Poincaré's Method for Systems with Impulse Effects: Application to Mechanical Biped Locomotion

J. W. Grizzle, **Franck Plestan** and **Gabriel Abba** [1]

Abstract

The existence and stability properties of periodic orbits are studied for nonlinear systems with impulse effects. This is achieved with an extension of the well-known method of Poincaré. The main result is then applied to a model of an under actuated, five degree of freedom biped robot with a torso in order to prove, for the first time, the existence of an asymptotically stable walking cycle.

1 Introduction

Limit cycles, that is, isolated periodic orbits, are ubiquitous. They can be desirable, such as a a periodic motion in a mechanical clock, or undesireable, such as an automatic transmission cycling between gears under steady state inputs. A common feature of these two examples is that they can be modeled as nonlinear systems with impulse effects [1, 28]. In such a model, the system evolves according to an ordinary differential equation until the state encounters a switching condition, at which time, a rapid change in the system state occurs. In order to avoid the "stiffness" associated with including a second differential equation to model the rapid evolution of the state at the switching time, a model with impulse effects collapses the change to an instant in time, and allows a discontinuity in the state. The effect is not unlike an impulse in a linear model, and hence the name.

The stability properties of equilibrium points in nonlinear systems with impulse effects have been studied extensively using Lyapunov methods [1, 28]. This communicaition extends the method of Poincaré sections to systems with impulse effects. The result is then used to study asymptotically stable walking in a planar biped robot model. Since regular walking can be viewed as a periodic orbit, the method of Poincaré sections is the natural means to study its asymptotic stability. However, due to the complexity of the associated dynamic models, this approach has only been applied successfully to Raibert's one-legged-hopper [17, 6, 8], and a biped robot without a torso [27, 9, 25]. A second contribution of the present work is to show that the control strategy can be designed in a way that greatly simplifies the application of the method of Poincaré to a class of biped models, and in fact, to reduce the stability assessment problem to the calculation of a continuous map from a sub-interval of $I\!R$ to itself.

[1] J.W. Grizzle is with the Control Systems Laboratory, Electrical Engineering and Computer Science Department, University of Michigan, Ann Arbor, MI 48109-2122,Tel: (734)-763-3598, FAX: (734)-763-8041, Email: grizzle@umich.edu.
F. Plestan and G. Abba are with GRAVIR-LSIIT, ENSPS-Université Louis Pasteur-CNRS, Blvd. Sébastien Brant, 67400 Illkirch-Graffenstaden, France, Tel: (0)3-88-65-50-87, Fax: (0)3-88-65-54-89, Email: {Franck.Plestan,Gabriel.Abba}@ensps.u-strasbg.fr

2 Background on Systems with Impulse Effects

Consider a differential equation

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{1}$$

where $x(t) \in \mathcal{X}$, a connected open subset of $I\!R^n$, $u(t) \in I\!R^m$, and f and the columns of g are continuous vector fields on $\mathcal{X}$. Let $S := \{x \in \mathcal{X} \mid H(x) = 0\}$, where $H : \mathcal{X} \to I\!R$. Finally, let $\Delta : S \to \mathcal{X}$. A *system with impulse effects* is a model of the form

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) & x^-(t) \notin S \\ x^+(t) = \Delta(x^-(t)) & x^-(t) \in S, \end{cases} \tag{2}$$

where $x^-(t) := \lim_{\tau \nearrow t} x(\tau)$ and $x^+(t) := \lim_{\tau \searrow t} x(\tau)$ denote, respectively, the left and right limits of the trajectory, $x(t)$. The mathematical meaning of a solution of the model will be given shortly. In simple words, a trajectory of the model is specified by the differential equation (1) until its state "impacts" the hyper surface S. At this point, the impulse model Δ compresses the impact event into an instantaneous moment of time, resulting in a discontinuity in the state trajectory. The ultimate result of the impact model is a new initial condition from which the differential equation model evolves until the next impact with S. In order to avoid the state having to take on two values at the "impact time", the impact event is, roughly speaking, described in terms of the values of the state "just prior to impact" at time "t^-", and "just after impact" at time "t^+". These values are represented by the left and right limits, x^- and x^+, respectively.

A function $\varphi : [t_0, t_f) \to \mathcal{X}$, $t_f \in I\!R \cup \{\infty\}$, $t_f > t_0$, is a *solution*[1] of (2) if 1) $\varphi(t)$ is right continuous on $[t_0, t_f)$,

[1] The definition is based on [28], except that solutions are taken to be right continuous instead of left continuous.

2) left limits exist at each point of (t_0, t_f), and 3) there exists a closed discrete subset $\mathcal{T} \subset [t_0, \infty)$ such that, a) for every $t \notin \mathcal{T}$, $\varphi(t)$ is differentiable and $\frac{d\varphi(t)}{dt} = f(\varphi(t)) + g(\varphi(t))u(t)$, and b) for $t \in \mathcal{T}$, $\varphi^-(t) \in S$ and $\varphi^+(t) = \Delta(\varphi^-(t))$. The condition that the set of impact times is closed and discrete simply means that there is no "chattering" about an impact point. A solution $\varphi(t)$ of (2) is *periodic* if there exists a finite $T > 0$ such that $\varphi(t+T) = \varphi(t)$ for all $t \in [t_0, \infty)$. A set $\mathcal{O} \subset \mathcal{X}$ is a *periodic orbit* of (2) if $\mathcal{O} = \{\varphi(t) \mid t \geq t_0\}$ for some periodic solution $\varphi(t)$. An orbit is *non-trivial* if it contains more than one point.

