A stabilization result with application to bipedal locomotion

A.R. Teel, R. Goebel, B. Morris, A.D. Ames, J.W. Grizzle

Abstract— **For general hybrid systems, we develop new stabilization results that can be used to achieve asymptotically stable locomotion for bipedal robots with series compliant actuation. The stabilization contributions build upon previous results involving partially rapidly exponentially decaying control Lyapunov functions. Such functions are useful when the dynamics that remain when the function is constrained to zero exhibits an asymptotically stable set and the solutions starting in this set have time domains that satisfy a uniform average dwell-time constraint. In a new result of independent interest, we establish that such an average dwell-time condition is robust; in particular, it degrades gracefully under perturbations and as the initial conditions move away from the asymptotically stable set. From this robustness result and the existence of a partially rapidly exponentially decaying control Lyapunov function, we establish local asymptotic stabilization. The result is then applied to robot locomotion. We conclude by showing that, because of the high-gain nature of the feedback, it is possible in some situations for the basin of attraction to become arbitrarily small as the gain becomes arbitrarily large. Future simulation studies will investigate whether this phenomenon occurs for the robot application.**

I. Introduction

This paper contributes new results on stabilization of hybrid systems through feedback controls in the flow map; as an application, it considers stabilization of a periodic gait for a bipedal robot with series compliant actuation. In the situation where controls appear only in the flow map, some type of average dwell-time constraint [12] on solutions is required to give the control signals suitable time to influence trajectories. Recent examples of hybrid stabilization in this context include the results in [22] and [7]. Our contribution has significant connections to the results in [22]. In the latter, a uniform average dwell-time condition is assumed *throughout* the state space and control is achieved by regulating, via high-gain output feedback, a *relative degree one* output with respect to which the hybrid system is *globally* minimum phase. A semi-global asymptotic stabilization result is achieved. In the current paper, outputs with *higher relative degree* are allowed and we assume only a *local* minimum-phase property. Moreover, the average dwell-time condition is imposed only on the attractor of the zero dynamics. With these relaxed assumptions, we achieve only local stabilization rather than semi-global stabilization. This result is expected in the absence of additional structural assumptions, especially in light of classical results on obstructions to semi-global stabilization for non-hybrid systems [21]; see also the example in Section V of this paper.

Among the many formalisms for posing and analyzing hybrid systems, we focus on two. Systems with impulsive effects, introduced in [2] and expanded in [26], [11], [17], [13] and [19], have been the basis for much of the work in the legged robotics literature. This formalism constructs solutions to hybrid systems by piecing together nontrivial solutions of ordinary differential equations, with the final value of one solution related to the initial condition of a subsequent solution by a transition map. Hybrid models permitting multiple unilateral contacts, by virtue of allowing more general solutions, have been studied in [3], [5], and [4], with potential applications to bipedal robots surveyed in [14]. This more general framework comes with more advanced stability analysis tools, but so far, control design tools have not kept pace. The other formalism we consider is the hybrid inclusions of [10], [8], [9]. An objective of this paper is to show how a Lyapunov-based control design method originally developed for systems with impulsive effects can be carried over to hybrid inclusions, where insights about robustness can be exploited to generate stronger guarantees on the properties of the closed-loop solutions.

Like in [22], our result pertains to the situation where the trajectories of the zero dynamics converge to a set rather than an equilibrium point. Periodic motion is ubiquitous and represents the next level of attractor complexity beyond equilibrium points. Arguably the quintessential example of periodic motion in hybrid systems is steady-state bipedal walking or running. The approach for bipedal locomotion developed in [24], [23] designs an output function satisfying a vector relative degree condition, a periodic orbit that lies in the zero set of the output function, and moreover, the periodic orbit is an exponentially stable solution of the hybrid zero dynamics associated with the output. Prior to the introduction of control Lyapunov functions to the hybrid setting in [1], the only results on stabilizing the periodic orbit in the full model (not just the invariant set given by the hybrid zero dynamics), was input-output linearization [16]. An important limitation of [1] is the restriction to outputs of vector relative degrees one or two, while the result of [16]

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applied more generally, but once again, only for input-output linearizing feedbacks. Here, we extend the definition of rapidly exponentially decaying control Lyapunov functions from [1] to remove the relative degree restriction of [1] and the input-output linearizing feedback requirement from [16].

The extensions achieved for bipedal locomotion are further enabled by casting the dynamic model in the hybrid systems framework of [10]. Within this framework, we can rely on a wealth of analysis machinery that has been developed (see [8], [9]) as well as a new result on robustness of average dwell-time constraints provided in Section III-A. Robustness of the average dwell-time condition is useful for passing from the natural average dwell-time condition that is consistent with a periodic gait to an average dwell-time condition close to but not exactly on the periodic gait.

The paper is organized as follows. In Section II, we review a modeling framework for hybrid systems, and define asymptotic stability and the average dwell-time condition. In Section III-A, we provide a new result on robustness of the average dwell-time condition. In Section III-B, we give a stability result for a parametrized class of hybrid systems. The result is expressed in terms of an average dwell-time condition and a rapidly exponentially decaying Lyapunov function where the decay rate can be tuned through the parameter that characterizes the class of systems. The result depends heavily on the result of Section III-A. Section III-C exploits the results of Section III-B to generate a "high-gain" stabilization result for a class of hybrid systems. This stabilization result is applied to bipedal locomotion for robots with series compliant actuation in Section IV. Finally, in Section V, we use an example to demonstrate that the basin of attraction may possibly shrink with the decay rate used in the stabilization algorithm. All proofs of theorems have been omitted due to space constraints.

