CONTROLLED PERIODIC MOTION IN A NONLINEAR SYSTEM WITH IMPULSE EFFECTS: WALKING OF A BIPED ROBOT

F. Plestan*, J.W. Grizzle+, E.R. Westervelt+, G. Abba×

Abstract

The goal of this paper is to demonstrate a means to prove asymptotically stable walking in a planar, under actuated, five-link biped robot model. The analysis assumes a rigid contact model when the swing leg impacts the ground and an instantaneous double-support phase. Under theses hypotheses, the robot is modeled by a dynamic nonlinear system and an impulse model. The controller is a an asymptotically finite-time stabilizing feedback one. In this case, the stability of the closed-loop system is analyzed by Poincaré's map whose the dimension equals one.

I. Introduction

This paper develops a provably, asymptotically stabilizing controller for the walking motion of a proto-type, five-link, planar biped robot consisting of a torso and two legs with knees but no feet; see Figure 1. The proto-type, named RABBIT[1], has four independent actuators: the axis between the torso and each thigh is actuated as is the axis of each knee. The actuators have been sized so that the robot is capable of generating motions of at least 5km/h when walking and 12km/h when running. These speeds compare well with the capabilities of humans [6]. Many of the technical considerations that went into the design of the robot are summarized in [6]. The principal motivations for constructing the proto-type were to study modeling (especially hybrid mechanical systems and compliant contact models), determination of optimal trajectories, limit cycles, stabilization of trajectories and the transition between walking and running [26]. RABBIT is limited to motion in the saggital plane by means of a radial bar. While the end of the robot's legs are fitted with wheels, these are provided so that radial movements of the contact points between the robot's leg and the floor are completely free; no mobility exists between the legs and the "feet" in the sagittal plane. The radius of the circular path imposed by the bar is approximately 3 m. The design of stabilizing controllers for the lateral motion of a walking robot has been addressed in [18], where it is shown that stability can be achieved by actively adjusting the lateral distance between the feet; this issue is not studied here. Section II develops the dynamical

***Corresponding author.** IRCCyN, Ecole Centrale de Nantes, UMR CNRS 6597, BP 92101, 1 rue de la Noe, 44321 Nantes cedex 03, France, Franck.Plestan@irccyn.ec-nantes.fr

+Control Systems Laboratory, Electrical Engineering and Computer Science Department, University of Michigan, Ann Arbor, MI 48109-2122, USA, {grizzle,ewesterv}@umich.edu

×LGIPM, Université de Metz, IUT de Thionville, Espace Cormontaigne, 57970 Yutz, France, abba@iut.univ-metz.fr

[1] The RABBIT prototype has been developed by the French Project "Commande de robots à pattes" of the "CNRS - GdR Automatique".

model of the robot and specifies all of the mechanical parameters. A rigid model is used for the contact between the swing leg and ground and the double support phase in assumed to be instantaneous. The contact between the support leg and the ground is modeled as a pivot, so the model has five degrees of freedom. Section III develops the controller. The work presented here is a natural continuation of [11], [12] where the asymptotic stability of the walking motion of a robot with a torso, two legs and no knees was fully proved. *The structure of the controller is motivated by the desire to render as tractable as possible the task of rigorously establishing the asymptotic stability of walking motions.* The principal idea is to design the controller so that the image of the Poincaré map has dimension one, which greatly simplifies the stability analysis problem. The controller's performance is evaluated by simulation in Section IV, under the hypotheses of a rigid contact model, an instantaneous double support phase and no slipping of the support leg. The simulations indicate that the controller induces an asymptotically stable walking motion of 0.7 $m.s^{-1}$, with peak torques of 110 $N.m$. The required torques are well within the capabilites of the proto-type. The actual stability of the induced walking motion is then proven in Section V under the above hypotheses.

II. Robot model

The robot is modeled as a planar biped. It consists of a torso, hips and two legs with knees, but no ankles (see Figure 1). It thus has 7 degrees of freedom (the five joint angles plus the Cartesian coordinates of the hips, for example). A torque is applied between each leg and the torso, and a torque is applied at each knee. It is assumed that the walking cycle takes place in the sagittal plane and consists of successive phases of single support. The complete model of the biped robot consists of two parts: the differential equations describing the dynamics of the robot during the swing phase (these equations are derived using the method of Lagrange [23]), and an impulse model of the contact event (the impact between the swing leg and the ground is modeled as a contact between two rigid bodies [15]). The contact between the stance leg and the ground is modeled as a pivot. As in [11], [12], the complete model can be expressed as a nonlinear system with impulse effects [25].

A. Swing phase model

The dynamic model of the robot between successive impacts is derived from the Lagrange formalism

$$D(q) \cdot \ddot{q} + C(q, \dot{q}) \cdot \dot{q} + G(q) \quad = \quad B \cdot u \qquad (1)$$

with $q = (q_{31}, q_{41}, q_{32}, q_{42}, q_1)'$ (see Figure 2) and $u = (u_1, u_2, u_3, u_4)'$ (see Figure 4 and Figure 5). The torques u_1, u_2, u_3 and u_4 are applied between the torso and the stance leg, the torso and the swing leg, at the knee of the stance leg and at the knee of the swing leg, respectively. Then, the model can be written in state space form by defining

$$\dot{x} := \frac{d}{dt}\begin{bmatrix} q \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \\ D^{-1}(q)\cdot(-C(q,\omega)\cdot\omega - G(q) - B\cdot u) \end{bmatrix} =: \ f(x) + g(x)\cdot u \tag{2}$$

where $\omega := \dot{q}$, and $x := (q', \omega')'$. The state space of the model will be restricted to physically reasonable values of q for walking. To define these bounds, it is convenient to introduce the coordinates $(p_{31}, p_{41}, p_{32}, p_{42})$ (see Figure 3) where

$$\begin{bmatrix} p_{31} \\ p_{41} \\ p_{32} \\ p_{42} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(q_{31}+q_{41}) \\ \pi + q_{41} - q_{31} \\ \frac{1}{2}(q_{32}+q_{42}) \\ \pi + q_{42} - q_{32} \end{bmatrix} \tag{3}$$

The variable p_{31} (resp. p_{32}) is the angle between the vertical axis and a "virtual" leg joining the hips to the foot of the stance leg (resp. the swing leg) and the variable p_{41} (resp. p_{42}) is the relative angle of the stance leg (resp. swing leg) knee. The state space for the system will be taken as $\mathcal{X} := \{(q', \omega')' \mid q \in M,\ \omega \in I\!R^5\}$, where $M = \{q \mid -\frac{\pi}{2} < q_1 < \frac{\pi}{2},\ \frac{3\pi}{4} < p_{31} < \frac{5\pi}{4},\ 0 < p_{41} < \pi,\ \frac{3\pi}{4} < p_{32} < \frac{5\pi}{4},\ 0 < p_{42} < \pi\}$ and the walking surface is taken as $\{(q,\omega) \in \mathcal{X} | z_1 = 0\}$.

