PROOF OF MAIN RESULT

The goal is to find a Lyapunov function V on S for the Poincaré map P^{ε} defined (locally) on the section S associated with the periodic orbit $\mathscr{O}$. I.e., we seek a Lyapunov function for the discrete-time system $(x_{k+1}, z_{k+1}) = P^{\varepsilon}(x_k, z_k)$ with fixed point $(0, z^*)$. (Here $x^* = 0$ because $\mathscr{O} = \iota_0(\mathscr{O}_Z)$.) Since $\mathscr{O}_Z$ is a periodic orbit transverse to $S \cap Z$, we can view $S \cap Z$ as the Poincaré section and consider the associated restricted Poincaré map $\rho : S \cap Z \to S \cap Z$ with $\rho(z^*) = z^*$; without loss of generality, and for notational simplicity, assume that $z^* = 0$. Before proving Theorem 1, we first state and prove a lemma establishing a bound of the Poincaré map in terms of the restricted Poincaré map and a bound on the time-to-impact function $T_I^{\varepsilon}(x,z)$ in terms of $T_\rho(z)$. In the following, $B_\delta(r)$ denotes an open ball of radius $\delta > 0$ centered on the point r, and $P_z^{\varepsilon}(x,z)$ is the z-component of $P^{\varepsilon}(x,z)$.

Lemma 1: *Let $\mathscr{O}_Z$ be a periodic orbit of the hybrid zero dynamics $\mathscr{H}|_Z$ transverse to $S \cap Z$ and assume there exists a RES-CLF V_ε for the continuous dynamics* (29) *of $\mathscr{H}\mathscr{C}$. Then there exist finite constants L_{T_I} and A_1 (both independent of ε) such that for all $0 < \varepsilon < 1$ and for all Lipschitz continuous $u_\varepsilon(x,z) \in K_\varepsilon(x,z)$ there exists a $\delta > 0$ such that for all $(x,z) \in B_\delta(0,0) \cap S$,*

$$\|T_I^{\varepsilon}(x,z) - T_\rho(z)\| \le L_{T_I}\|x\|, \tag{44}$$

$$\|P_z^{\varepsilon}(x,z) - \rho(z)\| \le A_1\|x\|. \tag{45}$$

Proof: In the first step of the proof, we construct an auxiliary time-to-impact function T_B that is Lipschitz continuous and independent of ε and then relate it to T_I^{ε}.

Recall that $h(x,z)$ is the guard. Let $\mu_1 \in \mathbb{R}^{n_x}$ and $\mu_2 \in \mathbb{R}^{n_z}$ be constant vectors and let $\phi_t^z(\Delta(0,z_0))$ be the solution of $\dot{z} = q(0,z)$ with $z(0) = \Delta_Z(0,z_0)$. Define

$$T_B(\mu_1,\mu_2,z) = \inf\{t \ge 0 : h(\mu_1, \phi_t^z(\Delta(0,z)) + \mu_2) = 0\},$$

wherein it follows that $T_B(0,0,z) = T_\rho(z)$. By construction, T_B is independent of ε and (by the same argument used for $T_I^{\varepsilon}(x,z)$) is Lipschitz continuous. Hence, in the norm $\|(\mu_1,\mu_2,z)\| := \|\mu_1\| + \|\mu_2\| + \|z\|$,

$$|T_B(\mu_1,\mu_2,z) - T_\rho(z)| \le L_B\left(\|\mu_1\| + \|\mu_2\|\right), \tag{46}$$

where L_B is the (local) Lipschitz constant.

Let $\varepsilon > 0$ be fixed and select a Lipschitz continuous feedback $u_\varepsilon \in K_\varepsilon$. We note that $T_I^{\varepsilon}(x,z)$ is continuous (since it is Lipschitz) and therefore there exists $\delta > 0$ such that for all $(x,z) \in B_\delta(0,0) \cap S$

$$0.9T^* \le T_I^{\varepsilon}(x,z) \le 1.1T^*, \tag{47}$$

where $T^* = T_\rho(0)$ is the period of the orbit $\mathscr{O}_Z$. Let $(x_1(t), z_1(t))$ satisfy $\dot{z}_1(t) = q(x_1(t), z_1(t))$ with $x_1(0) = \Delta_X(x,z)$ and $z_1(0) = \Delta_Z(x,z)$, and similarly, let $z_2(t)$ satisfy $\dot{z}_2(t) = q(0, z_2(t))$ with $z_2(0) = \Delta_Z(0,z)$.