In the following, it is assumed that $u(t)$ in (2) is identically zero, so that one may refer to (2) as being time-invariant. It is further assumed that solutions to (2), when they exist, are unique.

A periodic orbit $\mathcal{O}$ is *stable in the sense of Lyapunov* if for every $\epsilon > 0$, there exists an open neighborhood $\mathcal{V}$ of $\mathcal{O}$ such that for every $p \in \mathcal{V}$, there exists a solution $\varphi : [0, \infty) \to \mathcal{X}$ of (2) satisfying $\varphi(0) = p$ and $\mathrm{dist}(\varphi(t), \mathcal{O}) < \epsilon$ for all $t \geq 0$. $\mathcal{O}$ is *attractive* if there exists an open neighborhood $\mathcal{V}$ of $\mathcal{O}$ such that for every $p \in \mathcal{V}$, there exists a solution $\varphi : [0, \infty) \to \mathcal{X}$ of (2) satisfying $\varphi(0) = p$ and $\lim_{t\to\infty} \mathrm{dist}(\varphi(t), \mathcal{O}) = 0$. $\mathcal{O}$ is *asymptotically stable in the sense of Lyapunov* if it is both stable and attractive. From here on, the qualifier, "in the sense of Lyapunov", will be systematically assumed if it is not made explicit.

Finally, assume that in (2), $S = \{x \in \mathcal{X} \mid H(x) = 0\}$, where $H : \mathcal{X} \to I\!R$ is continuously differentiable. A periodic orbit $\mathcal{O}$ is *transversal* to S if its closure intersects S in exactly one point, and for $\bar{x} := \bar{\mathcal{O}} \cap S$, $L_f H(\bar{x}) := \frac{\partial H}{\partial x}(\bar{x}) f(\bar{x}) \neq 0$ (in words, at the intersection, $\bar{\mathcal{O}}$ is not tangent to S, where $\bar{\mathcal{O}}$ is the set closure of $\mathcal{O}$).

Remark: Note that a periodic orbit of a system with impulse effects may not be a closed set, since, for $t \in \mathcal{T}$, $\varphi^-(t) \notin \mathcal{O}$ (if solutions were assumed to be left continuous, instead of right continuous, then $\varphi^+(t) \notin \mathcal{O}$). Indeed, a periodic orbit is closed if, and only if, $\mathcal{T} = \emptyset$. If a periodic orbit does not impact the surface S, then standard forms of the method of Poincaré can be applied.

3 Method of Poincaré Sections

Consider a time-invariant system with impulse effects

$$\Sigma : \begin{cases} \dot{x}(t) &= f(x(t)) \quad x^-(t) \notin S \\ x^+(t) &= \Delta(x^-(t)) \quad x^-(t) \in S, \end{cases} \tag{3}$$

where the state space $\mathcal{X}$ is an open connected subset of $I\!R^n$. The method of Poincaré sections is extended to the above system for the case of nontrivial periodic orbits that are transversal to S, under the following hypotheses:

Hypotheses:

- H1) $f(x)$ is continuous on $\mathcal{X}$;
- H2) a solution of (4) from a given initial condition is unique and depends continuously on the initial condition;
- H3) there exists a differentiable function $H : \mathcal{X} \to I\!R$ such that $S = \{x \in \mathcal{X} \mid H(x) = 0\}$; moreover, for every $s \in S$, $\frac{\partial H}{\partial x}(s) \neq 0$.
- H4) $\Delta : S \to \mathcal{X}$ is continuous, where S is given the subset topology from $\mathcal{X}$.

Hypothesis H1 implies that at any point $x_0 \in \mathcal{X}$, a solution to (4) will exist over a sufficiently small interval of time [11]. This solution may not be unique, and may not depend continuously on the initial condition; whence Hypothesis H2. Hypothesis H3 implies that S is an embedded submanifold [14], when given the subset topology. Hypothesis H4 assures that the result of an impact varies continuously with respect to where it occurs on S.

As a point of notation, φ will be used to denote a solution of the system (3), while φ^f will denote a solution of the associated ordinary differential equation,

$$\dot{x} = f(x). \tag{4}$$

The point of introducing φ^f is that, firstly, a lot is known about solutions of ordinary differential equations with continuous right hand sides [11]; secondly, in view of the first point, it is convenient to prove properties of (3) in terms of properties of (4); thirdly, at times in the proofs, it is necessary to extend a solution of (4) "through" S, while this does not make sense for (3).

Define the *time to impact* function, $T_I : \mathcal{X} \to I\!R \cup \{\infty\}$, by

$$T_I(x_0) := \begin{cases} \inf\{t \geq 0 | \varphi^f(t, x_0) \in S\} & \text{if } \exists\, t \text{ such that } \varphi^f(t, x_0) \in S \\ \infty & \text{otherwise} \end{cases} \tag{5}$$

Lemma 1 *Suppose that Hypotheses H1-H3 hold. Then T_I is continuous at points x_0 where $0 < T_I(x_0) < \infty$ and $L_f H(\varphi^f(T_I(x_0), x_0)) \neq 0$.*