B. Impact model

An impact occurs when the swing leg touches the walking surface, also called the ground. The impact between the swing leg and the ground is modeled as a contact between two rigid bodies. The development of the impact model requires the full seven degrees of freedom of the robot. Let us add Cartesian coordinates (x_H, z_H) to the hips. One then obtains the following extended model

$$D_e(q_e)\cdot\ddot{q}_e + C_e(q_e,\dot{q}_e)\cdot\dot{q}_e + G_e(q_e) \ = \ B_e\cdot u + \delta F_{ext} \tag{4}$$

with $q_e = (q_1, q_{31}, q_{41}, q_{32}, q_{42}, x_H, z_H)'$. δF_{ext} represents the external forces acting on the robot at the contact point. The basic hypotheses are

- The contact of the swing leg with the ground results in no rebound and no slipping of the swing leg.
- At the moment of impact, the stance leg lifts from the ground without interaction.
- The impact is instantaneous.

- The external forces during the impact can be represented by impulses.
- The impulsive forces may result in an instantaneous change in the velocities, but there is no instantaneous change in the positions.
- The torques supplied by the actuators are not impulsional.

From these hypotheses, the angular momentum is conserved. One deduces

$$D_e(\dot{q}_e^+ - \dot{q}_e^-) \quad = \quad F_{ext} \tag{5}$$

where F_{ext} is the result of the contact impulse forces. $\dot{q}_e^+$ (resp. $\dot{q}_e^-$) is the velocity just after (resp. before) impact. An additional set of two equations is obtained by supposing that the stance leg does not rebound nor slip at impact. Then, from the condition that the swing leg does not rebound nor slip at impact, one gets

$$\frac{d}{dt}E(q_e) \quad = \quad \frac{\partial E}{\partial q_e} \cdot \dot{q}_e^+ = 0 \tag{6}$$

with $E(q_e) = (x_2, z_2)'$ the Cartesian coordinates of the end of the swing leg. The result of solving (5) and (6) yields an expression[2] for $\dot{q}_e^+$ in term of $\dot{q}_e^-$. The final result is an expression for $x^+ := (q^+, \omega^+)$ (state value just after the impact) in terms of $x^+ := (q^-, \omega^-)$ (state value just before the impact), which is expressed as

$$x^+ \quad = \quad \Delta(x^-) \tag{7}$$

C. Nonlinear system with impulse effects

The overall biped robot model can be expressed as a nonlinear system with impulse effects [1], [25]

$$\begin{array}{rclcl} \dot{x} & = & f(x) + g(x) \cdot u & \qquad & x^- \notin S \\ x^+ & = & \Delta(x^-) & \qquad & x^- \in S \end{array} \tag{8}$$

where,

$$S := \{(q, \omega) \in \mathcal{X} | z_2 = 0, L_{f+gu} z_2 < 0\}. \tag{9}$$

Solutions are taken to be right continuous (see [12] for details). If a solution attains the boundary of $\mathcal{X}$ in finite time it is stopped and no longer exists. With this convention, as long as the robot is initialized in $\mathcal{X}$ with the swing leg on or above the walking surface, all valid solutions of the model result in the robot remaining on or above the walking surface.

[2] The solvability of the equations is easily verified; see [12].

III. Feedback Controller Design

This section develops the extension of the controller of [11], [12] for the 5 link biped with knees. The fundamental idea is to encode walking in terms of a set of "posture conditions", which are in turn expressed as "holonomic constraints" on the position variables. These "constraints" are then used to construct outputs of the mechanical model and are "imposed" on the robot via feedback control. The controller is designed on the basis of the assumptions made in Section II, namely that the impact model is rigid and the double support phase instantaneous.

A. Output definition

In human walking, one observes that the torso is maintained at a nearly vertical angle, the hips remain roughly centered between the feet and at a nearly constant height above the walking surface, and the end of the swing leg traces an approximately parabolic trajectory. In addition, the knees are never hyper-extended (as opposed to a bird) and only slightly flexed (as opposed to a monkey). These observations have been used to build a set of control objectives through the following output functions:

$$\begin{array}{rcl} y_1 & = & k_1 \cdot (q_1 - q_{1d}) \\ y_2 & = & k_2 \cdot (d_1 + d_2) \\ y_3 & = & k_3 \cdot (z_H - z_{Hd}(d_1)) \\ y_4 & = & k_4 \cdot (z_2 - z_{2d}(d_1)) \end{array} \tag{10}$$

In the above, the Cartesian-coordinates of the hips, (x_H, z_H), and the end of the swing leg, (x_2, z_2), are expressed in the Cartesian-coordinates frame of the end of the stance leg, (x_1, z_1)

$$\begin{array}{rcl} x_1 & = & 0 \\ z_1 & = & 0 \\ x_H & = & L_3 \cdot \sin(q_{31}) + L_4 \cdot \sin(q_{41}) \\ z_H & = & -L_3 \cdot \cos(q_{31}) - L_4 \cdot \cos(q_{41}) \\ x_2 & = & x_H - L_3 \cdot \sin(q_{32}) - L_4 \cdot \sin(q_{42}) \\ z_2 & = & z_H + L_3 \cdot \cos(q_{32}) + L_4 \cdot \cos(q_{42}) \end{array} \tag{11}$$

