-1} \begin{bmatrix} \alpha_M \\ \theta_\alpha^- \end{bmatrix} \tag{19}$$

$$\omega_\alpha^- = H^{-1} \begin{bmatrix} \frac{M}{\theta_\alpha^- - \theta_\alpha^+} (\alpha_M - \alpha_{M-1}) \\ 1 \end{bmatrix}, \tag{20}$$

where $H := [H_0' \ c]'$, and the initial and final values of θ corresponding to this output are denoted by θ_α^+ and θ_α^-, respectively. Then $S \cap Z_\alpha = \{(q_\alpha^-, \dot q_\alpha^-) \mid \dot q_\alpha^- = a\,\omega_\alpha^-, a \in I\!R\}$ and is determined by the *last two columns* of the parameter matrix α. In a similar fashion $\Delta(S \cap Z_\alpha)$, the configuration, q_α^+, and velocity, $\dot q_\alpha^+$ of the robot at the beginning of a subsequent phase of single support, may be simply characterized and are determined by the *first two columns* of the parameter matrix α.

Let β also be a regular parameter value. Then using arguments almost identical to those in the proof of Theorem 4 of [9], it follows that

$$\begin{bmatrix} \beta_0 \\ \theta_\beta^+ \end{bmatrix} = HRH^{-1} \begin{bmatrix} a_M \\ \theta_\alpha^- \end{bmatrix} \tag{21}$$

implies $h(\cdot,\beta)\circ\theta\circ\Delta|_{(S\cap Z_\alpha)} = 0$, while, if $\dot q_\alpha^+ := \Delta_{\dot q}(q_\alpha^-)\,\omega_\alpha^-$, results in $c\dot q_\alpha^+ \neq 0$, then

$$\beta_1 = \frac{\theta_\beta^- - \theta_\alpha^+}{Mc\dot q_\alpha^+} H_0 \dot q_\alpha^+ + \alpha_0 \tag{22}$$

implies $L_f h(\cdot,\beta)\circ\theta\circ\Delta|_{(S\cap Z_\alpha)} = 0$. The key thing to note is that these two conditions involve, once again, only the *first two columns* of the parameter matrix β.

Taking $\beta = \alpha$, conditions (21) and (22) imply that $\Delta(S \cap Z_\alpha) \subset Z_\alpha$, in which case Z_α is then controlled-invariant for the full hybrid model of the robot. The resulting restriction dynamics is called the *hybrid zero dynamics*. Necessary and sufficient conditions can be given for the hybrid zero dynamics to admit an exponentially stable, periodic orbit, $\mathcal{O}_\alpha$ [9]. When these conditions are met, the matrix of parameters α is said to give rise to an exponentially stable walking motion. Under controller Γ_α, the exponentially stable orbit in the hybrid zero dynamics is also exponentially stable in the full order model, (5). The domain of attraction of $\mathcal{O}_\alpha$ in the full dimensional model cannot be easily estimated; however, its domain of attraction intersected with $S \cap Z_\alpha$, that is, the domain of attraction of the associated fixed-point of the restricted Poincaré map, $\rho_\alpha : S \cap Z_\alpha \to S \cap Z_\alpha$, is computed analytically in Section IV of [9]; denote this set by $\mathcal{D}_\alpha$, which is a subset of $S \cap Z_\alpha$.

Finally, define the average walking speed over a half-step, $\bar\nu$ (m/s), to be step length (m) divided by the time to impact, T_I (s), i.e., the elapsed time of a half-step. Since the controllers employed are not smooth, $\bar\nu : S \to I\!R$ is not a smooth function of the states. However, if α is a regular parameter value giving rise to a hybrid zero dynamics, $\Delta(S \cap Z_\alpha) \subset Z_\alpha$, then $\bar\nu$ restricted to $S \cap Z_\alpha$ depends smoothly on the states and the parameter values α used to define the outputs, (15); an explicit formula for $\bar\nu$ is given in Section VI of [9].