Proof: Let $\epsilon > 0$ be given. Define $\bar{x} := \varphi^f(T_I(x_0), x_0)$, and without loss of generality, suppose that $L_f H(\bar{x}) < 0$. Then, from the definition of T_I and H3, $H(\varphi^f(t, x_0)) > 0$ for all $0 \leq t < T_I(x_0)$. This in turn implies that, for any $0 < t_1 < T_I(x_0)$,

$$\mu(t_1) := \inf_{0 \leq t \leq t_1} \mathrm{dist}(\varphi^f(t, x_0), S) > 0, \tag{6}$$

since: (a) $\varphi^f(t, x_0)$ is continuous in t; (b) the interval $[0, t_1]$ is compact; and (c), by H3, S is closed and equals the zero level set of H. By H1, there exists $\bar{\epsilon} > 0$ such that φ^f can be continued on $[0, T_I(x_0) + \bar{\epsilon}]$, [11]. Moreover, since $L_f H(\bar{x}) < 0$, for $\bar{\epsilon} > 0$ sufficiently small, $t_2 := T_I(x_0) + \bar{\epsilon}/2$ and $x_2 := \varphi^f(t_2, x_0)$, result in $H(x_2) < 0$. From $H(x_2) < 0$, it follows that $\text{dist}(x_2, S) > 0$. If necessary, reduce $\bar{\epsilon}$ so that $0 < \bar{\epsilon} < \min\{\epsilon, T_I(x_0)\}$, and define $t_1 := T_I(x_0) - \bar{\epsilon}/2$ and $x_1 := \varphi^f(t_1, x_0)$. From (6), $\mu(t_1) > 0$. From H2, the solutions depend continuously on the initial conditions. Thus, there exists $\delta > 0$, such that, for all $x \in B_\delta(x_0)$, $\sup_{0 \le t \le t_2} ||\varphi^f(t, x) - \varphi^f(t, x_0)|| < \min\{\text{dist}(x_2, S), \mu(t_1)/2\}$. Therefore, for $x \in B_\delta(x_0)$, $t_1 < T_I(x) < t_2$, which implies that $|T_I(x) - T_I(x_0)| < \epsilon$, establishing the continuity of T_I at x_0. ■

Hence, under H1-H3, $\tilde{\mathcal{X}} := \{x \in \mathcal{X} \mid 0 < T_I(x) < \infty \text{ and } L_f H(\varphi^f(T_I(x), x)) \neq 0\}$ is open. If H4 also holds, then $\tilde{S} := \Delta^{-1}(\tilde{\mathcal{X}})$ is an open subset of S. It immediately follows that under H1-H4, the *Poincaré return map*, $P : \tilde{S} \to S$ by

$$P(x) := \varphi^f(T_I(\Delta(x)), \Delta(x)), \tag{7}$$

is well-defined and continuous. Next, note that under H1-H4, if $\mathcal{O}$ is any periodic orbit of (3) that is transversal to S, then $\mathcal{O} \subset \tilde{\mathcal{X}}$. This is essentially by definition. Thus, there exists $x_0 \in \tilde{S}$ that generates $\mathcal{O}$ in the sense that $\Delta(x_0) \in \mathcal{O}$; indeed, $x_0 = \bar{\mathcal{O}} \cap \tilde{S}$. It thus makes sense to denote the orbit by $\mathcal{O}(\Delta(x_0))$.

Theorem 1 (Method of Poincaré Sections for Systems with Impulse Effects) *Under H1-H4, the following statements hold:*

a) If $\mathcal{O}$ is a periodic orbit of (3) that is transversal to S, then there exists a point $x_0 \in \tilde{S}$ that generates $\mathcal{O}$.

b) $x_0 \in \tilde{S}$ is a fixed point of P if, and only if, $\Delta(x_0)$ generates a periodic orbit that is transversal to S.

c) $x_0 \in \tilde{S}$ is a stable equilibrium point of $x_{k+1} = P(x_k)$ if, and only if, the orbit $\mathcal{O}(\Delta(x_0))$ is stable in the sense of Lyapunov.

d) $x_0 \in \tilde{S}$ is an asymptotically stable equilibrium point of $x_{k+1} = P(x_k)$ if, and only if, the orbit $\mathcal{O}(\Delta(x_0))$ is asymptotically stable in the sense of Lyapunov.

The following is needed in the proof of the theorem. For $\mathcal{O}$ a given periodic orbit that is transversal to S, and $x \in \tilde{\mathcal{X}}$, define $d(x) := \sup_{0 \le t \le T_I(x)} \text{dist}(\varphi^-(t, x), \mathcal{O})$. Note that d vanishes on $\mathcal{O}$. Note also that for $0 \le t \le T_I(x)$, $\varphi^-(t, x) = \varphi^f(t, x)$.

Lemma 2 *Under H1-H3, $d : \tilde{\mathcal{X}} \to I\!R$ is well-defined and is continuous on $\mathcal{O}$.*