The output y_1 is chosen to maintain the angle of the torso at a desired constant value, say q_{1d}. The output y_2 ensures the advancement of the hips while the swing leg goes from behind the stance leg to in front of it (see Figure 2 for representation of d_1 and d_2). The output y_3 controls the hip height in such a way that the hips can rise and fall by a small amount in a natural way. The desired trajectory z_{Hd} of the hips is defined as a second order polynomial of d_1 such that $d_1 \in [-\mathrm{sld}/2, \mathrm{sld}/2]$, where sld is the desired step length, $z_{H\mathrm{MAX}}$ (resp. $z_{H\mathrm{MIN}}$) is the maximum (resp. minimum) desired value of z_H

over a step and

$$z_{Hd}(-\mathrm{sld}/2) = z_{H\mathrm{MIN}} \qquad z_{Hd}(0) = z_{H\mathrm{MAX}} \qquad z_{Hd}(\mathrm{sld}/2) = z_{H\mathrm{MIN}} \tag{12}$$

The output y_4 controls the trajectory of the end of the swing leg; the desired trajectory z_{2d} is defined as a second order polynomial of d_1 such that $d_1 \in [-\mathrm{sld}/2, \mathrm{sld}/2]$, where sld is the desired step length, $z_{2\mathrm{MAX}}$ is the maximum desired value of z_2 over a step and

$$z_{2d}(-\mathrm{sld}/2) = 0 \qquad z_{2d}(0) = z_{2\mathrm{MAX}} \qquad z_{2d}(\mathrm{sld}/2) = 0 \tag{13}$$

The gains k_1, k_2, k_3 and k_4 are constant values to be chosen later. Thus, with the same notation as in (8), the output vector reads as

$$y = \begin{bmatrix} k_1 \cdot (q_1 - q_{1d}) \\ k_2 \cdot (d_1(q) + d_2(q)) \\ k_3 \cdot (z_H(q) - z_{Hd}(d_1(q))) \\ k_4 \cdot (z_2(q) - z_{2d}(d_1(q))) \end{bmatrix} := h(q) \quad = \begin{bmatrix} h_1(q) \\ h_2(q) \\ h_3(q) \\ h_4(q) \end{bmatrix} \tag{14}$$

B. Controller synthesis

The control objective is to drive the outputs (14) to zero. Since the outputs (14) only depend on the generalized positions, q, and the dynamic model (2) is second order, the relative degree of each output component is either two or infinite. Using standard Lie derivative notation [16], [23], direct calculation yields

$$\ddot{y} \;\; = \;\; L_f^2 h(x) + L_g L_f h(x) \cdot u. \tag{15}$$

For the moment, it is supposed that the matrix $L_g L_f h$ is invertible on the region of interest. This will be confirmed later in the paper. The method of computed torque (or inverse dynamics) can then be used to define

$$v := L_f^2 h + L_g L_f h \cdot u, \tag{16}$$

resulting in four double integrators

$$\ddot{y}_i \;\; = \;\; v_i, \quad i = 1 \text{ to } 4. \tag{17}$$

One possible approach to control design would be to design asymptotically stabilizing controllers, such as $v_i = k_{i1} y_i + k_{i2} \dot{y}_i$, for the double integrators (17). In general, when such a feedback is applied to the full hybrid model (8), it is no longer able to drive the outputs (14) asymptotically to zero due to the impulsive effects of the impacts. A general means of trying to "overcome" this can be observed in the literature: for experimental as well as simulation based studies, the feedback 'gains' appear to be universally chosen sufficiently large so that the time constant for driving the outputs to zero is much less than the time interval of a single step. A biological basis for doing this is much more difficult to establish because the experiments are not easy to do well. Nevertheless, the evidence suggests that if a perturbation is deliberately introduced in a human's gait [7], [8], the subject's gait will recover to its original state in just a few cycles.

The use of high-gain control can be made to work quite well in simulation. The difficulty comes in mathematically analyzing the existence and stability of periodic orbits induced by the controller. Since we are dealing with periodic orbits, Poincaré's method is the appropriate tool. However, to apply it one must compute the induced discrete-time dynamics from a hyper-surface transversal to the orbit back to the hyper-surface [13], [20]. The induced discrete-time dynamics is called the Poincaré map. In the case of the model (8), the hyper-surface has dimension nine and the computation of the Poincaré map is impractical.

The key idea established in [12] is that for a mechanical system with N-degrees of freedom and m-independent inputs, the feedback control design can be carried out in a way that greatly simplifies the stability analysis problem: the dimension of the image of the Poincaré map can be reduced from $2N-1$ to $N-m$. For the biped considered here, this results in a *one-dimensional* analysis problem. The Poincaré map for this one-dimensional problem must still be computed numerically. The main points are that its numerical computation is very easy and it leads to *conclusive existence and stability properties* for periodic orbits.

The feedback design proceeds as follows. Define a continuous[3] feedback $v = v(y, \dot{y})$ on (15) so that each of the four double integrators $\ddot{y}_i = v_i$ is (globally) finite-time stabilized. The feedback functions used here come from [2]:

$$v \;= \Psi(y,\dot{y}) \;:= \frac{1}{\epsilon^2}\cdot\begin{bmatrix}\psi_1(y_1,\epsilon\cdot\dot{y}_1)\\ \psi_2(y_2,\epsilon\cdot\dot{y}_2)\\ \psi_3(y_3,\epsilon\cdot\dot{y}_3)\\ \psi_4(y_4,\epsilon\cdot\dot{y}_4)\end{bmatrix}. \tag{18}$$

Each function $\psi_i(y_i, \epsilon\cdot\dot{y}_i)$ $(1 \le i \le 4)$ is defined as

$$\psi_i(y_i,\epsilon\cdot\dot{y}_i) \;:=\; -\mathrm{sign}(\epsilon\cdot\dot{y}_i)\cdot|\epsilon\cdot\dot{y}_i|^{\alpha} - \mathrm{sign}(\phi_i(y_i,\epsilon\cdot\dot{y}_i))\cdot|\phi_i(y_i,\epsilon\cdot\dot{y}_i)|^{\frac{\alpha}{2-\alpha}} \tag{19}$$

[3]The theory in [12] does NOT allow the use a discontinuous feedback as is commonly used in sliding mode control.

with $0 < \alpha < 1$ and

$$\phi_i(y_i, \epsilon \cdot \dot{y}_i) \;=\; y_i + \tfrac{1}{2-\alpha}\mathrm{sign}(\epsilon \cdot \dot{y}_i) \cdot |\epsilon \cdot \dot{y}_i|^{2-\alpha} \tag{20}$$

The real parameter $\epsilon > 0$ allows the settling time of the controllers to be adjusted. The overall feedback applied to (8) is given by

$$u(x) \;:=\; (L_g L_f h(x))^{-1} \cdot \left(\Psi(h(x), L_f h(x)) - L_f^2 h(x)\right). \tag{21}$$

This is the method of computed torque with a finite-time stabilizing controller on each of the double integrators.