II. Hybrid systems review

Let $O \subset \mathbb{R}^n$ be open. A hybrid system has state $x \in O$ and data (C, F, D, G) where $C, D \subset O$, and $F : O \rightrightarrows \mathbb{R}^n$, $G : O \rightrightarrows O$ are set-valued mappings. This data satisfies the *basic conditions* if C and D are closed relative to O, F and G are outer semicontinuous and locally bounded (see [18, Chapter 5]), $F(x)$ is nonempty and convex for each $x \in C$, and $G(x)$ is nonempty for each $x \in D$. Formally, the hybrid system (C, F, D, G) is written

$$x \in C \qquad \dot{x} \in F(x) \tag{1a}$$

$$x \in D \qquad x^+ \in G(x). \tag{1b}$$

A solution to this hybrid system is a hybrid arc[1] $\phi : \mathrm{dom}\ \phi \to O$ such that

1) if $(t_1, j), (t_2, j) \in \mathrm{dom}\ \phi$ with $t_2 > t_1$ then $\phi(t, j) \in C$ and $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in [t_1, t_2]$;
2) if $(t, j), (t, j+1) \in \mathrm{dom}\ \phi$ then $\phi(t, j) \in D$ and $\phi(t, j+1) \in G(\phi(t, j))$.

For more details, see [10]. Given $x \in C \cup D$, the set of solutions starting from x is denoted $\mathcal{S}(x)$.

In the case where (C, D, G) satisfy the basic conditions and $f : C \to \mathbb{R}^n$ is a locally bounded function, *generalized solutions* are the solutions to (C, F, D, G) where F is the smallest set-valued mapping such that $f(x) \in F(x)$ for all $x \in C$ and such that (C, F, D, G) satisfies the basic conditions. In particular, F is the convex hull mapping applied to the outer semicontinuous hull mapping of f, the latter being the mapping whose graph is the closure of the graph of f; see [18, p. 154-155]. Equivalently, $F(x) := \mathrm{co} \bigcap_{\delta > 0} \overline{f((x + \delta\mathbb{B}) \cap C)}$, where $x + \delta\mathbb{B}$ is the closed ball of radius $\delta > 0$ centered at x. For more details, see [20]. At times we also consider solutions of a system with perturbed data $(C_\delta, F_\delta, D_\delta, G_\delta)$, $\delta > 0$, derived from (C, F, D, G) as

$$C_\delta := \{x \in O : (x + \delta\mathbb{B}) \cap C \neq \varnothing\} \tag{2a}$$

$$F_\delta(x) := \mathrm{co}F((x + \delta\mathbb{B}) \cap C) + \delta\mathbb{B} \tag{2b}$$

$$D_\delta := \{x \in O : (x + \delta\mathbb{B}) \cap D \neq \varnothing\} \tag{2c}$$

$$G_\delta(x) := (G((x + \delta\mathbb{B}) \cap D) + \delta\mathbb{B}) \cap O. \tag{2d}$$

A compact set $\mathcal{A} \subset O$ is said to be *stable* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $x \in \mathcal{A} + \delta\mathbb{B}$ and $\phi \in \mathcal{S}(x)$, $\phi(t, j) \in \mathcal{A} + \varepsilon\mathbb{B}$ for all $(t, j) \in \mathrm{dom}\ \phi$. A compact set $\mathcal{A}$ is *attractive* if there exists $\mu > 0$ such that, for each $x \in \mathcal{A} + \mu\mathbb{B}$, each $\phi \in \mathcal{S}(x)$ is bounded and if $\mathrm{dom}\ \phi$ unbounded then $\lim_{t+j\to\infty} |\phi(t, j)|_{\mathcal{A}} = 0$. A compact set is said to be *asymptotically stable* if it is stable and attractive. The *basin of attraction* for an asymptotically stable compact set is the set of points $x \in O$ such that each $\phi \in \mathcal{S}(x)$ is bounded and if $\mathrm{dom}\ \phi$ is unbounded then $\lim_{t+j\to\infty} |\phi(t, j)|_{\mathcal{A}} = 0$. When (C, F, D, G) satisfies the basic conditions, the basin of attraction of a compact set $\mathcal{A}$ contains $\mathcal{A}$ and is an open set. See [6, Theorem 3.14]. Let (C, F, D, G) satisfy the basic conditions, let $\mathcal{B}_\mathcal{A}$ denote the basin of attraction for an asymptotically stable compact set $\mathcal{A}$, and let $\omega : \mathcal{B}_\mathcal{A} \to \mathbb{R}_{\geq 0}$ be a continuous function that is zero on $\mathcal{A}$, positive outside of $\mathcal{A}$, and that grows unbounded as its argument approaches the boundary of $\mathcal{B}_\mathcal{A}$ or as its argument grows unbounded. Then there exists $\beta \in \mathcal{KL}$ such that, for all $x \in \mathcal{B}_\mathcal{A}$ and all $\phi \in \mathcal{S}(x)$, $\omega(\phi(t, j)) \leq \beta(\omega(x), t + j)$ for all $(t, j) \in \mathrm{dom}\ \phi$. For more details see [6, Proposition 7.3].