Proof: For any $x_0 \in \tilde{\mathcal{X}}$, $T_I(x_0)$ is finite, and $\varphi^f(t, x_0)$ is defined on $[0, T_I(x_0)]$. This and the continuity of $\varphi^f(t, x_0)$ with respect to t imply that $d(x_0)$ is finite. Next, let $x_0 \in \mathcal{O}$ and $\epsilon > 0$ be given. By definition of T_I, $\bar{x} := \varphi^f(T_I(x_0), x_0) \in S$. Without loss of generality, suppose that $L_f H(\bar{x}) < 0$. Let $\eta > 0$ be such that for all $0 < t < \eta$, $H(\varphi^f(t, \bar{x})) < 0$ and $||\bar{x} - \varphi^f(t, \bar{x})|| < \epsilon/2$. Such an η exists because: (1) H1 implies there exists $\eta > 0$ such that φ^f can be continued on $[0, T_I(x_0) + \eta]$, [11]; (2) $L_f H(\bar{x}) < 0$; and (3) $\varphi^f(t, \bar{x})$ depends continuously on t. Define $t_3 := T_I(x_0) + \eta$ and $x_3 := \varphi^f(t_3, x_0)$. By H2 and Lemma 1, there exists $\delta > 0$ such that for all $\tilde{x} \in B_\delta(x_0)$, $\sup_{0 \le t \le t_3} ||\varphi^f(t, x_0) - \varphi^f(t, \tilde{x})|| < \min\{\epsilon/2\}$ and $T_I(\tilde{x}) < t_3$. By the triangle inequality, $\text{dist}(\varphi^f(t, \tilde{x}), \mathcal{O}) \le \text{dist}(\varphi^f(t, \tilde{x}), \varphi^f(t, x_0)) + \text{dist}(\varphi^f(t, x_0), \mathcal{O})$. Hence, for $\tilde{x} \in B_\delta(x_0)$, $\sup_{0 \le t \le T_I(\tilde{x})} \text{dist}(\varphi^f(t, \tilde{x}), \mathcal{O}) \le \sup_{0 \le t \le t_3} \text{dist}(\varphi^f(t, \tilde{x}), \varphi^f(t, x_0)) + \sup_{0 \le t \le t_3} \text{dist}(\varphi^f(t, x_0), \mathcal{O}) \le \epsilon/2 + \epsilon/2$, which shows that $d(\tilde{x}) \le \epsilon$, and thereby the continuity of d at x_0. ■

Proof: (Theorem 1) The first and second statements are immediate. Since the sufficiency portions of the statement c) and d) are straightforward, only necessity is proven here. Suppose that $P(x_0) = x_0$, and let $\mathcal{O}$ be the periodic orbit of (3) corresponding to $\Delta(x_0)$. By b), the orbit is transversal to S. Let $\epsilon > 0$ be given. Since x_0 is a stable in the sense of Lyapunov, for any $\bar{\epsilon} > 0$, there exist $\delta(\bar{\epsilon}) > 0$ such that, for all $k \ge 0$, $\bar{x} \in B_{\delta(\bar{\epsilon})}(x_0) \cap S$, implies $P^k(\bar{x}) \in B_\epsilon(x_0) \cap S$, where P^k is P composed with itself k-times. In particular, this implies that for all $\bar{x} \in B_{\delta(\bar{\epsilon})}(x_0) \cap S$, there exists a solution $\varphi(t)$ of (3) defined on $[0, \infty)$, such that $\varphi(0) = \Delta(\bar{x})$. Moreover, an upper bound on how far the solution φ wanders from the orbit $\mathcal{O}$ is given by

$$\sup_{t \ge 0} \text{dist}(\varphi(t), \mathcal{O}) \le \sup_{x \in B_{\bar{\epsilon}}(x_0) \cap S} d \circ \Delta(x). \tag{8}$$

By Lemma 2, since $\mathcal{O}$ is transversal to S, and since $\Delta(x_0) \in \mathcal{O}$, $d \circ \Delta$ is continuous at x_0. Since $d \circ \Delta(x_0) = 0$, it follows that there exists $\bar{\epsilon} > 0$ such that $\sup_{x \in B_{\bar{\epsilon}}(x_0) \cap S} d \circ \Delta(x) < \epsilon$. This bound is valid for all initial conditions in $B_{\delta(\bar{\epsilon})}(x_0) \cap S$. It remains to produce an open neighborhood of $\mathcal{O}$ for which such a bound holds. But this is easily done by taking $\mathcal{V} := d^{-1}([0, \delta))$, which completes the proof of c). Assume in addition that $\delta(\bar{\epsilon}) > 0$ was chosen sufficiently small so that $\lim_{k \to \infty} P^k(\bar{x}) = x_0$. Then by continuity of d and Δ, $\lim_{k \to \infty} d \circ \Delta(P^k(\bar{x})) = d \circ \Delta(x_0) = 0$, from which it easily follows that $\lim_{t \to \infty} \text{dist}(\varphi(t), \mathcal{O}) = 0$, proving d). ■

4 Biped Model and Control Design

This section summarizes a model and a controller for use in an illustration of Poincaré's method. The goal is to study the stability of a walking cycle for an under actuated, planar, five degree of freedom, biped robot with a torso. To date, stability of a walking cycle has only been established for Raibert's one-legged-hopper [17, 6, 8], and a biped robot without a torso [27, 9, 25].

4.1 Biped model

The robot consists of a torso, hips, and two legs of equal length, with no ankles and no knees. It thus has five degrees of freedom. It is assumed that the walking cycle takes place in the (sagittal) plane. It is further assumed that the walking cycle consists of successive phases of single support (meaning only one leg is touching the ground), with the transition from one leg to another taking place in an infinitesimal length of time [24, 7]. This assumption entails the use of a rigid model to describe the impact of the swing leg with the ground. The model of the biped robot thus consists of two parts: the differential equations describing the dynamics of the robot during the swing phase, and an impulse model of the contact event. This leads naturally to a model as a nonlinear system with impulse effects.

During the swing phase of the motion, the stance leg is modeled as a pivot, and thus there are only three degrees of freedom. In order for the swing leg to move without touching the ground until the desired moment of contact, the idea of [19] is adopted here: the swing leg is assumed to move out of the plane of forward motion, and into the frontal (coronal) plane. This allows the swing leg to clear the ground and be posed in front of the stance leg (think of a person with a cast over their knee). It will be further assumed that the swing leg is designed to reenter the plane of motion when the angle of the stance leg attains a given value, θ_1^d. Alternate means of achieving leg clearance in rigid legged robots are discussed in [19, 7].

