An Analytical Approach to Asymptotically Stable Walking in Planar Biped Robots

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Abstract— **This paper summarizes recent research of the author, his colleagues, and graduate student on the control of a class of underactuated biped robots. Our goal is to develop a coherent mathematical framework for the rigorous analysis and synthesis of asymptotically stable walking motions. The presentation attempts to achieve a balance between being precise enough to be believable, and intuitive enough to be understood. No attempt is made to review the general literature. The reader is referred elsewhere for a complete bibliography and proofs. Two movies are included that illustrate the results of the paper applied to a five-link, underactuated, planar biped.**

I. Introduction

Control is an integral part of any biped, whether biological or mechanical. With the exception of certain passive robots that can walk un-assisted (i.e., without any control) down an inclined plane, bipeds are dynamically unstable. Said another way, without a properly functioning control system, a biped stumbles or falls. A primary goal of locomotion research is the discovery of the fundamental control principles that underlie bipedal locomotion. Biologists have made significant progress in the difficult task of constructing simplified mathematical models of the human musculoskeletal and neuro systems, and have used them to explain many features of normal and pathological gaits. Engineers have designed and constructed bipedal machines capable of an amazing array of walking, running and jumping patterns. With few exceptions, the primary methods for determining whether a control law yields a stable walking or running motion are either to simulate a detailed mathematical model or to build the robot and implement the control law. Until just recently, for the important cases of underactuated biped models with torsos, none of the control designs proposed in the literature had been analytically proven to yield asymptotically stable motions.

A central objective of the author's research on robotic, bipedal locomotion is the development of a coherent mathematical framework in which asymptotically stabilizing feedback controllers for biped robots may be rigorously analyzed and synthesized. More precisely, for the closed-loop system consisting of a biped robot, its environment, and a given feedback controller, the goal is to be able to determine if periodic orbits exist, and if they exist, whether they are asymptotically stable. Once asymptotic stability is settled, the goal is to achieve some modest performance objectives, such as minimal peak actuator torques or walking with a given average speed. This paper will summarize the theoretical progress achieved to date. Because it was impossible to do justice to the literature without significantly lengthening this already-too-long paper, *no external references are included*. The reader is referred instead to the well-known web site [1] for an introduction to the biped research community and to the bibliographies in the author's papers [2], [3], [4] and [5] for many references on the control of bipedal robots.

Section II describes the class of robots that can be analyzed. The models are potentially high dimensional and hybrid (contain both continuous dynamics and a discrete impact map). In addition, it will be assumed that there is no actuation at the end of the stance leg. Thus the system is *underactuated* during walking, as opposed to fully actuated (a control at each joint and the contact point with the ground). A steady walking cycle is a non-trivial periodic motion. This means that standard stability tools for static equilibria do not apply. Instead, one must use tools appropriate for the study of periodic orbits, such as Poincaré return maps. It is of course well known how to use numerical methods to compute a Poincaré return map and to find fixed points of it. The drawback in such a direct approach, which for bipeds involves the numerical computation of a high-dimensional, nonlinear map, *is that it does not yield much insight for feedback design and synthesis.* In some sense, the goal of the remainder of the paper is to structure the feedback design process in such a way that Poincaré analysis can be incorporated into feedback synthesis. Section III opens with a somewhat philosophical discussion of a few key elements in control design that may help to improve the analytical tractability of the resulting closed-loop system. The section then summarizes a design methodology that is based upon using feedback to reduce the number of degrees of freedom of the robot through the approximate realization of holonomic constraints. Section IV then shows how the stability analysis of the resulting closed-loop system can be rigorously completed by the numerical computation of a one-dimensional map. Using this result, one is able to check with necessary and sufficient conditions whether or not the proposed feedback induces an asymptotically stable, periodic orbit. The stability analysis is performed, however, after the feedback is designed. The goal of Section V is to bring the stability criterion directly into the feedback synthesis process. It is shown how asymptotically stable walking motions can be designed on the basis of a one DOF invariant subsystem. Parametric optimization is used to achieve approximate optimality with respect to energy consumption, for example, while guaranteeing stability and meeting natural kinematic and dynamic constraints.