IV. Simulations

Consider the biped robot model (8) with the following values of parameters (see Figures 4, 5)

Mechanical parameters	Torso	Femur	Tibia
Mass(kg)	M_T=20	M_3=6.8	M_4=3.2
Length (m)	L_T=0.625	L_3=0.4	L_4=0.4
Position of the Center of Mass (m)	X_T=0.01, Z_T=0.2	Z_3=0.16	Z_4=0.128

Consider the feedback of Section III-B with the following parameters[4]

Output	Gain	Parameters
y_1	k_1=62.5	$q_{1d} = \pi/30$ rad
y_2	k_2=500	
y_3	k_3=1	$z_{H\mathrm{MIN}} = 0.745$ m, $z_{H\mathrm{MAX}} = 0.76$ m, sld=0.5 m
y_4	k_4=1	$z_{2\mathrm{MAX}} = 0.01$ m, sld=0.5 m

The initial velocity of the hips, v_H^-, equals 1.25 $m.s^{-1}$. In the feedback (21), $\epsilon = 0.05$ and $\alpha = 0.9$. The parameter $\epsilon > 0$ allows the settling time of the controller to be adjusted and $0 < \alpha < 1$ achieves a finite-settling time. The choice of the parameter values has been made with an eye towards keeping the magnitudes of the applied torques within the capabilities of the actuators of the proto-type.

Several aspects of the solution corresponding to the model and feedback with the above parameters are now discussed. Figure 6 displays the outputs, which go to zero before the impact. The walking motion of the biped robot is displayed as a series of stick figures in Figure 7 over about four steps. Figure 8 displays the applied torques over a few walking cycles (about four steps); note that the

[4]The torso appears to be leaning backwards because the center of mass is not located along the axis of the torso.

peak torque magnitude is around 110 Nm, which is compatible with the proto-type RABBIT. Figure 9 displays the normal and tangential forces acting on the stance leg end. Figure 10 displays the coordinates z_2 and z_H, which are key parameters in the definition of the outputs used to generate the feedback controller

V. Stability Proof

The purpose of this section is to prove the asymptotic stability or instability of trajectories resulting from the biped in closed loop with the controller (21). An important result from [12] is that stability (or instability) can be proven on the basis of the restriction of the Poincaré map to a one dimensional manifold. In the following, only the bare minimum of mathematical notation needed to use this tool will be introduced. The reader seeking a careful development of these ideas is referred to [12]. Let Z denote the zeros dynamics manifold, i.e. $Z = \{(q, \dot{q}) \in \mathcal{X} | h(q) = 0, L_f h(q) = 0\}$. The conditions required to define the reduced Poincaré map are

1. $S \cap Z$ is a smooth submanifold of $\mathcal{X}$;
2. the decoupling matrix $L_g L_f h$ is invertible;
3. the convergence time of the controllers is strictly less than the time of a single step of the robot.

A. Smoothness of $S \cap Z$

From standard results in [3], $S \cap Z$ will be a smooth one-dimensional manifold if the map

$$\begin{bmatrix} h(q) \\ L_f h(q, \dot{q}) \\ z_2(q) \end{bmatrix} \tag{22}$$

has constant rank[5] equal to nine on $S \cap Z$. A simple argument shows that this is equivalent to the rank of $[h(q)' \; z_2(q)]'$ being equal to five. Hence, define the 5×5 matrix

$$A = \begin{bmatrix} \frac{\partial h}{\partial q} \\ \frac{\partial z_2}{\partial q} \end{bmatrix} \tag{23}$$

whose determinant in the p coordinates is proportional to

$$2 \cdot \sin(p_{42}) \cdot \sin(p_{31}) \cdot \sin(\frac{p_{41}}{2}) \cdot \sin(p_{41}). \tag{24}$$

[5]Recall that the rank of a map at a point is by definition the rank of its Jacobian matrix evaluated at the same point.

On M, it is easily verified that the determinant vanishes only at $p_{31} = \pi$. However, if $q \in Z$ and $p_{31} = \pi$, then $z_2(q) = 0.01 \neq 0$, and thus $q \notin S$. Hence, the determinant of A is non-zero on $S \cap Z$. From its definition, it is clear that if $(q, \omega) \in S \cap Z$, then q is equal to a constant; call this value q_0. Furthermore, it follows that ω is parameterized by a single variable. This parameterization is developed next. Let

$$\Phi(q) \quad = \quad \begin{bmatrix} h(q) \\ x_H(q) \end{bmatrix}, \tag{25}$$

where x_H is the horizontal position of the robot's hips. It is straightforward to verify that Φ has full rank at q_0. On Z, it follows that $\frac{d}{dt}h(q) = L_f h(q, \omega) = 0$, and thus

$$\begin{bmatrix} 0 \\ v_H \end{bmatrix} = \frac{d}{dt}\Phi(q) = \frac{\partial \Phi}{\partial q} \cdot \omega. \tag{26}$$

Thus, $\sigma : I\!R \to S \cap Z$ by

$$\sigma(v_H^-) := \begin{bmatrix} q_0 \\ \left[\frac{\partial \Phi(q_0)}{\partial q}\right]^{-1} \cdot v_H^- \end{bmatrix} \tag{27}$$

is a diffeomorphism from $I\!R$ to $S \cap Z$, with v_H^- the hips horizontal velocity just before the impact.

B. Proof of the decoupling matrix invertibility

The complexity of the decoupling matrix, $L_g L_f h$, makes a direct proof of invertibility highly non-trivial. Moreover, since the point $q_{ext} = (\pi, \pi, \pi, \pi, 0)'$ is an extremum of the height of the hips, the decoupling matrix for the choice of outputs (14) is necessarily singular at q_{ext}. Hence, proof of the invertibility of the decoupling matrix must be local in q.