V. Provably Stable Composition of Walking Motions

Let α and β be two regular sets of parameters for the outputs (15) with corresponding zero dynamics manifolds, Z_α and Z_β. Suppose that $\Delta(S \cap Z_\alpha) \subset Z_\alpha$ and $\Delta(S \cap Z_\beta) \subset Z_\beta$, and that there exist exponentially stable periodic orbits[1], $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$; denote the corresponding controllers by Γ_α and Γ_β. The goal is to be able to

[1]Typically, these would correspond to walking at different average speeds.

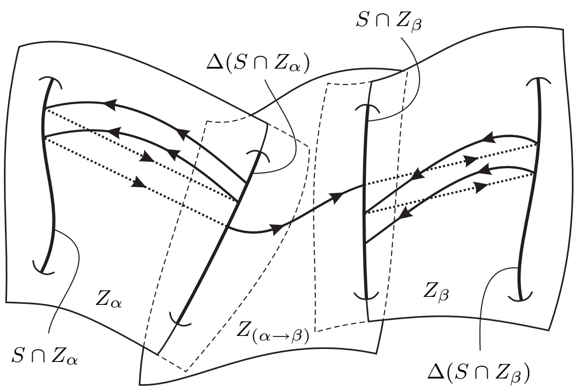

Fig. 4. Abstraction of the composition of two controllers Γ_α and Γ_β via transition controller $\Gamma_{(\alpha\to\beta)}$. Under the action of Γ_α the dynamics evolve on Z_α. Switching to $\Gamma_{(\alpha\to\beta)}$ when the state enters $\Delta(S\cap Z_\alpha)$ causes the dynamics to evolve along $Z_{(\alpha\to\beta)}$ to $S\cap Z_\beta$. Switching to Γ_β when the state enters $S\cap Z_\beta$ causes the dynamics to evolve on Z_β.

transition from $\mathcal{O}_\alpha$ to $\mathcal{O}_\beta$ without the robot falling (i.e., with stability guaranteed). If it were known that the domains of attraction of the two orbits had a non-empty intersection, then the method of [2] could be applied directly. Numerically evaluating the domains of attraction on the full-order model is unpleasant, so another means of assuring a stable transition is sought that is based on easily computable quantities, the domains of attraction of the restricted Poincaré maps associated with $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$.

Since in general $Z_\alpha\cap Z_\beta=\emptyset$, the method for providing a stable transition from $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ will be to introduce a one-step transition controller $\Gamma_{(\alpha\to\beta)}$ whose (swing phase) zero dynamics manifold $Z_{(\alpha\to\beta)}$ connects the zero dynamics manifolds Z_α and Z_β (see Figure 4). More precisely, switching will be synchronized with impact events and the zero dynamics manifold $Z_{(\alpha\to\beta)}$ will be chosen to map exactly from the one-dimensional manifold $\Delta(S\cap Z_\alpha)$ (i.e., the state of the robot just after impact with S under controller Γ_α) to the one-dimensional manifold $S\cap Z_\beta$ (i.e., the state of the robot just before impact with S under controller Γ_β).

From (19)–(22), any zero dynamics manifold $Z_{(\alpha\to\beta)}$ with parameters

$$\begin{aligned}
(\alpha\to\beta)_0 &= \alpha_0\\
(\alpha\to\beta)_1 &= \alpha_0-\frac{\theta_\beta^- -\theta_\alpha^+}{\theta_\alpha^- -\theta_\alpha^+}\,(\alpha_0-\alpha_1)\\
(\alpha\to\beta)_{M-1} &= \beta_M+\frac{\theta_\beta^- -\theta_\alpha^+}{\theta_\beta^- -\theta_\beta^+}\,(\beta_{M-1}-\beta_M)\\
(\alpha\to\beta)_M &= \beta_M\\
\theta^+_{(\alpha\to\beta)} &= \theta^+_\alpha,\quad \theta^-_{(\alpha\to\beta)} = \theta^-_\beta
\end{aligned}\tag{23}$$