II. Hybrid Robot Model

The robot is assumed to be planar and consist of N-links connected in such a way that there are at least a torso and two identical legs, with the legs connected at a common point called the hips; furthermore, all links have mass, are rigid, are connected in revolute joints, and all kinematic chains formed by the connections of links are assumed to be open. Figure 1 depicts an example of such a robot. All walking cycles will be assumed to take place in the sagittal plane and consist of successive phases of *single support* (meaning the stance leg is touching the walking surface and the swing leg is not) and *double support* (the swing leg and the stance leg are both in contact with the walking surface). During the single support phase, it is assumed that the stance leg acts as a pivot. It is further supposed that the walking gaits of interest are such that, in steady state, successive phases of single support are symmetric with respect to the two legs, involve motion from left to right, and the swing leg is posed in front of the stance leg (these assumptions rule out certain pathological gaits). Finally, actuation is applied at each joint, *but not between the stance leg and ground*. The robot is thus *underactuated* during the single support phase. It is worth noting that even if there were actuation between the stance leg end and ground, it would be worthwhile to first design a controller under the assumption of no "ankle" torque and then add an outer control loop to exploit the torque available at the ankle.

The two phases of the walking cycle naturally lead to a mathematical model of the biped consisting of two parts: the differential equations describing the dynamics during the single support phase, and a model of the contact event. A standard, rigid (*inelastic*)

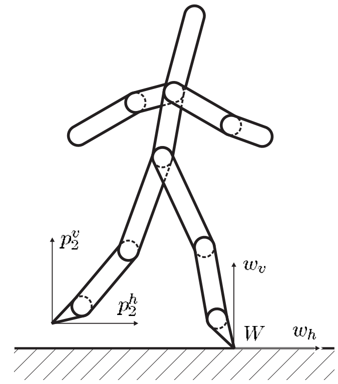

Fig. 1. Schematic illustrating the class of N-link robot models considered here. No actuation is applied between the stance leg and the ground, while all other joints are actuated.

contact model is assumed, which collapses the double support phase to an instant in time, yielding a discontinuity in the velocity component of the state, with the position remaining continuous. The biped model is thus *hybrid* in nature, consisting of a continuous dynamics and a re-initialization rule at the contact event.

Swing phase model: With N-links, the dynamic model of the robot during the swing phase has N-DOF. Let q be the set of coordinates describing the configuration of the robot with respect to a world reference frame W, whose origin is centered at the end of the stance leg. Since only symmetric gaits are of interest, the same model can be used irrespective of which leg is the stance leg if the coordinates are relabeled after each phase of double support. Using the method of Lagrange, the model is written in the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu. \tag{1}$$

Torques u_i, $i = 1$ to $(N-1)$, are applied between each connection of two links, but not between the stance leg and ground. The model is written in state space form by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \dot{q} \\ D^{-1}(q)\left[-C(q,\dot{q})\dot{q} - G(q) + Bu\right] \end{bmatrix} \\ &=: f(x) + g(x)u. \end{aligned} \tag{2}$$

where $x := (q', \dot{q}')'$. The state space of the model is taken as $T\mathcal{Q}$, where $\mathcal{Q}$ is a connected, open subset of $I\!R^N$ corresponding to physically reasonable configurations of the robot.

Impact model: An impact occurs when the swing leg touches the walking surface. Let $(p_2^h,\ p_2^v)$ denote the horizontal and vertical coordinates of the end of the stance leg. Define the walking surface as

$$S := \{(q,\dot{q}) \in T\mathcal{Q} \mid p_2^v = 0,\ p_2^h > 0\}, \tag{3}$$

also called the ground. The impact between the swing leg and the ground is modeled as an (inelastic) contact between two rigid bodies. In addition to modeling the change in state of the robot, the impact model also accounts for the relabeling of the robot's coordinates that occurs after each phase of double support. The impact model under standard hypotheses results in a smooth map $\Delta : S \to T\mathcal{Q}$,

$$x^+ = \Delta(x^-), \tag{4}$$

where $x^+ := (q^+, \dot{q}^+)$ (resp. $x^- := (q^-, \dot{q}^-)$) is the state value just after (resp. just before) impact.

Nonlinear system with impulse effects: The overall biped robot model is expressed as a nonlinear system with impulse effects

$$\Sigma : \begin{cases} \dot{x} &= f(x) + g(x)u \qquad x^- \notin S \\ x^+ &= \Delta(x^-) \qquad\qquad x^- \in S, \end{cases} \tag{5}$$

where, $x^-(t) := \lim_{\tau \nearrow t} x(\tau)$. Solutions are taken to be right continuous and must have finite left and right limits at each impact event (see [2] for details).