Let $\rho > 0$ and $N \in \mathbb{Z}_{\geq 1}$. The time domain of a hybrid arc ϕ is said to satisfy the (ρ, N) *average dwell-time condition* if every $(s, i), (t, j) \in \mathrm{dom}\ \phi$ with $t + j > s + i$ satisfies

$$j - i \leq \rho \cdot (t - s) + N.$$

Such a property was introduced in [12] and has been used extensively in the switched and hybrid systems literature.

III. Main results

A. Robustness of average dwell-time condition

We use a preliminary result, which is novel and of independent interest. It asserts that an average dwell-time

[1]That is, a function ϕ defined on a hybrid time domain and such that $t \mapsto \phi(t, j)$ is locally absolutely continuous for each $j \in \mathbb{Z}_{\geq 0}$. A compact hybrid time domain is a set of the form $\cup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$ for some real numbers $0 = t_0 \leq t_1 \leq \ldots \leq t_J$; a hybrid time domain is a set E such that, for each $(T, J) \in E$, $E \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain.

condition (on the time domains of solutions that remain in a compact set) degrades gracefully under perturbations.

Proposition 1: Suppose (C, F, D, G) satisfies the basic conditions, $K \subset O$ is compact, and $\rho > 0$ and $N \in \mathbb{Z}_{\geq 1}$ are such that the time domain of every solution ϕ of (C, F, D, G) with rge $\phi \subset K$ satisfies the (ρ, N) average dwell-time condition. Then, for each $\widehat{\rho} > \rho$, there exists $\delta > 0$ such that the time domain of every solution ϕ to $(C_\delta, F_\delta, D_\delta, G_\delta)$ with rge $\phi \subset K + \delta\mathbb{B}$ satisfies the $(\widehat{\rho}, N)$ average dwell-time condition. ■

B. Main stability result

We study the behavior of a family of hybrid systems with data (C, F^ε, D, G) parametrized by $\varepsilon > 0$.

Assumption 1: For each $\varepsilon > 0$, (C, F^ε, D, G) satisfies the basic conditions. ■

We are interested in additional conditions to guarantee local asymptotic stability of a compact set $\mathcal{A} \subset O$ that has a particular structure clarified in the next assumption.

Assumption 2: (Structure and coupling conditions) The following conditions hold.

1) $n_0, n_1 \in \mathbb{Z}_{\geq 1}$, $n_0 + n_1 = n$, $\mathcal{A}_0 \subset \mathbb{R}^{n_0}$ is compact, $\mathcal{A} := \mathcal{A}_0 \times \{0\} \subset O$, $\mathcal{R}_0 := \mathbb{R}^{n_0} \times \{0\}$, and there exist $L > 0$ and $\delta > 0$ such that
$$\left.\begin{array}{r} x \in D \cap (\mathcal{A} + \delta\mathbb{B}) \\ g \in G(x) \end{array}\right\} \implies |g|_{\mathcal{R}_0} \leq L|x|_{\mathcal{R}_0}.$$
2) There exist set-valued mappings $\widetilde{F}_0 : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n_0}$ and $F_1^\varepsilon : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n_1}$ such that $F^\varepsilon(x) = \widetilde{F}_0(x) \times F_1^\varepsilon(x)$ for all $x \in C \cap (\mathcal{A} + \delta\mathbb{B})$ and there exists $M > 0$ and for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for all $x \in C \cap (\mathcal{A} + \delta_\varepsilon\mathbb{B})$, $F_1^\varepsilon(x) \subset M\mathbb{B}$. ■

Based on the assumed structure of the set $\mathcal{A}$, we construct a "hybrid zero dynamics" and impose an asymptotic stability assumption on these dynamics. Recall the definition of $\mathcal{R}_0$ in Assumption 2, define $\Pi_0(x_0, x_1) := x_0$,

$$C_0 := \Pi_0 (C \cap \mathcal{R}_0) \tag{3a}$$
$$F_0(x_0) := \Pi_0(F^\varepsilon(x_0, 0)) = \widetilde{F}_0((x_0, 0)) \tag{3b}$$
$$D_0 := \Pi_0 (D \cap \mathcal{R}_0) \tag{3c}$$
$$G_0(x_0) := \Pi_0(G(x_0, 0)), \tag{3d}$$

and impose the following assumption.

Assumption 3: The compact set $\mathcal{A}_0$ of Assumption 2 is locally asymptotically stable for (C_0, F_0, D_0, G_0) defined in (3), and $\rho > 0$ and $N \in \mathbb{Z}_{\geq 1}$ are such that the time domain of each solution ϕ of (C_0, F_0, D_0, G_0) that starts in $\mathcal{A}_0$ satisfies the (ρ, N) average dwell-time condition. ■

Finally, we impose a Lyapunov condition on the flow data (C, F^ε), corresponding to what results from a partially rapidly exponentially stabilizing control Lyapunov function as developed in the next subsection.

Assumption 4: There exist strictly positive real numbers p, c_1, c_2, c_3 and, for each $\varepsilon > 0$, a continuously differentiable function $V_\varepsilon : O \to \mathbb{R}$ and $\delta_\varepsilon > 0$ such that, for all $x \in (C \cup D) \cap (\mathcal{A} + \delta_\varepsilon\mathbb{B})$,
$$c_1|x|^2_{\mathcal{R}_0} \leq V_\varepsilon(x) \leq \frac{c_2}{\varepsilon^p}|x|^2_{\mathcal{R}_0}$$
(where $\mathcal{R}_0$ was defined in Assumption 2) and, for all $x \in C \cap (\mathcal{A} + \delta_\varepsilon\mathbb{B})$ and $f \in F^\varepsilon(x)$,
$$\langle \nabla V_\varepsilon(x), f\rangle \leq -\frac{c_3}{\varepsilon}V_\varepsilon(x).$$
■

Our main stability result is stated next.