The definition of the angular coordinates and the disposition of the masses of the legs, hips and torso are indicated in Figure 1. In particular, note that all masses are lumped, and positive angles are computed clockwise with respect to the indicated vertical lines. Two torques, u_1 and u_2, are applied between the torso and the stance leg, and the torso and the swing leg, respectively. The system is thus under actuated. The dynamic model of the robot between successive impacts is easily derived using the method of Lagrange [26], and results in a standard second order system

$$D(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = Bu, \qquad (9)$$

where $u = (u_1, u_2)'$, and $\theta = (\theta_1, \theta_2, \theta_3)'$: θ_1 parameterizes the stance leg, θ_2 the swing leg and θ_3 the torso. The matrices D, C, and G are deduced from the Lagragian [26], which is given in the Appendix. The second order system (9) is written in state space form by defining

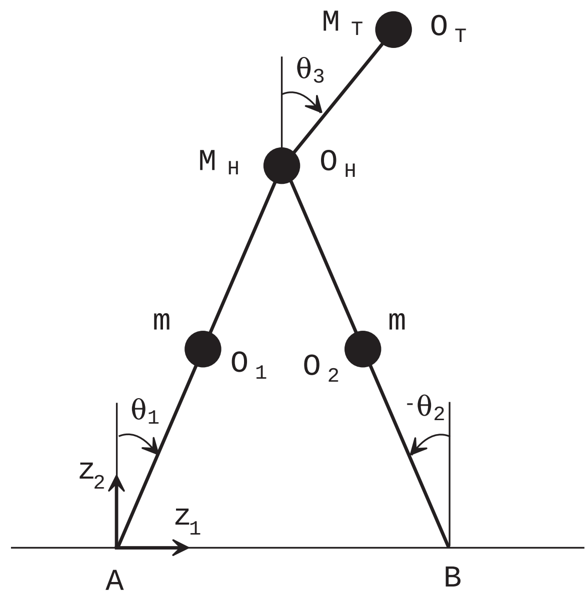

Figure 1: Schematic indicating the defintion of the generalized coordinates and the mechanical data of the biped robot. All masses are lumped. The legs are symmetric, with length r equal to the length of the line segment $A - O_H$ (also, $B - O_H$). The mass of each leg is lumped at $r/2$. The distance from the center of gravity of the hips to the center of gravity of the torso, denoted by l, is the distance from O_H to O_T.

$$\dot{x} := \frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \\ D^{-1}(\theta)\left(-C(\theta,\omega)\omega - G(\theta) + Bu\right) \end{bmatrix}$$
$$=: f(x) + g(x)u. \qquad (10)$$

The state space for the system is taken as $\mathcal{X} := \{x := (\theta', \omega')' \mid \theta \in M,\ \omega \in I\!R^3\}$, where $M = (-\pi, \pi)^3$.

The second part of the model involves the impact between the swing leg and the ground. This is modeled as a contact between two rigid bodies, and the standard model from [13] is used. The premises underlying this model are that: (a) the impact takes place over an infinitesimally small period of time; (b) the external forces during the impact can be represented by impulses; (c) impulsive forces may result in an instantaneous change in the velocities of the generalized coordinates, but the positions remain continuous; and (d) the torques supplied by the actuators are not impulsional. If one further assumes that the contact of the swing leg with the ground results in no rebound and no slipping of the swing leg, and the stance leg naturally lifting from the ground without interaction, an expression for x^+, the position and velocity just after the impact, can be computed from x^-, the position and velocity just before the impact [13]. Finally, since the coordinate defintion assumes that θ_1 corresponds to the stance leg and θ_2 to the swing leg, it is necessary to do a coordinate transformation after the impact, which amounts to swapping the first two position coordinates, and the first two velocity coordinates, respectively. The final result is expressed as $\Delta : S \rightarrow \mathcal{X}$, where

$$S := \{(\theta, \omega) \in \mathcal{X} \mid \theta_1 = \theta_1^d\}, \tag{11}$$

which clearly satisfies H3 with $H(x) = \theta_1 - \theta_1^d$, and

$$x^+ = \Delta(x^-) := \begin{bmatrix} \theta_2^- \\ \theta_1^- \\ \theta_3^- \\ \omega_2^+(x^-) \\ \omega_1^+(x^-) \\ \omega_3^+(x^-) \end{bmatrix}, \tag{12}$$

with ω_1^+, ω_2^+ and ω_3^+ specified in the Appendix. It is clear that H4 is satisfied.

4.2 Controller design

The goal of the control design is twofold: (1) to induce an asymptotically stable periodic orbit, and (2), to facilitate the verification of the existence and stability properties of the orbit. At its most basic level, walking consists of two things [22]: posture control, that is, maintaining the torso in a semi-erect position, and swing leg advancement, that is, causing the swing leg to come from behind the stance leg, pass it by a certain amount, and prepare for contact with the ground. The simplest version of posture control is to maintain the angle of the torso at some constant value, say θ_3^d, while the simplest version of swing leg advancement is to command the swing leg to behave as the mirror image of the stance leg, that is, $\theta_2 = -\theta_1$. Thus the "behavior" of walking will be "encoded" into the dynamics of the robot by defining outputs [16, 12, 15, 20].