One method of local proof is to demonstrate sign definiteness of the decoupling matrix's determinant in an open set about the biped's trajectories. Sign definiteness implies the determinate never equals zero in that set and, hence, in that set, the decoupling matrix is invertible. This is the method used here. The proof is carried out in two steps. In the first step, the decoupling matrix is simplified by the application of an invertible feedback [21] to the model[6]. In the second step, elementary bounds on the individual terms appearing in the determinant of the decoupling matrix are determined and used to compute upper and lower bounds on the determinant of the decoupling matrix.

To apply the technique of [21], it is easiest to work in relative coordinates

$$\bar{q} := (\bar{q}_{31}, \bar{q}_{41}, \bar{q}_{32}, \bar{q}_{42}, q_1) \tag{28}$$

[6]By standard results in [16], the invertibility of the decoupling matrix is invariant under the application of invertible feedbacks.

where

$$\begin{bmatrix} \bar{q}_{31} \\ \bar{q}_{41} \\ \bar{q}_{32} \\ \bar{q}_{42} \end{bmatrix} = \begin{bmatrix} q_{31} - q_1 \\ q_{31} - q_{41} \\ q_{32} - q_1 \\ q_{32} - q_{42} \end{bmatrix}. \tag{29}$$

Denote the dynamic model (1) in these new coordinates as

$$\bar{D}(\bar{q}) \cdot \ddot{\bar{q}} + \bar{C}(\bar{q}, \dot{\bar{q}}) \cdot \dot{\bar{q}} + \bar{G}(\bar{q}) = \bar{B} \cdot u. \tag{30}$$

It is easily shown that $\bar{B}$ has the form

$$\bar{B} = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{31}$$

Next, partition the coordinates into

$$q_a = (\bar{q}_{31}, \bar{q}_{41}, \bar{q}_{32}, \bar{q}_{42})' \quad \text{and} \quad q_b = q_1, \tag{32}$$

the "actuated" coordinates and "un-actuated" coordinates, respectively. Write (30) as

$$\bar{D}_{11}(\bar{q})\ddot{q}_a + \bar{D}_{12}(\bar{q})\ddot{q}_b + \bar{C}_1(\bar{q}, \dot{\bar{q}})\dot{q}_a + \bar{G}_1(\bar{q}) = u \tag{33}$$

$$\bar{D}_{21}(\bar{q})\ddot{q}_a + \bar{D}_{22}(\bar{q})\ddot{q}_b + \bar{C}_2(\bar{q}, \dot{\bar{q}})\dot{q}_a + \bar{G}_2(\bar{q}) = 0, \tag{34}$$

and solve (34) for $\ddot{q}_b$ as

$$\ddot{q}_b = -\bar{D}_{22}(\bar{q})^{-1}\left(\bar{D}_{21}(\bar{q})\ddot{q}_a + \bar{C}_2(\bar{q}, \dot{\bar{q}})\dot{q}_a + \bar{G}_2(\bar{q})\right). \tag{35}$$

Substituting (35) into (33) yields

$$\hat{D}(\bar{q})\ddot{q}_a + \hat{C}(\bar{q}, \dot{\bar{q}})\dot{q}_a + \hat{G}(\bar{q}) = u \tag{36}$$

where

$$\hat{D}(\bar{q}) = \bar{D}_{11}(\bar{q}) - \bar{D}_{12}(\bar{q})\bar{D}_{22}^{-1}(\bar{q})\bar{D}_{21}(\bar{q}) \tag{37}$$

$$\hat{C}(\bar{q}, \dot{\bar{q}}) = \bar{C}_1(\bar{q}, \dot{\bar{q}}) - \bar{D}_{12}(\bar{q})\bar{D}_{22}^{-1}(\bar{q})\bar{C}_2(\bar{q}, \dot{\bar{q}}) \tag{38}$$

$$\hat{G}(\bar{q}) = \bar{G}_1(\bar{q}) - \bar{D}_{12}(\bar{q})\bar{D}_{22}^{-1}(\bar{q})\bar{G}_2(\bar{q}). \tag{39}$$

Applying the partial linearizing feedback[7]

$$u = \hat{D}(\bar{q})v + \hat{C}(\bar{q}, \dot{\bar{q}})\dot{q}_a + \hat{G}(\bar{q}) \tag{40}$$

[7]The invertibility of $\bar{D}_{22}$ is assured by the positive definiteness of D.

to (33) allows (33) and (34) to be re-written as

$$\ddot{q}_a = v \tag{41}$$

$$\ddot{q}_b = -\bar{D}_{22}(\bar{q})^{-1}\left(\bar{D}_{21}(\bar{q})\ddot{q}_a + \bar{C}_2(\bar{q},\dot{\bar{q}})\dot{q}_a + \bar{G}_2(\bar{q})\right). \tag{42}$$

The model (41) and (42) is feedback equivalent to the original system. It can be expressed in state space form with the same choice of x as before to obtain

$$\dot{x} = \hat{f}(x) + \hat{g}(x)v. \tag{43}$$

Since the rank of the decoupling matrix is invariant under invertible feedback, the decoupling matrices for systems (2) and (43) have the same rank. The determinant of the decoupling matrix for (43) can be directly computed and shown to be of the form[8]

$$\det L_{\hat{g}}L_{\hat{f}}h(\bar{q}) = \frac{\mathrm{Num}(\bar{q})}{\mathrm{Den}(\bar{q})} \tag{44}$$

with

$$\mathrm{Num}(\bar{q}) = \sum_{i=1}^{114} k_i^N g_i^N\left(c_i^N\bar{q}\right) \quad \text{and} \quad \mathrm{Den}(\bar{q}) = \sum_{i=1}^{11} k_i^D g_i^D\left(c_i^D\bar{q}\right) \tag{45}$$

where the k_i's are constants, g_i's are sine and cosine functions, and c_i's are row vectors in $I\!R^5$. For a given subset $\mathcal{O} \subset M$ (recall that M is the allowed set for the configuration variables), upper and lower bounds on the determinant of the decoupling matrix can be found via calculation of the minimum and maximum of each of the 125 terms of the numerator and denominator over $\mathcal{O}$. For example, if the denominator in (44) is positive, then