Defining

$$\begin{aligned}\mu_1 &= x_1(t)\big|_{t=T_I^{\varepsilon}(x,z)} \\ \mu_2 &= z_1(t)\big|_{t=T_I^{\varepsilon}(x,z)} - z_2(t)\big|_{t=T_I^{\varepsilon}(x,z)},\end{aligned} \tag{48}$$

results in

$$T_B(\mu_1,\mu_2,z) = T_I^{\varepsilon}(x,z) \tag{49}$$

because T_I^{ε} and T_B are locally unique solutions where the guard vanishes (follows from Implicit Function Theorem). We will establish (44) by bounding μ_1 and μ_2 and substituting into (46) by virtue of (49), as follows.

Using the fact that V_ε is rapidly exponentially stabilizing, we have the bound from (35) given by

$$\|x_1(t)\| \le \sqrt{\frac{c_2}{c_1}}\frac{1}{\varepsilon}e^{-\frac{c_3}{2\varepsilon}t}\|x_1(0)\|. \tag{50}$$

Note that $\Delta_X(0,z) = 0$ and therefore $\|x_1(0)\| = \|\Delta_X(x,z) - \Delta_X(0,z)\| \le L_{\Delta_X}\|x\|$. Then making use of (47), we have

$$\begin{aligned}\|\mu_1\| &= \|x_1(t)\|_{t=T_I^{\varepsilon}(x,z)} \\ &\le \sqrt{\frac{c_2}{c_1}}\frac{1}{\varepsilon}e^{-\frac{c_3}{2\varepsilon}0.9T^*}L_{\Delta_X}\|x\| \\ &\le \frac{2e^{-1}}{0.9T^*c_3}\sqrt{\frac{c_2}{c_1}}L_{\Delta_X}\|x\|.\end{aligned}$$

The next step is to bound $\|\mu_2\|$ using a Gronwall-Bellman argument. We first note that

$$z_1(t) - z_2(t) = z_1(0) - z_2(0) + \int_0^t q(x_1(\tau), z_1(\tau)) - q(0, z_2(\tau))d\tau$$

and thus

$$\begin{aligned}\|z_1(t) - z_2(t)\| &\le L_{\Delta_Z}\|x\| + \int_0^t L_q\left(\|x_1(\tau)\| + \|z_1(\tau) - z_2(\tau)\|\right)d\tau \\ &\le L_{\Delta_Z}\|x\| + \frac{2}{c_3}\sqrt{\frac{c_2}{c_1}}L_qL_{\Delta_X}\|x\| \\ &\quad + \int_0^t L_q\left(\|z_1(\tau) - z_2(\tau)\|\right)d\tau,\end{aligned}$$

where (50) has been substituted, integrated, and bounded. Hence, by the Gronwall-Bellman inequality,

$$\|z_1(t) - z_2(t)\| \le \left(L_{\Delta_Z} + \frac{2}{c_3}\sqrt{\frac{c_2}{c_1}}L_qL_{\Delta_X}\right)\|x\|e^{L_qt}, \tag{51}$$

and therefore $\|\mu_2\| \le C_1e^{1.1L_qT^*}\|x\|$, where C_1 is the term in parentheses in (51). The proof of (44) is then completed by substituting the bounds for $\|\mu_1\|$ and $\|\mu_2\|$ into (46) and grouping terms.

To establish (45), we first define

$$C_2 = \max_{.9T^* \le t \le 1.1T^*}\|q(0, z_2(t))\|.$$

It then follows from (44), (47) and (51) that

$$\begin{aligned}&\|P_z^{\varepsilon}(x,z) - \rho(z)\| \\ &\quad\le \|z_1(0) - z_2(0)\| \\ &\qquad + \int_0^{T_I^{\varepsilon}(x,z)}\|q(x_1(\tau), z_1(\tau)) - q(0, z_2(\tau))\|d\tau \\ &\qquad + \left|\int_{T_I^{\varepsilon}(x,z)}^{T_\rho(z)}\|q(0, z_2(\tau))\|d\tau\right| \\ &\quad\le \left(C_1e^{1.1L_qT^*} + C_2L_{T_I}\right)\|x\|,\end{aligned}$$

which establishes (45). ∎

We now have the necessary framework in which to prove Theorem 1.