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := \begin{bmatrix} h_1(\theta) \\ h_2(\theta) \end{bmatrix} := \begin{bmatrix} \theta_3 - \theta_3^d \\ \theta_1 + \theta_2 \end{bmatrix}, \tag{13}$$

with the control objective being to drive the outputs to zero. Of course, the idea of building in a dynamic behavior of a system through the judicious definition of a set of outputs, which when nulled yields a desireable internal behavior, is not novel in control [14] nor robotics [5, 16, 12, 4, 15, 20, 23]. The result for the biped is essentially to use the system itself as its own trajectory generator, as opposed to tracking pre-computed reference trajectories. This idea seems to be an essential step for *proving* anything about the trajectories of the closed-loop system [5, 4, 20, 23].

Since the system (10) comes from the second order model (9), and the outputs (13) only depend upon θ, it follows that the relative degree of each output component is either two or infinite. Direct computation gives that [14, 18, 21]

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u \tag{14}$$

and the determinant of the decoupling matrix, $L_g L_f h$, is further computed to be

$$-r\left(rM_H + rm + rM_T + lM_T \cos(\theta_1 - \theta_3)\right).$$

Thus, the decoupling matrix is invertible for all $x \in \mathcal{X}$ as long as $0 < lM_T < r(m + M_T + M_H)$, which imposes a very mild constraint on the position of the center of gravity of the upper body of the robot in relation to the length of its legs. This leads to the following hypothesis.

Hypothesis CH1): The decoupling matrix is globally invertible.

From now on, it is supposed that CH1 is met. Therefore, stabilizing dynamics for the output of system (10) can be assigned. The easiest way to do this is to first decouple the system [14, 21, 18] and then impose a desired dynamic response. In preparation for doing this, note that $\Phi : M \rightarrow I\!R^3$ by

$$\Phi(\theta) := \begin{bmatrix} y_1 \\ y_2 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \theta_3 - \theta_3^d \\ \theta_1 + \theta_2 \\ \theta_1 \end{bmatrix} \tag{15}$$

is a diffeomorphism onto its range. With this coordinate transformation, and upon defining

$$v := L_f^2 h + L_g L_f h u, \tag{16}$$

the system can be written in the decoupled-form

$$\begin{bmatrix} \ddot{y} \\ \ddot{\theta}_1 \end{bmatrix} = \begin{bmatrix} v \\ \zeta_0(y, \dot{y}, \theta_1, \dot{\theta}_1) + \zeta_1(y, \dot{y}, \theta_1, \dot{\theta}_1) v \end{bmatrix}. \tag{17}$$

The next step is to impose a continuous feedback $v = v(y, \dot{y})$ on (17), and thus on (2), so that the pair of double integrators $\ddot{y} = v$ is globally finite-time stabilized [2, 3]. This will collapse the image of the Poincaré return map to a one-dimensional set.

Hypotheses: The closed-loop pair of double integrators, $\ddot{y} = v(y, \dot{y})$, satisfies the following conditions:

CH2) solutions globally exist on $I\!R^4$, and are unique;

CH3) solutions depend continuously on the initial conditions;

CH4) the origin is globally asymptotically stable, and convergence is achieved in finite time;

CH5) the settling time function[2], $T_{set} : I\!R^4 \rightarrow I\!R$ by

$$T_{set}(y_0, \dot{y}_0) := \inf\{t > 0 \mid (y(t), \dot{y}(t)) = (0, 0), (y(0), \dot{y}(0)) = (y_0, \dot{y}_0)\}$$

depends continuously on the initial condition, $(y_0, \dot{y}_0)$.

[2]That is, the time it takes for a solution initialized at $(y_0, \dot{y}_0)$ to converge to the origin. The terminology is taken from [2].

Hypotheses CH2-CH4 correspond to the definition of finite-time stability [10, 2]; CH5 will also be needed, but is not implied by CH2-CH4 [3]. These requirements rule out traditional sliding mode control, with its well-known discontinuous action. A means of meeting these four objectives can be found in [2, 3]. Let $\psi_i(x_1, x_2)$, $i = 1, 2$, be any feedbacks for the pair of double integrators in (17) so that, with

$$v := \Psi(y, \dot{y}) := \left[\begin{array}{c} \psi_1(y_1, \dot{y}_1) \\ \psi_2(y_2, \dot{y}_2) \end{array} \right], \tag{18}$$

CH2-CH5 are satisfied for $\ddot{y} = v$. Define a feedback on (10), and hence on (2) as well, by

$$u(x) := (L_g L_f h(x))^{-1} \left(\Psi(h(x), L_f h(x)) - L_f^2 h(x) \right), \tag{19}$$

and denote the right-hand side of the closed-loop by

$$f_{cl}(x) := f(x) + g(x)u(x). \tag{20}$$

Finally, define

$$T_{set}^{cl}(x) := \max\{T_{set}(h_1, L_f h_1), T_{set}(h_2, L_f h_2)\} \tag{21}$$

in the obvious way. It follows that $T_{set}^{cl}(x)$ is a continuous function of x.

5 Application of Poincaré's Method to the Biped Robot

The model of the biped robot in closed loop with the controller is thus:

$$\Sigma_{cl} : \left\{ \begin{array}{rcll} \dot{x}(t) & = & f_{cl}(x(t)) & x^-(t) \notin S \\ x^+(t) & = & \Delta(x^-(t)) & x^-(t) \in S. \end{array} \right. \tag{22}$$

In this section, the method of Poincaré sections will be applied to analyze the existence and stability of periodic orbits. The finite-time convergence property of the controller will be exploited to deduce properties of the solutions of (22) by studying the solutions of

$$\dot{x}(t) \;=\; f_{cl}(x(t)) \tag{23}$$

corresponding to a one-dimensional subset of initial conditions.