In simple words, a trajectory of the model is specified by the swing phase model (2) until its state "impacts" the hypersurface S. At this point, the impulse model Δ compresses the impact event into an instantaneous moment of time, resulting in a discontinuity in the state trajectory. The result of the impact model is a new initial condition from which the swing phase model evolves until the next impact with S. In order to avoid the state having to take on two values at the "impact time", the impact event is, roughly speaking, described in terms of the values of the state "just prior to impact" at time "t^-", and "just after impact" at time "t^+". These values are represented by the left and right limits, x^- and x^+, respectively. A *half-step* of the robot is defined to be a solution of (5) that starts with the robot in double support, with the swing leg behind the stance leg, ends in double support with the swing leg in front of the stance leg, and contains no other impact event.

III. Control Design for Analytical Tractability

A. *Philosophy: If stability analysis is too difficult, then it will not be incorporated into feedback synthesis.*

One can distinguish several control design approaches in the biped literature. By far, the most common approach to control is through the tracking of pre-computed reference trajectories. The trajectories may be determined via analogy, either with human motion, or with simpler, passive, mechanical biped systems; they may be generated by an external oscillator, such as van der Pol's oscillator, or computed through optimization of various cost criteria, such as minimum expended control energy over a walking cycle. Within the context of tracking, many different control methods have been explored, including continuous-time methods based on PID controllers, computed torque (also called inverse dynamics) and sliding mode control, or essentially discrete-time methods, where control actions are applied at each contact event.

The are several drawbacks associated with control based on the tracking of time trajectories. First of all, the resulting closed-loop system is then time-varying, in addition to being nonlinear and hybrid, further complicating its analysis. Secondly, in the case of an underactuated system, it is not always possible to track a given periodic solution of the model in an asymptotically stable manner. Thirdly, if a disturbance affects the robot and causes its motion to be retarded, for example, with respect to a reference trajectory, the feedback system is obliged to play catch up and regain synchrony with the reference trajectory. Presumably, what is much more important is the orbit of the robot, that is, the path in state space traced out by its motion, and not the slavish notion of time imposed by a reference trajectory. A preferable situation, therefore, would be for the robot in response to a disturbance to converge back to the periodic orbit, but not to attempt otherwise resynchronizing itself with time. One way to achieve this is by parameterizing the orbit (i.e., the walking motion) with respect to a scalar-valued function of the states of the robot. In this way, for example, when a disturbance retards the motion of the robot, it may also automatically "retard" the function that parameterizes the orbit. Consequently, the feedback controller will not have to play catch up, and can focus solely on maintaining posture and relative limb velocities.

A second key ingredient for achieving analytical tractability is to reduce the dimension of the problem. This is not a novel idea in control, in general, nor biped locomotion, in particular. Biped robots tend to have many degrees of freedom. Several authors have sought to cast their control problem in terms of designing a controller for a lower dimensional target system. These include: regulating angular momentum; controlling total energy; and approximating the robot as an inverted pendulum, resulting in the reduced task of regulating its center of mass. Others have achieved a reduction in complexity through a proposed set of walking principles, such as maintaining the torso at a constant angle and the hips at a constant height above the ground while moving one foot in front of the other. Principles of this sort can be viewed as establishing kinematic constraints on the system, which, as explained in [4], induce a low order target dynamics, namely, the hybrid zero dynamics of the bipedal walker.

While dimension reduction for the purpose of controller design has been recognized as being important, its consequences for analysis have not been fully explored. One of the principal objectives of the author's work is to show that freedom in the control strategy can be exploited in a way that greatly simplifies the stability analysis of the closed-loop system. Once stability analysis becomes sufficiently simple, it can be incorporated into feedback synthesis. Time-invariance and dimension reduction are two important aspects of simplification.

B. Locomotion objectives that reduce the number of degrees of freedom

The control designs studied in [2], [3], [4] and [5] involve the judicious choice of a set of holonomic constraints that are imposed on the robot via feedback control. This is accomplished by interpreting the constraints as output functions depending only on the configuration variables of the robot, and then combining ideas from finite-time stabilization and computed torque. The desired posture of the robot is encoded into the set of outputs in such a way that the nulling of the outputs is equivalent to achieving the desired posture.

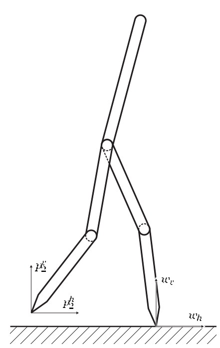

Fig. 2. A specific five link biped. As in the more general robot, there is no actuation between the stance leg and the ground, while all other joints are actuated.