$$\max_{\bar{q}\in\mathcal{O}} \det L_{\hat{g}}L_{\hat{f}}h(\bar{q}) \leq \frac{\max_{\bar{q}\in\mathcal{O}}\mathrm{Num}(\bar{q})}{\min_{\bar{q}\in\mathcal{O}}\mathrm{Den}(\bar{q})} \leq \frac{\max_{i\in I}\max_{\bar{q}\in\mathcal{O}_i}\mathrm{Num}(\bar{q})}{\min_{i\in I}\min_{\bar{q}\in\mathcal{O}_i}\mathrm{Den}(\bar{q})}, \tag{46}$$

where, $\mathcal{O} \subset \cup_{i\in I}\mathcal{O}_i$, and the $\mathcal{O}_i$ are closed and bounded. The max and min operations in (46) are especially trivial to evaluate if the sets $\mathcal{O}_i$ are selected to be of the form

$$\begin{aligned}\mathcal{O}_i := \Big\{x \mid\ & \bar{q}_{31,i}^{min} \leq \bar{q}_{31} \leq \bar{q}_{31,i}^{max},\ \bar{q}_{41,i}^{min} \leq \bar{q}_{41} \leq \bar{q}_{41,i}^{max},\\ & \bar{q}_{32,i}^{min} \leq \bar{q}_{32} \leq \bar{q}_{32,i}^{max},\ \bar{q}_{42,i}^{min} \leq \bar{q}_{42} \leq \bar{q}_{42,i}^{max}, q_{1,i}^{min} \leq q_1 \leq q_{1,i}^{max}\Big\}.\end{aligned} \tag{47}$$

The above technique was applied to the apparent limit-cycle of Section of IV. Individual closed sets $\mathcal{O}_i$ were determined by dividing the time trajectory into disjoint pieces, and over bounding the

[8]It is straightforward to check that the decoupling matrix depends only upon the configuration variables, q, and not on the angular velocities.

configuration variables so that over the $i-th$ time interval, the trajectory of the configuration variables lies strictly in the interior of $\mathcal{O}_i$. As an illustration, Figure 13 shows the result of this process for $\bar{q}_{31}$. Division of the trajectories in time into pieces over which the determinant could be proven to be sign definite was accomplished with a simple binary search algorithm. The results of this process are presented in Table I, which gives the upper and lower bounds of the determinate of the decoupling matrix as well as the minimum and maximum of the determinant over each subset, and the beginning and end of each set's division in time. It should be noted that: (1) this process could be iterated to prove the decoupling matrix's invertibility over a larger subset of the biped's state space, and, (2) the fact that this method works is not an accident. Results from real analysis can be used to show that the decoupling matrix is invertible on an open set about the configuration variable trajectories if and only if a there exists a set $\mathcal{O}$ which is the interior of a union of a *finite* number of closed sets $\mathcal{O}_i$ as described above.

C. The reduced Poincaré map

The Poincaré surface will be taken to be S, the impact surface. Let $P : S \to S$ be the usual Poincaré[9] map. A trajectory for which the convergence time of the controller (21) is less than the time to make a single step will converge to the zero dynamics manifold, Z, before impacting S. In this case, P takes values in $S \cap Z$. The *reduced Poincaré map* is defined to be $\rho : S \cap Z \to S \cap Z$ by $\rho(x) = P(x)$; that is $\rho := P_{|S \cap Z}$. To compute ρ, it is easiest to use the identification of $S \cap Z$ with $I\!R$ given by (27). Thus, define $\lambda : I\!R \to I\!R$ by $\lambda := \sigma^{-1} \circ \rho \circ \sigma$. The function λ can be computed in a straightforward manner:

Reduced Poincaré map: $\lambda : I\!R \to I\!R$

1. Let $v_H^- > 0$ denote the horizontal velocity of the robot's hips just before impact (the restriction to positive velocities corresponds to the robot walking from left to right). Compute $x^- := \sigma(v_H^-) \in S \cap Z$, the position of the robot just before impact.

2. Apply the impact model to x^-, that is, compute $x^+ := \Delta(x^-)$.

3. Use x^+ as the initial condition in (2) controlled by (21), the robot in closed loop with the controller, and simulate until one of the following happens:

[9]Since not every initial condition in S will result in the robot making a successful step, P is in general only a partial map; that is, its domain of definition is not all of S. The same is true, of course, for the reduced Poincaré map.

a. There exists a (first) time $T > 0$ where $z_2(T) = 0$. If T is greater than the settling time of the controller, then $\lambda(v_H^-) := v_H^+(T)$; else, $\lambda(v_H^-)$ is undefined at this point.

b. There does not exist a $T > 0$ such that $z_2(T) = 0$; in this case, it is also true that $\lambda(v_H^-)$ is undefined at this point. ■

D. Stability results

To determine if the closed-loop system is stable under the controller (21), the function λ is computed for $v_H^- \in [1, 2]$. Figure 11 displays the function λ. One deduces that λ is undefined for v_H^- less than 1.05 $m.s^{-1}$ and more than 1.55 $m.s^{-1}$. A fixed point appears at approximately 1.25 $m.s^{-1}$, and corresponds to an asymptotically stable walking cycle. Figure 12 displays the limit cycle over several steps with the previous parameters of simulation, in particular $v_H^- = 1.25$ $m.s^{-1}$; it appears that the trajectory is very close to the limit cycle at the beginning of the simulation, which confirms that the point $v_H^- = 1.25$ $m.s^{-1}$ is fixed.

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Set		1	2	3	4	5
$\bar{q}_{31}$	min	200.5959	195.6506	176.6136	168.6505	164.5881
	max	206.2664	204.6483	199.6031	180.1816	172.0576
$\bar{q}_{41}$	min	21.3857	30.1298	31.3344	25.3116	21.3857
	max	30.7385	35.4825	36.7536	31.9674	25.8229
$\bar{q}_{32}$	min	162.6050	167.1885	183.9608	200.8606	201.3176
	max	170.5660	187.6771	204.9184	205.9873	205.9672
$\bar{q}_{42}$	min	18.7805	23.8975	34.3898	26.6355	21.3857
	max	24.3803	38.7004	41.1356	35.0845	27.1736
q_1	min	5.6967	5.7436	5.9302	5.9400	5.9400
	max	6.0600	6.0500	6.0600	6.0600	6.0600
$\det L_{\hat{g}} L_{\hat{f}} h(x)$	min	-310.7850	-531.0551	-626.6833	-420.7033	-265.5934
	max	-0.3874	-0.2203	-0.1184	-0.2959	-4.2181
time	start	0.0000	0.0807	0.2256	0.5786	0.6466
	stop	0.0807	0.2256	0.5786	0.6466	0.6718