5.1 Analysis à la Poincaré with a finite-time stabilizing controller

The first step in the analysis is to verify that Hypotheses H1-H4 hold for the closed-loop system (22). It can be readily shown that continuity of the feedback (18) plus Hypotheses CH1-CH3 imply H1 and H2. Hypotheses H3 and H4 were verified in Section 4. Thus Theorem 1 is applicable. The second step in the analysis is to simplify the application of the theorem. This is achieved by studying the image of the Poincaré return map in the case that the controller has had sufficient time to converge. Convergence of the controller is equivalent to the outputs, (13), being identically zero. The internal dynamics of the system (10) compatible with the output (13) being identically zero is called the zero dynamics [14], and the state space on which the zero dynamics evolves is called the zero dynamics manifold. For the biped model under study, the zero dynamics manifold is computed from (17) to be

$$\begin{array}{rcl} Z & = & \big\{(\theta, \omega) \in \mathcal{X} \mid \theta_3 = \theta_3^d,\ \theta_1 + \theta_2 = 0,\ \omega_3 = 0, \\ & & \omega_1 + \omega_2 = 0,\ -\pi < \theta_1 < \pi,\ \omega_1 \in I\!R\big\}. \end{array} \tag{24}$$

Note that the feedback (19) makes Z an invariant manifold of (10), while the same feedback does not render Z invariant for (2) since Δ does not map $Z \cap S$ into Z. The zero dynamics itself will not be computed here since it is not needed directly in the stability analysis.

Lemma 3 *Under Hypotheses CH1-CH5, and H3-H4*

1. *The set*

$$\begin{array}{rcl} \hat{S} & := & \{x_0 \in S \mid T_{set}(x_0) < T_I(x_0) < \infty, \\ & & L_f H(\phi^f(T_I(x_0), x_0)) \neq 0\} \end{array} \tag{25}$$

   is an open subset of $\tilde{S}$.

2. *Let $P : \tilde{S} \to S$ be the Poincaré return map. Then $P : \hat{S} \to S \cap Z$.*

The straightforward proof is skipped. Note that in terms of the original coordinates (θ, ω) of the robot,

$$\begin{array}{rcl} S \cap Z & = & \big\{(\theta, \omega) \in \mathcal{X} \mid \theta_3 = \theta_3^d,\ \theta_1 + \theta_2 = 0,\ \omega_3 = 0, \\ & & \omega_1 + \omega_2 = 0,\ \theta_1 = \theta_1^d,\ \omega_1 \in I\!R\big\}, \end{array}$$

a one-dimensional (embedded) submanifold of $\mathcal{X}$. Define

$$\rho : \hat{S} \cap Z \to S \cap Z \text{ by } \rho(x) := P(x). \tag{26}$$

For $x^* \in \hat{S}$, $P(x^*) \in S \cap Z$. Thus, by the definition of ρ, $P(x^*) = x^*$ if, and only if, $x^* \in \hat{S} \cap Z$ and $\rho(x^*) = x^*$. Suppose that for some $x_0 \in \hat{S}$, the sequence $x_{k+1} := P(x_k)$ is well-defined for $k \geq 0$, and remains in some open neighborhood of x_0. Then for all $k \geq 1$, $x_{k+1} = \rho(x_k)$. It follows that $x^* \in \hat{S}$ is a stable (resp., asymptotically stable) equilibrium point of P if, and only if, it is a stable (resp., asymptotically stable) equilibrium point of ρ. Thus, the determination of the existence and stability properties of periodic orbits that are transversal to $\hat{S}$ can be reduced to the analysis of a one-dimensional map. These results are summarized in the following theorem. A numerical illustration on the biped robot is given immediately in the next subsection.

Theorem 2 (Method of Poincaré for Finite-Time Control) *Consider the biped robot model of Section 4. Define outputs such that Hypothesis CH1 is met. Suppose that a continuous, finite-time stabilizing feedback is applied, and that Hypotheses CH2-CH4 are met. Define Z, $\hat{S}$ and ρ as in (24), (25) and (26), respectively. Then,*

1. *A periodic orbit is transversal to $\hat{S}$ if, and only if, it is transversal to $\hat{S} \cap Z$.*

2. *$x^* \in \hat{S} \cap Z$ gives rise to a periodic orbit of (22) if, and only if, $\rho(x^*) = x^*$.*

3. *$x^* \in \hat{S} \cap Z$ gives rise to a stable (resp., asymptotically stable) periodic orbit of (22) if, and only if, x^* is a stable (resp., asymptotically stable) equilibrium point of ρ.*