Theorem 1: If Assumptions 1-4 hold then the compact set $\mathcal{A}$ defined in Assumption 2 is locally asymptotically stable for the hybrid system (C, F^ε, D, G) for each sufficiently small $\varepsilon > 0$, in particular, for each $\varepsilon > 0$ satisfying
$$\exp\left(-\frac{c_3}{\varepsilon\rho}\right)\frac{1}{\varepsilon^p}\frac{c_2 L^2}{c_1} < 1 \tag{4}$$
with $L > 0$ from Assumption 2, $\rho > 0$ from Assumption 3, and p, c_1, c_2, and c_3 from Assumption 4. ■

C. Partially rapidly exponentially stabilizing control Lyapunov functions

Consider a control system with state $x := (x_0, x_1) \in \mathbb{R}^{n_0} \times O_1$, where $O_1 \subset \mathbb{R}^{n_1}$ is open, and control $u \in U \subset \mathbb{R}^m$ of the form
$$x \in C \quad \begin{cases} \dot{x}_0 \in \widetilde{F}_0(x) \\ \dot{x}_1 = f_1(x, u) \end{cases} \tag{5a}$$
$$x \in D \qquad x^+ \in G(x) \tag{5b}$$
under the following assumption.

Assumption 5: The following conditions hold:

1) $(C, \widetilde{F}_0, D, G)$ satisfy the basic conditions, U is closed, and $f_1 : C \times U \to \mathbb{R}^{n_1}$ is continuous.
2) Condition 1) of Assumption 2 holds.
3) Assumption 3 holds. ■

In order to guarantee condition 2) of Assumption 2 and Assumption 4, we adapt the definition of a rapidly exponentially stabilizing control Lyapunov function given in [1].

Definition 1: For the system (5), a one-parameter family of continuously differentiable functions $V_\varepsilon : O_1 \to \mathbb{R}$ is said to be a *partially, rapidly exponentially stabilizing control Lyapunov function (PRES-CLF) with the locally bounded control property* if there exist positive real numbers $c_1, c_2, c_3, p, \widehat{M}$ and for each $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, 1]$ such that for all $x = (x_0, x_1) \in (C \cup D) \cap (\mathcal{A} + \delta_\varepsilon\mathbb{B})$,
$$c_1|x_1|^2 \leq V_\varepsilon(x_1) \leq \frac{c_2}{\varepsilon^p}|x_1|^2$$
and, for all $x = (x_0, x_1) \in C \cap (\mathcal{A} + \delta_\varepsilon\mathbb{B})$,
$$\inf_{u \in U \cap \widehat{M}\mathbb{B}} \left[\langle \nabla V_\varepsilon(x_1), f_1(x, u)\rangle + \frac{c_3}{\varepsilon}V_\varepsilon(x_1)\right] \leq 0. \tag{6}$$
■

This definition above can be extended to allow V_ε to depend on x_0; such an extension is not needed for our application. Consider the set-valued mapping $K_\varepsilon : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined as
$$K_\varepsilon(x) := \left\{u \in U \cap \widehat{M}\mathbb{B} : \langle \nabla V_\varepsilon(x_1), f_1(x, u)\rangle + \frac{c_3}{\varepsilon}V_\varepsilon(x_1) \leq 0\right\}. \tag{7}$$

Theorem 2: Under Assumption 5, if the family of continuously differentiable functions V_ε is a PRES-CLF with the locally bounded control property then, considering generalized solutions generated by any control selection $u_\varepsilon(x) \in K_\varepsilon(x)$, the set $\mathcal{A}$ is locally asymptotically stable. ■

IV. Application to Biped Locomotion

In the analysis of biped locomotion we naturally encounter robotic hardware and feedback controllers that are appropriately modeled as hybrid control systems (5). For the purpose of illustration, we apply the main stability result of Theorem 1 to the planar bipedal robot with series compliant actuation shown in Figure 1 and studied previously in [16, Sect. V] with a different controller. The dynamic model is based on adding series compliance to the actuation of the robot Rabbit; see [23, Table 6.3] or [25, Table 1] for complete details on the dynamic model, including parameter values.

A. Dynamic Model of Stance

For $q_a := (q_1, q_2, q_3, q_4)$ and $q_m := (q_{m_1}, q_{m_2}, q_{m_3}, q_{m_4})$, let $q := (q_a, \theta)$ be the vector of generalized coordinates that determine the posture of the robot. We define the configuration vector as $(q, q_m) \in \mathcal{Q}$, where $\mathcal{Q}$ is the (closed) subset of $\mathbb{R}^9$ such that the swing 'point' foot has a vertical height ($\mathbf{y_2}$ in Figure 1) greater than or equal to zero. Let $u \in U$ be a vector of control inputs, with U being a closed subset of $\mathbb{R}^4$. The swing phase dynamics of the robot can now be written as

$$\begin{array}{rcl} M(q)\ddot{q} + H(q,\dot{q}) - BK(q_m - q_a) & = & 0 \\ J\ddot{q}_m + K(q_m - q_a) & = & u. \end{array} \tag{8}$$