In order to be more concrete at this point, consider the 5-link, 4-actuator, planar bipedal walker shown in Figure 2. Several different constraint choices have been explored, with the common element being that four outputs are controlled since the robot has four actuators. In [3], a largely "Cartesian" view is taken. The choice is made to regulate the angle of the torso, the height of the hips, and the position of the end of the swing leg (both horizontal and vertical components). Over each half-step of a normal walking gait, the horizontal position of the hips is monotonically increasing. Hence, along a half-step, the *desired* torso angle, hip height and swing leg end position are expressed as functions of the horizontal position of the hip. These four functions are chosen so that, as the hips advance, the torso is erect at a nearly vertical angle, the height of the hips rises and falls naturally during the step, the swing leg advances from behind the stance leg to in front of it, tracing a parabolic trajectory. In [5], the desired motion of the robot is described in terms of the evolution of relative joint angles. The four outputs are selected to be co-located with the actuators: the (two) relative angles of the torso with the femurs and the (two) relative angles of the knees. The angle of the "virtual leg", that is, the line connecting the stance leg end to the hips, is clearly monotonic over a half-step whenever the horizontal component of the hips is monotonic. Consequently, the four relative angles are expressed as desired functions of the angle of the virtual leg.

In the general case of an N-link bipedal walker, $N-1$ outputs of the form

$$\begin{aligned} y &= h(q) \\ &= h_0(q) - h_d(\theta(q)) \end{aligned} \tag{6}$$

are constructed. The function h_0 specifies $(N-1)$ independent quantities that are to be controlled, while $\theta(q)$ is a scalar function[1] of the configuration variables that is "independent" of h_0 and is monotonically increasing along a (non-pathological) half-step. The function $h_d(\theta(q))$ specifies the desired evolution[2] of the controlled quantities in the sense that $y \equiv 0$ would assure the desired posture of the robot during a half-step. In short, the function h_d is taking the place of a desired time trajectory and the quantity $\theta(q)$ is playing the role of time, so that the evolution of the robot will be "synchronized" to an internal variable.

Achieving $y \equiv 0$ would reduce the number of degrees of freedom to one. But since the only means to null y is through feedback control, $y = 0$ can only be achieved asymptotically. This raises the question: *Can the asymptotic nulling of y be done in a way that leads to a reduced dimensional, and hence, simplified, analysis problem?*

C. Achieving dimension reduction through finite-time feedback control

Since the output $y = h(q)$ in (6) depends only on the configuration variables, the first derivative of the output along solutions of (2) does not depend directly

[1]For example, the horizontal position of the hips or the angle of the virtual leg.

[2]For the design of h_d, one is free to use essentially any of the methods known for creating desirable time trajectories for tracking; see [2].

on the inputs. Hence the relative degree is at least two. Differentiating the output once again computes the accelerations, resulting in

$$\frac{d^2y}{dt^2} =: L_f^2h(q,\dot{q}) + L_gL_fh(q)u. \tag{7}$$

The matrix $L_gL_fh(q)$ is called the *decoupling matrix* and depends only on the configuration variables. Assume its invertibility[3]:

CH1-a) there exists open set $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ such that for each point $q \in \tilde{\mathcal{Q}}$, the decoupling matrix $L_gL_fh(q)$ is square and invertible (i.e., h has vector relative degree $(2,\ldots,2)'$); and

CH1-b) there exists at least one point in $\tilde{\mathcal{Q}}$ where h vanishes.

Then designing a feedback controller to asymptotically drive the output to zero is a standard problem in control and robotics, $\cdots$ *or is it?* The catch is the hybrid nature of the biped model. For example, a common feedback controller may be

$$u = -(L_gL_fh)^{-1}\left(L_f^2h + K_DL_fh + K_ph\right), \tag{8}$$

with K_D and K_P positive definite matrices, which yields

$$\ddot{y} + K_D\dot{y} + K_Py = 0, \tag{9}$$

during the swing phase. However, at each impact event of (5), the solution of (9) must be re-initialized to $y^+ = h \circ \Delta(x^-)$ and $\dot{y}^+ = L_fh \circ \Delta(x^-)$, respectively. Conceptually, this is equivalent to a persistent, impulsive disturbance being applied to (9), and hence, convergence of $y(t)$ to zero is not guaranteed. Continuing with the disturbance analogy, since achieving $y = 0$ is equivalent to achieving the desired robot posture during a half-step, a natural desire may be to attenuate the effects of the disturbances by increasing the controller gains, thereby inducing rapid convergence of $y(t)$ toward zero during a single step. Of course, a drawback of high gain is large torques when the output is "far" away from zero. In response to this, other researchers have considered sliding-mode control, which achieves convergence in finite time. However, this leads to chattering and poses very real difficulties for rigorous analysis of solutions of the closed-loop system. A viable alternative is to include a "square root-like" action in the feedback so that modest controls are applied when y and $\dot{y}$ are large and finite-time convergence to zero takes place once y and $\dot{y}$ become small. With a little care, such continuous, though non-Lipschitz continuous, feedbacks can be used with great advantage. This is developed next.