Proof: [of Theorem 1] The results of Lemma 1 and the exponential stability of $\mathscr{O}_Z$ imply that there exists a $\delta > 0$ such that $\rho : B_\delta(0) \cap (S \cap Z) \to B_\delta(0) \cap (S \cap Z) \to$ is well-defined for all $z \in B_\delta(0) \cap (S \cap Z)$ and $z_{k+1} = \rho(z_k)$ is (locally) exponentially stable, i.e., $\|z_k\| \leq N\alpha^k \|z_0\|$ for some $N > 0$, $0 < \alpha < 1$ and all $k \geq 0$. Therefore, by the converse Lyapunov theorem for discrete-time systems, there exists a Lyapunov function V_ρ, defined on $B_\delta(0) \cap (S \cap Z)$ for some $\delta > 0$ (possibly smaller than the previously defined δ), and positive constants r_1, r_2, r_3, r_4 satisfying

$$\begin{aligned} r_1\|z\|^2 \leq V_\rho(z) \leq r_2\|z\|^2, \\ V_\rho(\rho(z)) - V_\rho(z) \leq -r_3\|z\|^2, \\ |V_\rho(z) - V_\rho(z')| \leq r_4\|z - z'\|(\|z\| + \|z'\|). \end{aligned} \tag{52}$$

For the RES-CLF V_ε, denote its restriction to the switching surface S by $V_{\varepsilon,X} = V_\varepsilon|_S$. With these two Lyapunov functions (motivated by the construction from [16] for singularly perturbed systems) we define the following candidate Lyapunov function

$$\bar{V}_\varepsilon(x,z) = V_\rho(z) + \sigma V_{\varepsilon,X}(x)$$

defined on $B_\delta(0,0) \subset S$, where $\sigma > 0$ is any constant such that $\sigma > \overline{\sigma} > 0$. (We will define $\overline{\sigma}$ explicitly later.) By (32) and (52), it is clear that

$$\min\{\sigma c_1, r_1\}\|(x,z)\|^2 \leq \bar{V}_\varepsilon(x,z) \leq \max\{\sigma \frac{c_2}{\varepsilon^2}, r_2\}\|(x,z)\|^2.$$

Noting that $\|(x,z)\|^2 = \|x\|^2 + \|z\|^2 + 2\|x\|\|z\| \geq \|x\|^2 + \|z\|^2$, we therefore need to establish that

$$\bar{V}_\varepsilon(P^\varepsilon(x,z)) - \bar{V}_\varepsilon(x,z) \leq -\kappa(\|x\|^2 + \|z\|^2), \tag{53}$$

for some $\kappa > 0$. Since $P^\varepsilon(x,z) \in S \subset X \times Z$, denote the X and Z components of P^ε by $P_x^\varepsilon(x,z)$ and $P_z^\varepsilon(x,z)$, respectively. With this notation,

$$\begin{aligned} \bar{V}_\varepsilon\big(P^\varepsilon(x,z)\big) - \bar{V}_\varepsilon(x,z) = V_\rho\big(P_z^\varepsilon(x,z)\big) - V_\rho(z) \\ + \sigma\big(V_{\varepsilon,X}(P_x^\varepsilon(x,z)) - V_{\varepsilon,X}(x)\big). \end{aligned} \tag{54}$$

We begin by noting that, because V_ε is a RES-CLF and $u(x,z) \in K_\varepsilon(x,z)$, and since $P_x^\varepsilon(x,z) = \big(\phi^\varepsilon_{T_I^\varepsilon(x,z)}(\Delta(x,z))\big)_x$, it follows that

$$\begin{aligned} V_{\varepsilon,X}(P_x^\varepsilon(x,z)) &\leq \frac{c_2}{\varepsilon^2} e^{-\frac{c_3}{\varepsilon} T_I^\varepsilon(x,z)} \|\Delta_X(x,z)\|^2 \\ &\leq \frac{c_2}{\varepsilon^2} L_{\Delta_X}^2 e^{-\frac{c_3}{\varepsilon} T_I^\varepsilon(x,z)} \|x\|^2, \end{aligned} \tag{55}$$

where the last inequality follows from the fact that $\Delta_X(0,z) = 0$ and therefore

$$\|\Delta_X(x,z)\|^2 = \|\Delta_X(x,z) - \Delta_X(0,z)\|^2 \leq L_{\Delta_X}^2 \|x\|^2,$$