To design feedback controllers, define a vector of control parameters $\beta \in B$, where B is a closed subset of $\mathbb{R}^p$ on which a feedback law may depend. During the stance phase, the vector of parameters β is held constant: $\dot{\beta} = 0$. Thus, for the stance phase, the flow set is contained in the closed set

$$\widehat{C} := \mathcal{Q} \times \mathbb{R}^9 \times B \subset \mathbb{R}^{18+p}.$$

The gait design procedure discussed in Section IV-C below provides a set of parameterized output functions of the form

$$y = h(x) := q_a - h_d(\theta, \beta) \tag{9}$$

for $h_d : \mathbb{R} \times B \to \mathbb{R}^4$. Let $\mathcal{R}_0 \subset \mathbb{R}^{18+p}$ be the set where these outputs vanish, and let $O \subset \mathbb{R}^{18+p}$ be the open set on which the outputs uniform vector relative degree 4 with respect to the dynamics (8). At all points $x \in O$ we then have a change of coordinates[2] such that

$$\begin{array}{rl} x & := (x_0, x_1) \\ x_0 & := (\theta, \sigma, \beta) \\ x_1 & := (y, \dot{y}, \ddot{y}, y^{(3)}), \end{array} \tag{10}$$

where σ is the angular momentum of the linkage about the stance foot. Let the flow set be $C := \widehat{C} \cap O$, so that the stance phase dynamics becomes

$$x \in C \quad \begin{cases} \dot{x}_0 = \widetilde{F}_0(x) \\ \dot{x}_1 = f_1(x, u), \end{cases} \tag{11}$$

[2]See Section 3.4.5 of [23] for further discussion of this coordinate system.

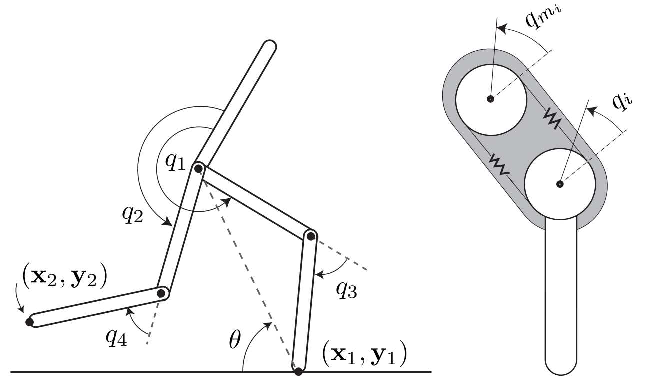

Fig. 1. **(Left)** A coordinate diagram of an example of the class of N-link biped robot models considered. **(Right)** A schematic of a rotational joint with series compliant actuation.

where

$$\widetilde{F}_0 = \begin{bmatrix} \dot{\theta} \\ \dot{\sigma} \\ 0 \end{bmatrix}, f_1(x,u) = \begin{bmatrix} L_f h(x) \\ L_f^2 h(x) \\ L_f^3 h(x) \\ L_f^4 h(x) + L_g L_f^3 h(x) u \end{bmatrix}. \tag{12}$$

B. Dynamic Model of Collisions

When a rigid link with nonzero velocity encounters the ground plane, the interaction is modeled as a fully plastic, instantaneous collision, as in [15]. The post-impact joint state $(q^+, \dot{q}^+, q_m^+, \dot{q}_m^+)$ can be calculated from the pre-impact joint state $(q, \dot{q}, q_m, \dot{q}_m)$ by the following equations

$$\begin{array}{ll} q^+ = R \circ q & q_m^+ = R_m \circ q_m \\ \dot{q}^+ = R \circ \Delta_{\dot{q}}(q, \dot{q}) & \dot{q}_m^+ = R_m \circ \dot{q}_m \end{array} \tag{13}$$

where R and R_m are appropriately defined joint relabeling operators. The impact dynamics (13) and stance dynamics (8) can together be written as a hybrid system,

$$x \in C \quad \begin{cases} \dot{x}_0 & = \widetilde{F}_0(x) \\ \dot{x}_1 & = f_1(x, u) \end{cases} \tag{14a}$$

$$x \in D \quad x^+ \in \widetilde{G}(x, v) \tag{14b}$$

For the open loop reset map $\widetilde{G}$, the parameter vector $\beta^+ = v$ for a reset input $v : D \to B$. All other elements of x^+ are found by applying a change of coordinates to the impact equations (13) and are thus independent of the reset input v. Let y_2 be the vertical clearance between the ground and tip of the swing leg. Let the jump set be $D := \widehat{D} \cap O$, where

$$\widehat{D} = \{x \in \widehat{C} : y_2(x) = 0, \dot{y}_2(x) \leq 0\}$$

For outputs of the form (9) and rigid impacts of the form (14), a discrete parameter update function $v : D \to B$ is typically needed to satisfy the impact invariance property

$$\forall x \in D \cap \mathcal{R}_0, \quad \widetilde{G}(x, v(x)) \in \mathcal{R}_0. \tag{15}$$

A continuously differentiable parameter update v satisfying this condition can be constructed using output augmentation polynomials described in [16]. Define the closed-loop reset map $G(x) := \widetilde{G}(x, v(x))$.