The application of the pre-feedback to (2)

[3]A method for checking this is given in [3].

$$u(x) = (L_gL_fh(x))^{-1}(v - L_f^2h(x)) \tag{10}$$

results in the chain of $N-1$ double integrators,

$$\frac{d^2y}{dt^2} = v; \tag{11}$$

see (7). Let $v(y,\dot{y})$ be any feedback controller on (11) satisfying conditions CH2 to CH5 of [2], that is,

Controller Hypotheses: for the closed-loop chain of double integrators, $\ddot{y} = v(y,\dot{y})$,

CH2) solutions globally exist on $I\!R^{2N-2}$, and are (forward) unique;

CH3) solutions depend continuously on the initial conditions;

CH4) the origin is globally asymptotically stable, and convergence is achieved in finite time;

CH5) the settling time function[4], $T_{set} : I\!R^{2N-2} \to I\!R$ by

$$\begin{aligned} T_{set}(y_0,\dot{y}_0) \quad := \quad & \inf\{t>0 \mid (y(t),\dot{y}(t)) = (0,0), \\ & (y(0),\dot{y}(0)) = (y_0,\dot{y}_0)\} \end{aligned}$$

depends continuously on the initial condition, $(y_0,\dot{y}_0)$. Hypotheses CH2-CH4 correspond to the definition of finite-time stability; CH5 is also needed, but is not implied by CH2-CH4. These requirements rule out traditional sliding mode control, with its well-known discontinuous action. A particular choice for the feedback function $v(y,\dot{y})$ is highlighted in [2].

Consider now the feedback

$$u(x) := (L_gL_fh(x))^{-1}\left(v(h(x), L_fh(x)) - L_f^2h(x)\right). \tag{12}$$

The hybrid model of the biped robot (5) in closed loop with the above feedback is then

$$\Sigma_{cl}: \left\{ \begin{array}{rcll} \dot{x}(t) &=& f_{cl}(x(t)) & x^-(t) \notin S \\ x^+(t) &=& \Delta(x^-(t)) & x^-(t) \in S, \end{array} \right. \tag{13}$$

where,

$$f_{cl}(x) := f(x) + g(x)u(x). \tag{14}$$

IV. Determining Existence and Stability of Periodic Orbits on the Basis of a One-Dimensional Map

Define the *time to impact* function, $T_I : T\mathcal{Q} \to I\!R \cup \{\infty\}$, by

$$T_I(x_0) := \left\{ \begin{array}{ll} \inf\{t \ge 0 | \varphi^{f_{cl}}(t,x_0) \in S\} & \text{if } \exists\, t \text{ such} \\ & \text{that } \varphi^{f_{cl}}(t,x_0) \in S \\ \infty & \text{otherwise} \end{array} \right.$$

[4]That is, the time it takes for a solution initialized at $(y_0,\dot{y}_0)$ to converge to the origin.

where $\varphi^{f_{cl}}(t, x_0)$ is the solution of $\dot{x}(t) = f_{cl}(x(t))$ initialized at x_0. It can be shown that

$$\hat{S} := \{x_0 \in S \mid T_{set}(x_0) < T_I(x_0) < \infty, \\ L_f p_2^v(\varphi^{f_{cl}}(T_I(x_0), x_0)) \neq 0\} \quad (15)$$

is an open subset of S.

Take the Poincaré section as S, the walking surface, and define the *Poincaré return map*, as the partial map $P : S \to S$ by, if $T_I(\Delta(x)) < \infty$,

$$P(x) := \varphi^{f_{cl}}(T_I(\Delta(x)), \Delta(x)). \quad (16)$$

The return map represents the evolution of the robot just before an impact with the walking surface, to just before the next impact, assuming that a next impact does occur. If it does not, that is, the robot falls due to the preceeding impact, the point being analyzed is not in the domain of definition of the return map. The Poincaré return map is continuous on $\hat{S}$.