satisfies $Z_{(\alpha\to\beta)}\cap\Delta(S\cap Z_\alpha)=\Delta(S\cap Z_\alpha)$ and $\Delta(S\cap Z_{(\alpha\to\beta)})=\Delta(S\cap Z_\beta)$. The choice of the intermediate parameter values, $(\alpha\to\beta)_i$, $i=2$ to $M-2$ affects the walking motion, and one could choose their values through optimization, for example, in order to minimize the torques required to evolve along the surface $Z_{(\alpha\to\beta)}$. However, the simple choice

$$(\alpha\to\beta)_i=(\alpha_i+\beta_i)/2,\ i=2\ \text{ to }\ M-2,\tag{24}$$

has proven effective in examples worked by the authors. The reason for this seems to be intimately linked the use of Bézier polynomials; see Section V.B of [9].

Assume that the parameter matrix given in (23) and (24) is regular and let $\Gamma_{(\alpha\to\beta)}$ be an associated controller; then $\Gamma_{(\alpha\to\beta)}|_{Z_{(\alpha\to\beta)}}$ is uniquely determined by the matrix of parameters $(\alpha\to\beta)$. The goal now is to determine under what conditions $\Gamma_{(\alpha\to\beta)}$ will effect a transition from $\mathcal{O}_\alpha$ to $\mathcal{O}_\beta$.

Let $P_{(\alpha\to\beta)}:S\to S$ be the Poincaré return map of the model (5) in closed loop with $\Gamma_{(\alpha\to\beta)}$ and consider $P_{(\alpha\to\beta)}|_{(S\cap Z_\alpha)}$. By construction of $Z_{(\alpha\to\beta)}$, $\Delta(S\cap Z_\alpha)\subset Z_{(\alpha\to\beta)}$. Since $Z_{(\alpha\to\beta)}$ is invariant under $\Gamma_{(\alpha\to\beta)}$, it follows that $P_{(\alpha\to\beta)}(S\cap Z_\alpha)\subset S\cap Z_{(\alpha\to\beta)}$. But by construction, $S\cap Z_{(\alpha\to\beta)}=S\cap Z_\beta$. Thus, the restriction of the Poincaré return map to $S\cap Z_\alpha$ induces a (partial) map

$$\rho_{(\alpha\to\beta)}:S\cap Z_\alpha\to S\cap Z_\beta.\tag{25}$$

In Section IV.A of [9], a closed-form expression for $\rho_{(\alpha\to\beta)}$ is computed on the basis of the two-dimensional zero dynamics associated with $Z_{(\alpha\to\beta)}$.

Let $\mathcal{D}_\alpha\subset S\cap Z_\alpha$ and $\mathcal{D}_\beta\subset S\cap Z_\beta$ be the domains of attraction of the restricted Poincaré maps $\rho_\alpha:S\cap Z_\alpha\to S\cap Z_\alpha$ and $\rho_\beta:S\cap Z_\beta\to S\cap Z_\beta$ associated with the orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, respectively. (Since the existence of exponentially stable, periodic orbits has been assumed, these domains are non-empty and open.) It follows that $\rho^{-1}_{(\alpha\to\beta)}(\mathcal{D}_\beta)$ is precisely the set of states in $S\cap Z_\alpha$ that can be steered into the domain of attraction of $\mathcal{O}_\beta$ under the control law $\Gamma_{(\alpha\to\beta)}$. In general, from stability considerations, one is more interested in $\mathcal{D}_\alpha\cap\rho^{-1}_{(\alpha\to\beta)}(\mathcal{D}_\beta)$, the set of states in the domain of attraction of $\mathcal{O}_\alpha$ that can be steered into the domain of attraction of $\mathcal{O}_\beta$ in one step under the control law $\Gamma_{(\alpha\to\beta)}$.