TABLE I

$\mathcal{O}$ SET DEFINITION AND DETERMINANT VALUE

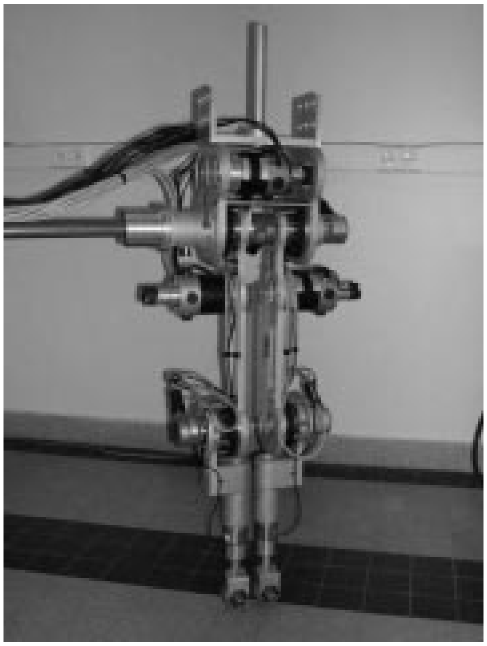

Fig. 1. Photo of RABBIT proto-type.

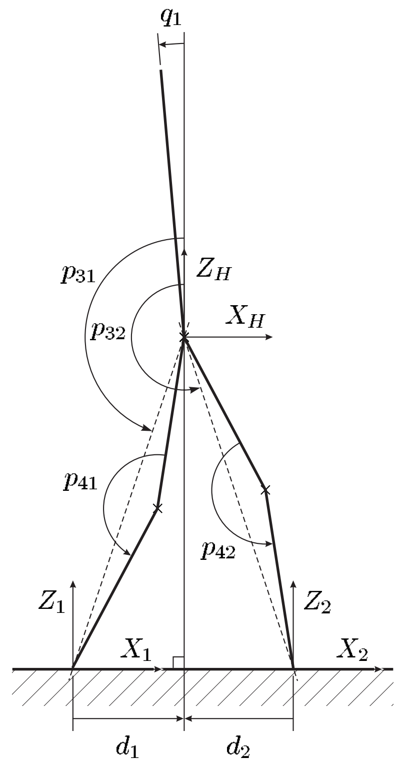

Fig. 3. Schematic of biped robot; relative angles

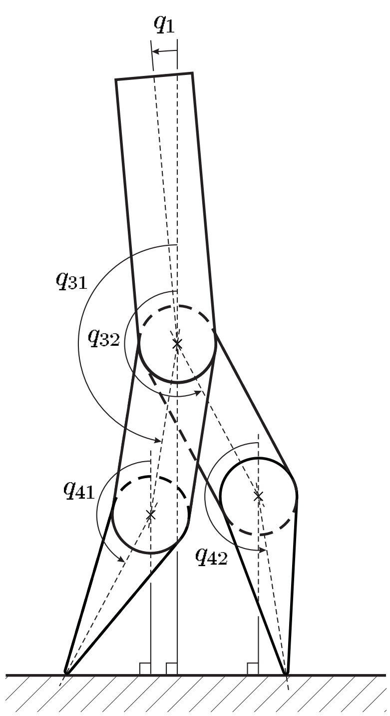

Fig. 2. Schematic of biped robot; absolute angles

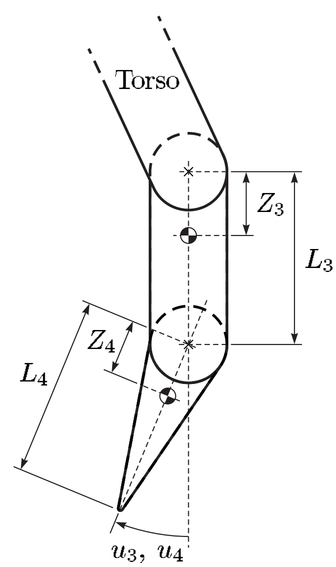

Fig. 4. Schematic of biped robot leg

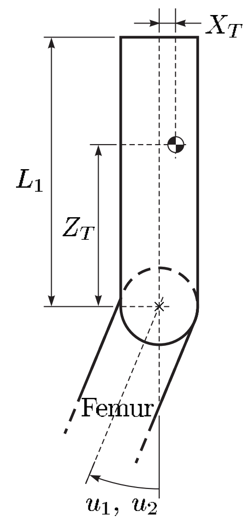

Fig. 5. Schematic of biped robot torso

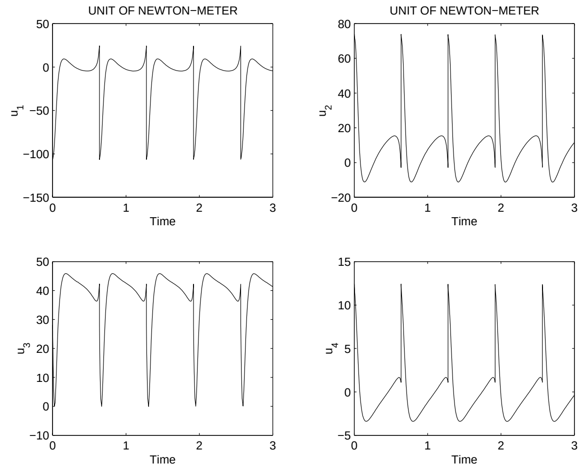

Fig. 8. Plot of applied torques versus time; unit of newton-meter.

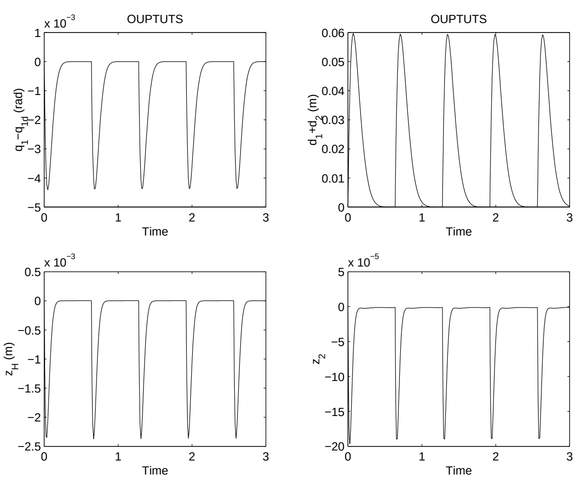

Fig. 6. Plot of outputs versus time.