Hypothesis CH1 ensures that

$$Z := \{x \in T\tilde{\mathcal{Q}} \mid h(x) = 0, L_f h(x) = 0\} \quad (17)$$

is a smooth two dimensional submanifold of $T\mathcal{Q}$, called the *zero dynamics manifold*. For $x \in \hat{S}$, $P(x) \in S \cap Z$ since the settling time is less than the time for an impact. Define a reduced Poincaré map by

$$\rho : \hat{S} \cap Z \to S \cap Z \text{ by } \rho(x) := P(x). \quad (18)$$

Thus, for $\hat{x}^* \in \hat{S}$, $P(x^*) = x^*$ if, and only if, $x^* \in \hat{S} \cap Z$ and $\rho(x^*) = x^*$. This fact results in the existence and stability properties of periodic orbits being characterized in terms of a one-dimensional map. An example is given right after the statement of the theorem.

THEOREM 1: **(Method of Poincaré for a System with Impulse Effects under Finite-Time Control [2])** Consider the biped robot model of Section II, written in the form of a system with impulse effects, (5). Define outputs such that Hypothesis CH1 is met. Suppose that a continuous, finite-time stabilizing feedback is applied, and that Hypotheses CH2-CH4 are met. Define Z, $\hat{S}$ and ρ as in (17), (15) and (18), respectively. Then,
1. A periodic orbit is transversal to $\hat{S}$ if, and only if, it is transversal to $\hat{S} \cap Z$.
2. $x^* \in \hat{S} \cap Z$ gives rise to a periodic orbit of (13) if, and only if, $\rho(x^*) = x^*$.
3. $x^* \in \hat{S} \cap Z$ gives rise to a stable (resp., asymptotically stable) periodic orbit of (13) if, and only if, x^* is a stable (resp., asymptotically stable) equilibrium point of ρ. □

The above results have been used to rigorously prove the stability of feedback control designs for a

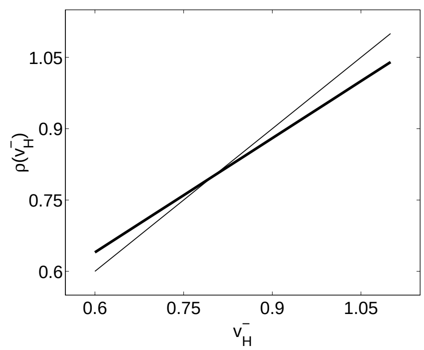

Fig. 3. The reduced Poincaré return map, ρ (bold line), as a function of the horizontal velocity of the hips just before impact, v_H^-, (m/s) and, for visualization purposes, the identity function (thin line). This proves that there exists a periodic orbit and that it is asymptotically stable.

3-link biped (torso, and two legs without knees) in [2] and a 5-link biped (torso, two legs with revolute knees) in [3]. A typical reduced Poincaré map, ρ, is shown in Figure 3. The set $S \cap Z$ was parameterized by the horizontal velocity of the hips, and the Poincaré map generated numerically in a few seconds. The plot shows the existence of a fixed point (where the Poincaré map intersects the identity function) and its asymptotic stability (note the slope of the intersection). An example closed-loop walking motion[5] is included as a movie on the CD-ROM with the paper (Movie_1_grizzle.mpg). The average walking speed is approximately 0.75 m/s and the peak torque is around 77 Nm (the robot's total mass is 40 kg).

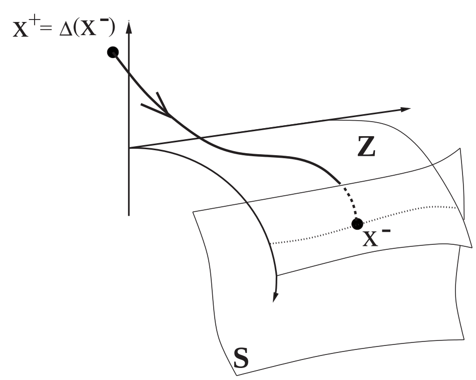

Fig. 4. Representation of a typical periodic orbit without additional invariance assumptions placed on the output function. If the condition $\Delta(S \cap Z) \subset Z$ held, then the orbit would be entirely in Z.

[5]The function h_0 was selected to control the torso angle, hip height and swing leg position; $\theta(q)$ was taken as the angle of the virtual leg. The function h_d was designed to induce tracking of an orbit that approximately minimized integral-squared torque per step length; the orbit was determined through open-loop trajectory optimization.

Figure 4 shows a projection of the phase portrait of a typical periodic orbit. The trajectory converges in finite time to the set Z where the output is identically zero, evolves along Z until the impact of the swing leg with the walking surface S occurs, and then is bumped off of Z by the impact map, Δ. This raises the question, *is it possible to shape the output so that the result of the impact map lands the system back on Z?* Intuitively, this property would seem to reduce transients, and hence, in many cases, reduce peak torque requirements.