C. Remarks on Gait Design

As developed above, the full dynamic model of walking is a hybrid system with data (C, F, D, G). We are interested in stabilizing a periodic gait that involves one jump per period with ordinary time period $T^* > 0$ and that is contained in the set $\mathcal{R}_0$. Such a periodic gait is generated by a hybrid arc ϕ^* and stabilization corresponds to stabilizing the set compact $\mathcal{A} := \cup_{(t,j)\in \text{dom } \phi^*} \{\phi^*(t, j)\}$. Note that $\mathcal{A} = \mathcal{A}_0 \times \{0\}$. The solutions from this set are assumed to be unique and thus the time domain of every solution that starts in this set satisfies the average dwell-time condition with $\rho = 1/T^*$ and $N = 1$.

A periodic walking gait and an accompanying set of output functions (9) can be found using a variety of optimization procedures, such as those in Sections 3.6.2 and 6.5.1 of [23]. A stability constraint can be included in such an optimization procedure to ensure that any resulting periodic orbit will have an associated compact set $\mathcal{A}_0$ is asymptotically stable in the restricted hybrid system with data (C_0, F_0, D_0, G_0).

The next subsection outlines the construction of a parameterized family of PRES-CLFs, each defined in an open neighborhood of $\mathcal{A}$.

D. Control Lyapunov Functions

Choose constants k_0, k_1, k_2, k_3 such that the matrix

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -k_0 I & -k_1 I & -k_2 I & -k_3 I \end{bmatrix}. \quad (16)$$

is Hurwitz. Then, for any $Q = Q^T > 0$ there exists $P = P^T > 0$ satisfying $A^T P + PA = -Q$. By the Raleigh-Ritz inequality there exists some $\gamma > 0$ such that

$$A^T P + PA + \gamma P \leq 0. \quad (17)$$

For such a value of P and for $0 < \varepsilon < 1$, define a candidate control Lyapunov function

$$V_\varepsilon(x_1) := \eta_\varepsilon(x_1)^T P \eta_\varepsilon(x_1) \quad (18)$$

where

$$\eta_\varepsilon(x_1) := \text{col}\left[\left(\frac{1}{\varepsilon^3}\right) y, \left(\frac{1}{\varepsilon^2}\right) \dot{y}, \left(\frac{1}{\varepsilon}\right) \ddot{y}, y^{(3)}\right].$$

For each $x \in C$, where the coordinates (10) are well-defined, $c_1|x_1|^2 \leq V_\varepsilon(x_1) \leq \frac{c_2}{\varepsilon^p}|x_1|^2$, where $p = 6$, $c_1 = \lambda_{min}(P)$, and $c_2 = \lambda_{max}(P)$. For V_ε to be a PRES-CLF, the set-valued control mapping (7) must be non-empty at all points of interest. We prove this non-emptiness by construction.

Consider the feedback

$$\begin{aligned} u_\varepsilon(x) = & \; L_g L_f^3 h(x)^{-1} \left(-L_f^4 h(x) + v_\varepsilon(x)\right) \\ v_\varepsilon(x) = & \; -(k_0/\varepsilon^4) h(x) - (k_1/\varepsilon^3) L_f h(x) \ldots \\ & -(k_2/\varepsilon^2) L_f^2 h(x) - (k_3/\varepsilon) L_f^3 h(x), \end{aligned}$$

applied to the open loop subsystem (12). Due to the definition of $\mathcal{A}$ and (10), there exists $M > 0$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|v_\varepsilon(x)| \leq M$ for all $x \in \mathcal{A} + \delta\mathbb{B}$. The resulting closed loop dynamics $\dot{\eta}_\varepsilon(x_1) = \frac{1}{\varepsilon} A \eta_\varepsilon(x_1)$ can be substituted into the PRES-CLF condition (6),

$$\begin{aligned} &\langle \nabla V_\varepsilon(x_1), f_1(x, u)\rangle + \frac{c_3}{\varepsilon} V_\varepsilon(x_1) = \\ &\qquad \frac{1}{\varepsilon} \eta_\varepsilon^T(x_1)(A^T P + PA)\eta_\varepsilon(x_1) + \frac{c_3}{\varepsilon} V_\varepsilon(x). \end{aligned}$$

Applying (17) with $c_3 = \gamma$ shows that there exists $\widehat{M} > 0$ and for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that the set-valued control mapping (7) is non-empty on $\mathcal{A} + \delta_\varepsilon \mathbb{B}$ and that V_ε is a PRES-CLF on this open neighborhood of $\mathcal{A}$.

Apply Theorems 1 and 2 together to conclude that for the family of partially rapidly exponentially stabilizing control Lyapunov functions V_ε, for any $\varepsilon > 0$ satisfying

$$\exp\left(\frac{-\gamma T^*}{\varepsilon}\right) \frac{1}{\varepsilon^6} \left(\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\right) L^2 < 1 \quad (19)$$

the control selection $u_\varepsilon(x) \in K_\varepsilon(x)$ will render the compact set $\mathcal{A}$ locally asymptotically stable in the full hybrid system with data (C, F, D, G).