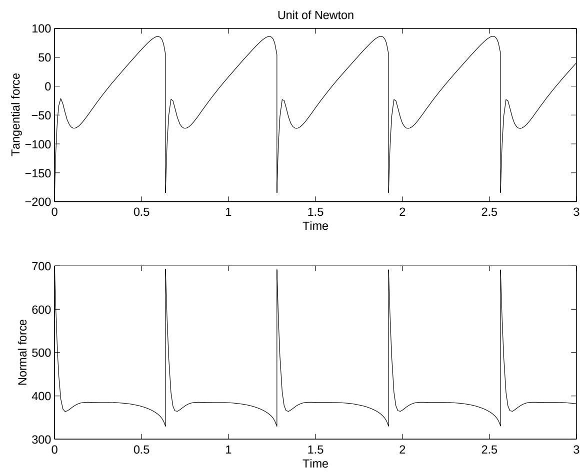

Fig. 9. Plot of normal and tangential forces acting on the stance leg end versus time; unit of newton.

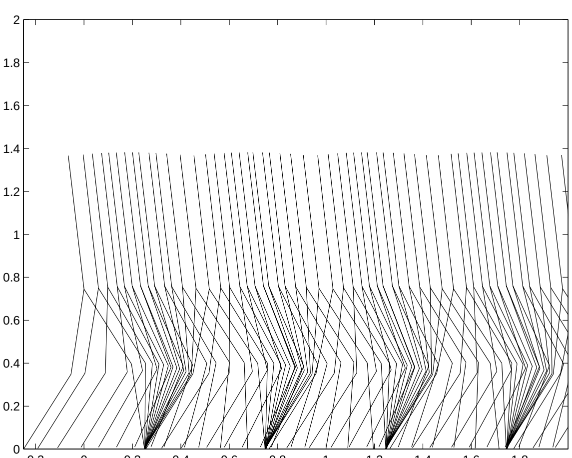

Fig. 7. Plot of walking of biped robot; unit of meter

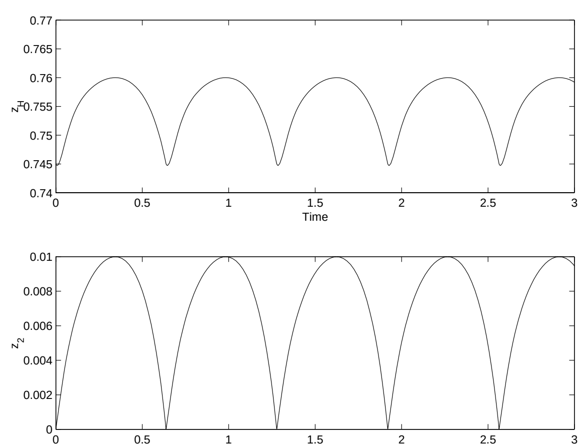

Fig. 10. Plot of z_H and z_2; unit of meter

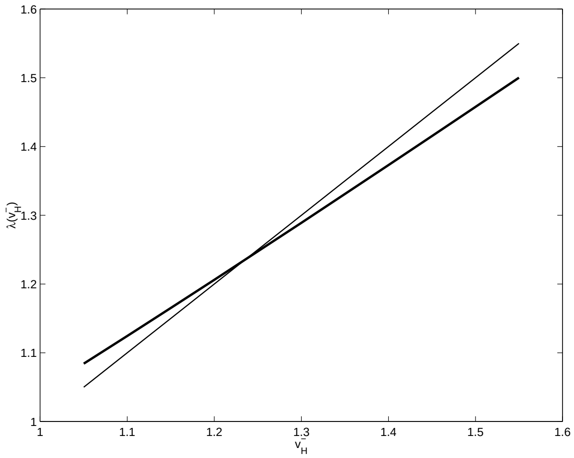

Fig. 11. Function λ (bold line) and identity function (thin line) versus v_H^-

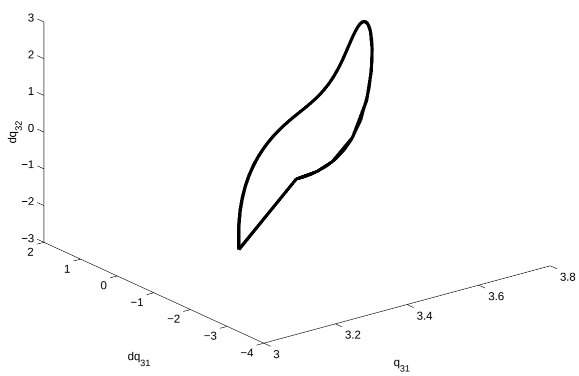

Fig. 12. Limit cycle

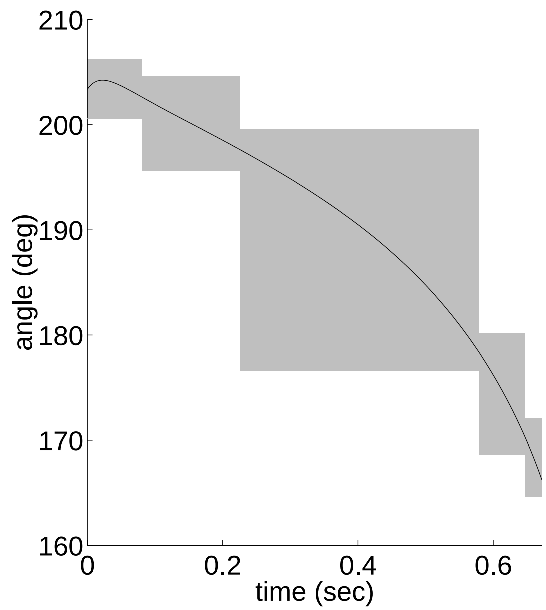

Fig. 13. Projection of the determined subsets $\mathcal{O}_i$ onto the $\bar{q}_{31}$ trajectory. Note that $\bar{q}_{31}(t)$ lies strictly within the union of the interiors of the subsets. The same is true of each of the other configuration variables.