In summary, using the results of this section, one is able to *verify* with necessary and sufficient conditions whether or not a proposed set of outputs leads to a feedback controller that induces an asymptotically stable, periodic orbit. The goal of the next section is to bring the stability criterion directly into the feedback (or output) synthesis problem.

V. Designing Asymptotically Stable, Approximately Optimal, Orbits on the Basis of a One-DOF Invariant Subsystem

This section summarizes the main results of [4] and [5], which lead to a design procedure for creating *asymptotically stable orbits that are approximately optimal with respect to energy minimization.* Due to space limitations, the development will be brief. Familiarity with the zero dynamics of non-hybrid models is assumed and many informative, intermediate steps are skipped.

The additional required technical assumptions[6] are [4]:

HH3) there exists a smooth real valued function $\theta(q)$ such that $\Phi : \tilde{\mathcal{Q}} \to I\!R^N$ by $\Phi(q) := (h(q)', \theta(q))'$ is a diffeomorphism onto its image;

HH4) there exists a unique point $q_0^- \in \tilde{\mathcal{Q}}$ such that $(h(q_0^-), p_2^v(q_0^-)) = (0, 0)$ and the rank of $[h',\ p_2^v]'$ at q_0^- equals N.

For a system modeled by ordinary, non-hybrid, differential equations, the *maximal internal dynamics of the system that are compatible with the output being identically zero* is called the *zero dynamics*. The feedback control

$$u^*(x) = -(L_g L_f h(x))^{-1} L_f^2 h(x) \tag{19}$$

renders the zero dynamics manifold Z in (17) invariant under the swing phase dynamics[7] in the sense that, every $z \in Z$, $f_{zero}(z) := f(z) + g(z)u^*(z) \in T_z Z$. The autonomous system

$$\dot{z} = f_{zero}(z) \tag{20}$$

is called the *swing phase zero dynamics*. In [4], the notion of the zero dynamics was extended to include the impact map common in many biped models. This is necessary because a solution of (20) is in general not a solution of the hybrid model, (5). Indeed, the problem is that $z^-(t) \in S$ does not necessarily imply that $z^+(t) := \Delta(z^-(t)) \in Z$; that is, applying the impact model when the solution of the swing phase zero dynamics impacts the walking surface does not result, in general, in a new initial condition on Z, the state space of the zero dynamics. Consequently, to continue the solution from the new initial condition, the full model, (5) is required. This leads to the following additional requirement when designing the output function:

HH5) $\Delta(S \cap Z) \subset Z$.

This condition renders the swing phase dynamics *invariant under the impact map*, which results in the *hybrid zero dynamics*,

$$\Sigma_{zero} : \begin{cases} \dot{z} = f_{zero}(z) & z^- \notin S \cap Z \\ z^+ = \Delta(z^-) & z^- \in S \cap Z. \end{cases} \tag{21}$$

The following can be shown:

- along all solutions of (21), the output h is identically zero, and hence this is a valid zero dynamics for the hybrid model;
- $S \cap Z$ is diffeomorphic to $I\!R$;
- the function θ (see, HH3), when evaluated along any half-step of the zero dynamics, is a strictly monotonic function of time and thus achieves its maximum and minimum values at the end points;
- the extrema of $\theta(q)$ over a half step are $\theta^- := \theta(q_0^-)$ and $\theta^+ := \theta(\Delta(q_0^-, 0))$, (see, HH4) and thus, without loss of generality, it can be assumed that $\theta^+ < \theta^-$; that is, that along any half-step of the hybrid zero dynamics, θ is *monotonically increasing.*

In appropriate local coordinates[8] (ξ_1, ξ_2), the zero dynamics have the form

$$\begin{aligned} \dot{\xi}_1 &= \alpha(\xi_1)\xi_2 \\ \dot{\xi}_2 &= \beta(\xi_1) \end{aligned} \tag{22}$$

where $\alpha(\xi_1) \neq 0$ for all ξ_1. Furthermore, the impact portions of the model, $S \cap Z$ and $\Delta : (\xi_1^-, \xi_2^-) \to (\xi_1^+, \xi_2^+)$, simplify to

$$S \cap Z = \left\{(\xi_1^-, \xi_2^-) \mid \xi_1^- = \theta^-, \xi_2^- \in I\!R\right\} \tag{23}$$

$$\xi_1^+ = \theta^+ \tag{24}$$

$$\xi_2^+ = \delta_{zero} \cdot \xi_2^-, \tag{25}$$

where δ_{zero} is a constant that may be computed directly from the impact map, Δ, and the output, h. In

[6] Conditions HH1 and HH2 of [4] are subsumed by previous hypotheses.