Theorem 1: (Serial composition of stable walking motions) Assume that α and β are regular parameters for the output (15) and that $(\alpha\to\beta)$ defined by (23) and (24) is also regular. Suppose furthermore that
1. $\Delta(S\cap Z_\alpha)\subset Z_\alpha$ and $\Delta(S\cap Z_\beta)\subset Z_\beta$;
2. there exist exponentially stable, periodic orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ in Z_α and Z_β, respectively, so that the domains of attraction $\mathcal{D}_\alpha\subset S\cap Z_\alpha$ and $\mathcal{D}_\beta\subset S\cap Z_\beta$ of the associated restricted Poincaré maps are non-empty and open.

Then the set of states in $\mathcal{D}_\alpha$ that can be steered into $\mathcal{D}_\beta$ in one step under any control law $\Gamma_{(\alpha\to\beta)}$ satisfying assumptions CH2-CH5 of Section IV.C of [9] is equal to $\mathcal{D}_\alpha\cap\rho^{-1}_{(\alpha\to\beta)}(\mathcal{D}_\beta)$. □

Proof: This follows directly from the definition of $\rho_{(\alpha\to\beta)}$. ■

An example is given in the next section.

VI. Example

This section demonstrates the technique for serial composition developed in Section V. Exponentially stable walking controllers designed via optimization as described

Step num.	Controller	T_I (s)	$\bar{\nu}$ (m/s)
1	0.75	0.5186	0.7500
2	$0.75 \rightarrow 0.85$	0.5593	0.7604
3	0.85	0.5320	0.7993
4	0.85	0.5194	0.8187
5	0.85	0.5120	0.8306
6	0.85	0.5075	0.8379
7	0.85	0.5048	0.8424
8	0.85	0.5031	0.8453
9	$0.85 \rightarrow 0.65$	0.4481	0.8379
10	0.65	0.4839	0.7759
11	0.65	0.5073	0.7401
12	0.65	0.5258	0.7140
13	0.65	0.5401	0.6952
14	0.65	0.5507	0.6818
15	0.65	0.5585	0.6723
16	0.65	0.5641	0.6656
17	0.65	0.5682	0.6608
18	0.65	0.5710	0.6575

TABLE II

Composition example result (Notes: controllers are identified by their average walking rate fixed points, and $A \rightarrow B$ indicates a transition controller from a controller at average walking rate A to average walking rate B).

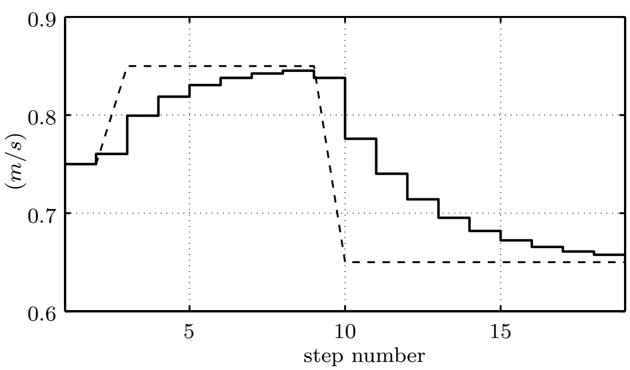

Fig. 5. Command (dashed) verses actual (solid) average speed (Note: the sloped portions of the command curve correspond to transition controllers).

in [8] are used as well as transition controllers designed according to (23) and (24).

For this example, the robot starts at the fixed point of an exponentially stable walking controller with fixed point 0.75 m/s, transitions to a controller with fixed point 0.85 m/s, takes several steps under that controller, transitions to a controller at 0.65 m/s, and, finally, takes several steps under that controller. Table II lists result of this composition experiment: the step number; which controller was applied; the time to impact (step duration), T_I; and the average speed, $\bar{\nu}$. Figure 5 gives the command verses actual average walking speed. For an animation of this example as well as supporting plots, see [1].

Acknowledgments

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