with L_{Δ_X} the Lipschitz constant for Δ_X. Defining $\beta_1(\varepsilon) = \frac{c_2}{\varepsilon^2} L_{\Delta_X}^2 e^{-\frac{c_3}{\varepsilon} .9T^*}$ (with T^* defined as in the proof of Lemma 1), we have established that

$$\sigma\big(V_{\varepsilon,X}(P_x^\varepsilon(x,z)) - V_{\varepsilon,X}(x)\big) \leq \sigma(\beta_1(\varepsilon) - c_1)\|x\|^2, \tag{56}$$

where, clearly, $\beta_1(0^+) := \lim_{\varepsilon \searrow 0} \beta_1(\varepsilon) = 0$.

As a result of Lemma 1 and the assumption that the origin is an exponentially stable equilibrium for $z_{k+1} = \rho(z_k)$, we have the following inequalities:

$$\begin{aligned} &\|P_z^\varepsilon(x,z) - \rho(z)\| \leq A_1\|x\|, \\ &\|P_z^\varepsilon(x,z)\| = \|P_z^\varepsilon(x,z) - \rho(z) + \rho(z) - \rho(0)\| \leq A_1\|x\| + L_\rho\|z\|, \\ &\|\rho(z)\| \leq N\alpha\|z\|, \end{aligned}$$

where L_ρ is the Lipschitz constant for ρ. Thus, using (52),

$$V_\rho(P_z^\varepsilon(x,z)) - V_\rho(\rho(z)) \leq r_4 A_1^2 \|x\|^2 + r_4 A_1(L_\rho + N\alpha)\|x\|\|z\|.$$

Setting $\beta_2 = r_4 A_1^2$ and $\beta_3 = r_4 A_1(L_\rho + N\alpha)$ for notational simplicity, it follows that

$$\begin{aligned} &V_\rho(P_z^\varepsilon(x,z)) - V_\rho(z) \\ &\quad = V_\rho(P_z^\varepsilon(x,z)) - V_\rho(\rho(z)) + V_\rho(\rho(z)) - V_\rho(z) \\ &\quad \leq \beta_2\|x\|^2 + \beta_3\|x\|\|z\| - r_3\|z\|^2. \end{aligned} \tag{57}$$

Therefore, combining (54), (56), and (57), we have

$$\begin{aligned} \bar{V}_\varepsilon(P^\varepsilon(x,z)) - \bar{V}_\varepsilon(x,z) &\leq (\beta_2 + \sigma(\beta_1(\varepsilon) - c_1))\|x\|^2 \\ &\quad + \beta_3\|x\|\|z\| - r_3\|z\|^2 \\ &= -\begin{bmatrix} \|z\| & \|x\| \end{bmatrix} \Lambda(\varepsilon) \begin{bmatrix} \|z\| \\ \|x\| \end{bmatrix}, \end{aligned}$$

with

$$\Lambda(\varepsilon) = \begin{bmatrix} r_3 & -\frac{1}{2}\beta_3 \\ -\frac{1}{2}\beta_3 & \sigma(c_1 - \beta_1(\varepsilon)) - \beta_2 \end{bmatrix}.$$

Therefore, the goal is to find $\sigma > 0$ such that for $\varepsilon > 0$ sufficiently small, $\Lambda(\varepsilon)$ is positive definite or, more specifically, $\det(\Lambda(\varepsilon)) > 0$. With this in mind, consider

$$\lim_{\varepsilon \searrow 0} \det(\Lambda(\varepsilon)) = -\frac{\beta_3^2}{4} - \beta_2 r_3 + \sigma c_1 r_3.$$

Therefore, pick

$$\sigma > \frac{\beta_3^2 + 4\beta_2 r_3}{4 c_1 r_3} =: \overline{\sigma},$$

wherein by the continuity of $\Lambda(\varepsilon)$ with respect to ε, there exists an $\overline{\varepsilon} > 0$ such that for all $0 < \varepsilon < \overline{\varepsilon}$, $\det(\Lambda(\varepsilon)) > 0$. Therefore (53) is satisfied with $\kappa = \lambda_{\max}(\Lambda(\varepsilon))$, the largest eigenvalue of $\Lambda(\varepsilon)$, and we have established the local exponential stability of $\mathscr{O}$. ■