[7] In plain words, for every initial condition $x_0 \in Z$, the solution $x(t, x_0)$ of the feedback system $f(x) + g(x)u^*(x)$ remains in Z.

[8] $\xi_1 = \theta(q)$ and ξ_2 has the interpretation of angular momentum.

these coordinates, the Poincaré map of the hybrid zero dynamics can be explicitly computed, which leads to a necessary and sufficient condition for the existence of an asymptotically stable, periodic orbit.

THEOREM 2: **(Poincaré Analysis of the Hybrid Zero Dynamics [4])** Assume that a smooth output function, h, on (5) satisfies CH1, HH3, HH4 and HH5. For $\theta^+ \leq \xi_1 \leq \theta^-$, define

$$\kappa(\xi_1) := \int_{\theta^+}^{\xi_1} \frac{\beta(\xi)}{\alpha(\xi)} \cdot d\xi \qquad (26)$$

$$K := \min_{\theta^+ \leq \xi_1 \leq \theta^-} \kappa(\xi_1). \qquad (27)$$

The hybrid zero dynamics admit an asymptotically stable, periodic orbit if, and only if, the following two conditions hold:
a) $\delta_{zero}^2 < 1$; and
b) $\frac{\delta_{zero}^2}{1-\delta_{zero}^2}\kappa(\theta^-) + K > 0$.
Moreover, under the feedback

$$u(x) := (L_g L_f h(x))^{-1} \left(v(h(x), L_f h(x)) - L_f^2 h(x) \right), \qquad (28)$$

where v satisfies CH2-CH5, an asymptotically stable, periodic orbit of the hybrid zero dynamics is also an asymptotically stable, periodic orbit of the full-order hybrid model, (5). □

This result provides a powerful method for *analyzing* and *designing* asymptotically stable walking motions. In [5], Bézier polynomials are used to parameterize the output (6), yielding

$$y = h_0(q) - h_d(\theta(q), a), \qquad (29)$$

where a is a vector of real coefficients. The Bézier polynomials make it very easy to satisfy the invariance condition, HH5. Along solutions of the zero dynamics, the feedback (28) reduces to (19) and thus is *independent* of the choice of v. A cost function of the form

$$J(a) := \frac{1}{p_2^h(T^-)} \int_0^{T^-} ||u^*(t)||^2 dt, \qquad (30)$$

is posed, where T^- is the time for a half-step, $p_2^h(T^-)$ corresponds to step length, and $u^*(t)$ is the result of evaluating (19) along a solution of the hybrid zero dynamics. A sequential, quadratic programming package is used to minimize $J(a)$ with respect to a, subject to a number of constraints:
- periodicity of the orbit;
- the stability conditions (a) and (b) of Theorem 2;
- the desired walking rate;
- minimum normal ground reaction force;
- maximum ratio of tangential to normal ground reaction forces experienced by the stance leg end (i.e., maximum allowed static friction coefficient);
- maximum deflection of stance leg and swing leg knees; etc.

Whenever the optimization problem has a feasible solution, the result is an *asymptotically stable, closed-loop system* that meets natural kinematic and dynamic constraints. Technically, once the optimization is completed, one must verify that all of the conditions of Theorem 2 are met. At least on the five-link biped, this has proven to be straightforward. The most critical condition, CH1-a, the invertibility of the decoupling matrix, is essentially guaranteed whenever $J(a)$ is finite, since singularities in $L_g L_f h$ will normally result in $u^*(t)$ taking on unbounded values; however, a simply connected, open set about the periodic orbit where the decoupling matrix is invertible can be *explicitly* computed by a method developed in [3].

As a numerical example [5], consider once again the biped of Figure 2. Table I summarizes the result of optimizing for a desired rate of 1.05 m/s. The walking motion is asymptotically stable since $\delta_{zero}^2 < 1$ and $\frac{\delta_{zero}^2}{1-\delta_{zero}^2}\kappa(\theta^-) + K \approx 364.4 > 0$. The peak torque is around 48 Nm (recall that the robot's total mass is 40 kg). A movie of the walking motion is included on the CD-ROM with the paper (Movie_2_grizzle.mpg).

$J(a)$ (N^2m)	δ_{zero}^2 -	$\kappa(\theta^-)$ $(kgm^2/s)^2$	K $(kgm^2/s)^2$
36.79	0.638	354.4	−260.4

TABLE I

Optimization results

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