5.2 Numerical illustration

Consider the biped model with the following values of the parameters: $m = 5, M_H = 15, M_T = 10, r = 1, l = 0.5$, corresponding to the mass of the legs, the mass of the hips, the mass of the torso, the length of the legs and the distance between the center of mass of the hips and the center of mass of the torso. The units are kilograms and meters. With the outputs defined as in (13), Hypothesis CH1 is met. Suppose that the desired inclination angle of the torso is $\theta_3^d = \pi/6$ and that impact occurs with the walking surface when $\theta_1^d = \pi/8$. In the feedback (19), suppose that

$$\Psi(x) := \begin{bmatrix} \frac{1}{\epsilon^2}\psi_\alpha(y_1, \epsilon\dot{y}_1) \\ \frac{1}{\epsilon^2}\psi_\alpha(y_2, \epsilon\dot{y}_2) \end{bmatrix} \qquad (27)$$

is used, with $\epsilon = 0.1$ and $\alpha = 0.9$, where $\psi_\alpha(x_1, x_2)$ is given by [2] $\psi_\alpha(x_1, x_2) := -sign(x_2)|x_2|^\alpha - sign(\phi_\alpha(x_1, x_2))|\phi_\alpha(x_1, x_2)|^{\frac{\alpha}{2-\alpha}}$, where $\phi_\alpha(x_1, x_2) := x_1 + \frac{1}{2-\alpha} sign(x_2)|x_2|^{2-\alpha}$. The parameter $\epsilon > 0$ allows the settling time of the controller to be adjusted. With this feedback, CH2-CH5 hold [2, 3].

To determine if this choice of parameters results in an asymptotically stable orbit that is transversal to $\hat{S}$, that is, the orbit is transversal to S and the finite-time stabilizing feedback has had enough time to converge over the walking cycle, the function ρ of Theorem 2 must evaluated. This is conveniently done as follows. Define $\sigma : I\!R \to \hat{S} \cap Z$ by $\sigma(\omega_1^-) := (\theta_1^d, -\theta_1^d, \theta_3^d, \omega_1^-, -\omega_1^-, 0)$, where ω_1^- denotes the angular velocity of the stance leg just before impact. Define $\lambda := \sigma^{-1} \circ \rho \circ \sigma$.

Figure 2 displays the function λ; it also displays the related function $\delta\lambda(\omega_1^-) := \lambda(\omega_1^-) - \omega_1^-$, which represents the change in velocity over successive cycles, from just before an impact to just before the next one. It is seen that λ is undefined for ω_1^- less than approximately 1.32 radians/second (for initial ω_1^- less than this value, the robot fell backwards). The plot was truncated at 2 radians/second because nothing interesting occurs beyond this point (except an upper bound on its domain of existence will eventually occur due to the controller not having enough time to settle over one walking cycle). A fixed point occurs at approximately 1.6 radians/second, and, from the graph of λ, it clearly corresponds to an asymptotically stable walking cycle. This is supported by simulation.

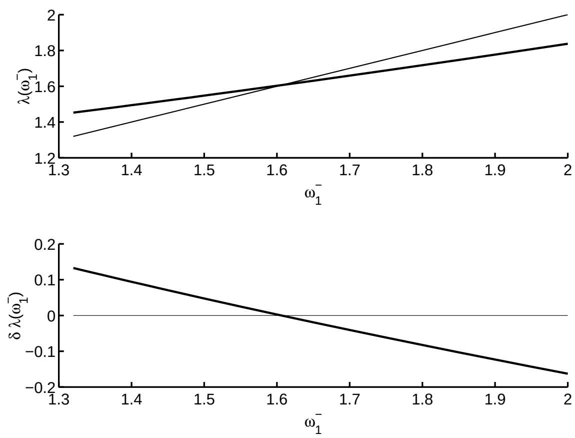

Figure 2: From either graph, it is seen that there exists a periodic orbit and that it is asymptotically stable.

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6 Appendix

$L = (\frac{5}{8}m + M_H)r^2{\omega_1}^2 + \frac{1}{8}mr^2{\omega_2}^2 + \frac{1}{2}M_T l^2{\omega_3}^2 - \frac{1}{2}mr^2\omega_1\omega_2\cos(-\theta_1 + \theta_2) + M_T r l\omega_1\omega_3\cos(-\theta_1 + \theta_3) - g(\frac{3}{2}m + M_H + M_T)r\cos(\theta_1) + \frac{1}{2}gmr\cos(\theta_2) - gM_T l\cos(\theta_3)$

The matrix B is given by $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$.

$\omega_1^+(x) = \frac{1}{den}\,[\, m\omega_1 - (4m + 4M_H + 2M_T)\omega_1\cos(2\theta_1 - 2\theta_2) + 2M_T\omega_1\cos(2\theta_1 - 2\theta_3) + 2m\omega_2\cos(\theta_1 - \theta_2)\,]$

$\omega_2^+(x) = \frac{1}{den}\,[\, 2M_T\omega_1\cos(-\theta_1+2\theta_3-\theta_2) - (2m+4M_H+2M_T)\omega_1\cos(\theta_1 - \theta_2) + m\omega_2\,]$

$\omega_3^+(x) = \frac{1}{l den}\,[\,(2mr + 2M_H r + 2M_T r)\omega_1\cos(\theta_3 + \theta_1 - 2\theta_2) - 2M_H r\omega_1\cos(-\theta_1 + \theta_3) - (2mr + 2M_T r)\omega_1\cos(-\theta_1 + \theta_3) + mr\omega_1\cos(-3\theta_1 + 2\theta_2 + \theta_3) - rm\omega_2\cos(-\theta_2 + \theta_3) - (3ml + 4M_H l + 2M_T l)\omega_3 + 2ml\omega_3\cos(2\theta_1 - 2\theta_2) + 2M_T l\omega_3\cos(-2\theta_2 + 2\theta_3)\,]$

$den = -3m - 4M_H - 2M_T + 2m\cos(2\theta_1 - 2\theta_2) + 2M_T\cos(-2\theta_2 + 2\theta_3)$