V. On the Basin of Attraction

The state component x_1 in the systems that we have analyzed typically exhibits the so-called "peaking phenomenon". See [21]. As long as Assumptions 2 and 4 hold even for large values of x_1 and the average dwell-time condition does not degrade significantly for large x_1, there exists a neighborhood of $\mathcal{A}$ that is contained in the basin of attraction of $\mathcal{A}$ for all sufficiently small $\varepsilon > 0$. Without such extra assumptions, it is possible to construct an example where the basin of attraction does not contain points arbitrarily close to $\mathcal{A}$ for $\varepsilon > 0$ arbitrarily small. For example, define

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

and consider the situation where $n_0 = 2$, $n_1 = 2$,

$$C := (\mathbb{R}_{\geq 0} \times \mathbb{R}) \times (\mathbb{R} \times (-\infty, 1]) \quad (20a)$$

$$F^\varepsilon(x) := \text{diag}(J, \text{diag}(1, \varepsilon^{-1}) \cdot A \cdot \text{diag}(\varepsilon^{-1}, 1))x \quad (20b)$$

$$\widehat{D}_0 := (\{0\} \times \mathbb{R}_{\leq 0}) \times \mathbb{R}^2 \quad (20c)$$

$$\widehat{D}_1 := (\mathbb{R}_{\geq 0} \times \mathbb{R}) \times (\mathbb{R} \times [1, \infty)) \quad (20d)$$

$$D := \widehat{D}_0 \cup \widehat{D}_1 \quad (20e)$$

$$\widehat{g}_0(x) := \text{diag}(0.5, 0.5, 2, 2)x + (0, 0.5, 0, 0)^T \quad (20f)$$

$$\widehat{g}_1(x) := 2x \quad (20g)$$

$$G(x) := \begin{cases} \widehat{g}_0(x) & x \in \widehat{D}_0 \backslash \widehat{D}_1 \\ \widehat{g}_1(x) & x \in \widehat{D}_1 \backslash \widehat{D}_0 \\ \widehat{g}_0(x) \cup \widehat{g}_1(x) & x \in \widehat{D}_0 \cap \widehat{D}_1. \end{cases} \quad (20h)$$

It is immediate that Assumption 1 holds. For any compact set $\mathcal{A}_0 \subset \mathbb{R}^2$, condition 1) of Assumption 2 holds with $L = 2$ and any $\delta > 0$, since $|g|_{\mathcal{R}_0} = 2|x|_{\mathcal{R}_0}$ for all $x \in D$ and $g \in G(x)$, and it is also easy to see that condition 2) of Assumption 2 holds with $\delta_\varepsilon = \varepsilon^2$ and some $M > 0$ independent of ε. The definitions in (3) give

$$C_0 = \mathbb{R}_{\geq 0} \times \mathbb{R}, \qquad F_0(x_0) = J x_0$$

$$D_0 = \{0\} \times \mathbb{R}_{\leq 0}, \quad G_0(x_0) = \text{diag}(0.5, 0.5)x_0 + (0, 0.5)^T$$

With $\mathcal{A}_0 := \mathbb{S}^1$ (the unit circle), Assumption 3 holds with $N = 1$ and $\rho = 1/\pi$. Moreover, Assumption 4 holds with $V_\varepsilon(x) := x^T P_\varepsilon x$ where

$$P = P^T > 0, \quad Q = Q^T > 0, \quad A^T P + PA = -Q$$
$$P_\varepsilon := \mathrm{diag}(0, 0, \mathrm{diag}(\varepsilon^{-1}, 1) \cdot P \cdot \mathrm{diag}(\varepsilon^{-1}, 1))$$

and $p = 2$, $c_1 = \lambda_{min}(P)$, $c_2 = \lambda_{max}(P)$, $c_3 = \lambda_{min}(Q)$.

To see that the basin of attraction of the set $\mathcal{A} := \mathcal{A}_0 \times \{0\}$ does not contain some points arbitrarily close to $\mathcal{A}$ for $\varepsilon > 0$ arbitrarily small, consider the coordinate transformation $\widetilde{x} := \mathrm{diag}(1, 1, 1, \varepsilon)x$. In the new coordinates, we get the data

$$\widetilde{C} := (\mathbb{R}_{\geq 0} \times \mathbb{R}) \times (\mathbb{R} \times (-\infty, \varepsilon]) \tag{21a}$$
$$\widetilde{F}^\varepsilon(\widetilde{x}) := \mathrm{diag}(J, \varepsilon^{-1} A)\widetilde{x} \tag{21b}$$
$$\widetilde{D}_0 := (\{0\} \times \mathbb{R}_{\leq 0}) \times \mathbb{R}^2 \tag{21c}$$
$$\widetilde{D}_1 := (\mathbb{R}_{\geq 0} \times \mathbb{R}) \times (\mathbb{R} \times [\varepsilon, \infty)) \tag{21d}$$
$$D := \widetilde{D}_0 \cup \widetilde{D}_1 \tag{21e}$$
$$\widetilde{g}_0(\widetilde{x}) := \mathrm{diag}(0.5, 0.5, 2, 2)\widetilde{x} + (0, 0.5, 0, 0)^T \tag{21f}$$
$$\widetilde{g}_1(\widetilde{x}) := 2\widetilde{x} \tag{21g}$$
$$G(\widetilde{x}) := \begin{cases} \widetilde{g}_0(\widetilde{x}) & \widetilde{x} \in \widetilde{D}_0 \backslash \widetilde{D}_1 \\ \widetilde{g}_1(\widetilde{x}) & \widetilde{x} \in \widetilde{D}_1 \backslash \widetilde{D}_0 \\ \widetilde{g}_0(\widetilde{x}) \cup \widetilde{g}_1(\widetilde{x}) & \widetilde{x} \in \widetilde{D}_0 \cap \widetilde{D}_1. \end{cases} \tag{21h}$$

Let $a \neq 0$ be such that from the point $(a, 0)$, the solution of the differential equation $\dot{\xi} = A\xi$ reaches a point (b, c) such that $c > 1$. Such a value $a \neq 0$ exists since the eigenvalues of A are complex. Consider the initial condition $x = (0, 1, \varepsilon a, 0)^T$, i.e., $\widetilde{x} = (0, 1, \varepsilon a, 0)^T$. which satisfies $|x|_{\mathcal{A}} = \varepsilon a$. Then it is straightforward to verify that the solution of the hybrid system with data in (21) reaches $\widetilde{D}_1$, from which point there is a solution that forever jumps, with each jump scaling the state by the factor 2. Thus the point $(0, 1, \varepsilon a, 0)$ is not in the basin of attraction for (20).

VI. Conclusions

We have established a local asymptotic stabilization result for a class of hybrid systems that includes dynamic models of bipedal locomotion for robots with series compliant actuation. The key assumptions, which hold for the robot application, is that the hybrid trajectories that start on the attractor of the zero dynamics satisfy a uniform average dwell-time condition. We have established that such a condition is robust, namely that it holds close to attractor as well. This fact permits establishing a local asymptotic stabilization result. We have given an example to show that, in the absence of additional structural assumptions, it is possible that the basin of attraction is quite small and even vanishes with a parameter in the feedback control algorithm. Future work will investigate, through simulations, the size of the basin of attraction achieved for the bipedal robot model using the control laws provided here.

References

[1] A. D. Ames, K. Galloway, and J. W. Grizzle. Control Lyapunov functions and hybrid zero dynamics. In *51st IEEE Conference on Decision and Control*, 2012.

[2] D.D. Bainov and P.S. Simeonov. *Systems with Impulse Effects : Stability, Theory and Applications*. Ellis Horwood Limited, Chichester, 1989.

[3] B. Brogliato. *Nonsmooth Impact Dynamics: Models, Dynamics and Control*, volume 220 of *Lecture Notes in Control and Information Sciences*. London, 1996.

[4] B. Brogliato. *Nonsmooth Mechanics: Models, Dynamics and Control*. New York, 1999.

[5] B. Brogliato, S.-I. Niculescu, and P. Orhant. On the control of finite-dimensional mechanical systems with unilateral constraints. 42(2):200–215, February 1997.

[6] C. Cai, A.R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems. Part II: (Pre-)asymptotically stable compact sets. *IEEE Transactions on Automatic Control*, 53(3):734 – 748, April 2008.

[7] F. Forni, A.R. Teel, and L. Zaccarian. Follow the bouncing ball: Global results on tracking and state estimation with impacts. *IEEE Transactions on Automatic Control*, 2013. to appear.

[8] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, April 2009.

[9] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid dynamical systems*. Princeton University Press, 2012.

[10] R. Goebel and A. R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42:573–587, 2006.

[11] J. W. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. 46:51–64, January 2001.

[12] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proc. 38th IEEE Conference on Decision and Control*, pages 2655–2660, 1999.

[13] I. A. Hiskens. Stability of hybrid limit cycles: application to the compass gait biped robot. In *40th IEEE Conf. on Decision and Control*, pages 774–779, Orlando, FL, December 2001.

[14] Y. Hürmüzlü, F. Génot, and B. Brogliato. Modeling, stability and control of biped robots – a general framework. 40(10):1647–1664, 2004.

[15] Y. Hürmüzlü and D. B. Marghitu. Rigid body collisions of planar kinematic chains with multiple contact points. 13(1):82–92, 1994.

[16] B. Morris and J.W. Grizzle. Hybrid invariant manifolds in systems with impulse effects with application to periodic locomotion in bipedal robots. *IEEE Transactions on Automatic Control*, 54(8):1751 – 1764, 2009.

[17] S. G. Nersesov, V. Chellaboina, and W. M. Haddad. A generalization of Poincaré's theorem to hybrid and impulsive dynamical systems. *Int. J. Hybrid Systems*, 2:35–51, 2002.

[18] R. T. Rockafellar and R. J-B Wets. *Variational Analysis*. Springer, 1998.

[19] A. V. Roup, D. S. Bernstein, S. G. Nersesov, W. M. Haddad, and V. Chellaboina. Limit cycle analysis of the verge and foliot clock escapement using impulsive differential equations and Poincare maps. In *Proc. Amer. Contr. Conf., Arlington, VA*, pages 3245–3250, June 2001.

[20] R. G. Sanfelice, R. Goebel, and A. R. Teel. Generalized solutions to hybrid dynamical systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(4):699–724, 2008.

[21] H.J. Sussmann and P.V. Kokotovic. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 36(4):424–440, 1991.

[22] A.R. Teel and L. Marconi. Stabilization for a class of minimum phase hybrid systems under an average dwell-time constraint. *Int. J. of Robust and Nonlinear Control*, 21:1178–1192, 2011.

[23] E. R. Westervalt, J. W. Grizzle, C. Chevallereau, J. H. Choi, and B. Morris. *Feedback Control of Dynamic Bipedal Robot Locomotion*. Taylor & Francis/CRC, 2007.

[24] E. Westervelt, J.W. Grizzle, and D.E. Koditschek. Hybrid zero dynamics of planar biped walkers. *IEEE Transactions on Automatic Control*, 48(1):42–56, January 2003.

[25] Eric R. Westervelt, Gabriel Buche, and Jessy W. Grizzle. Experimental validation of a framework for the design of controllers that induce stable walking in planar bipeds. *The International Journal of Robotics Research*, 23(6):559–582, June 2004.

[26] H. Ye, A.N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Transactions on Automatic Control*, 43(4):461–474, 1998.