On Observer-Based Feedback Stabilization of Periodic Orbits in Bipedal Locomotion

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***Abstract*— This communication develops an observer-based feedback controller for the stabilization of periodic orbits arising in bipedal robots. The robot is modeled as a system with impulse effects and it is assumed that the robot's configuration variables are measured and that the moment of impact of the swing leg with the ground can be detected as well. It is shown that if a continuously differentiable static state variable feedback controller exists that induces an exponentially stable periodic orbit, then the same orbit can be exponentially stabilized with a continuously differentiable output feedback controller.**

I. Introduction

The lack of velocity measurements is as ubiquitous a problem in the design of feedback controllers for bipedal robots as it is for robotic manipulators. The presence of impacts at leg contact with the ground is a fundamental issue that must be faced in legged robots when attempting to reconstruct velocity estimates on the basis of position measurements. The impulsive forces at impact cause jumps in the velocity variables of the robot [17], [16]. In the study of walking or running gaits, one is often seeking a periodic solution, in which case the impacts cannot be treated as an infrequent disturbance. The objective of this communication is to prove that the observer designs proposed in [22], [11] can be tuned to preserve exponential stability of periodic walking and running gaits created with the full-state feedback designs of [13], [31], [33], [8], [9], [34], when implemented with a smooth feedback controller, as in [23].

An extensive literature exists addressing the modeling, analysis and control of systems with impacts; see [4], [5], [26], [15], [28], [10] and references therein. Only relatively recently, however, have the design and analysis of observers for systems with impacts been undertaken. The papers [21], [22], [11] consider mechanical systems with repeated rigid impacts and assume that the configuration variables are measured, but not the velocity variables. Conditions are established under which the error in the estimated velocities converges to zero.

In the context of bipedal robots, closed-loop stability of observer-based feedback has been addressed in [19], [20], where only the shape configuration variables are assumed measured; in particular, the orientation of the robot with respect to a world frame has to be estimated as well as all velocity components. The employed observer is non-smooth and converges in finite time.

The present communication provides a continuously differentiable observer-based feedback controller for achieving exponentially stable periodic gaits in bipedal robots. It is assumed that the configuration variables are measured (i.e., shape variables and absolute orientation) *and that the moment of impact is detected as well*. The latter assumption is very reasonable for legged robots. Indeed, because it is crucial that any controller quickly take into account the respective change in the roles of the two legs (i.e., stance and swing phases), a force sensor or a contact switch [6], [33], [24] is usually provided to detect impacts of the feet with the ground.

A few example robots to which the theory of the paper applies are depicted in Fig. 1. Though not explicitly addressed here, the feedback controllers of [7], [30], [1] can be modified to include velocity estimation as well.

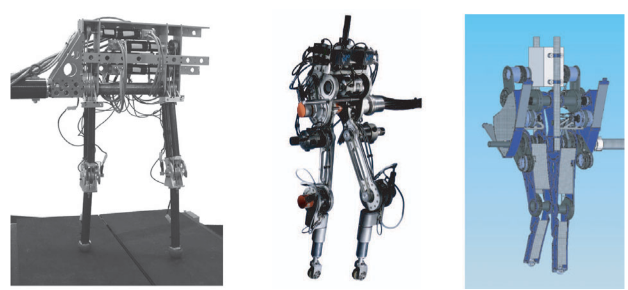

Fig. 1. Example robots to which the observer theory developed in this communication may be applied. On the left is Ernie [32], in the center is RABBIT [6] and on the right is a robot under construction by the University of Michigan and Carnegie Mellon University [18].

II. System Model

We nominally consider any robot model that can be expressed as a system with impulse effects

$$\Sigma : \begin{cases} \dot{x} &= f(x) + g(x)u \quad x^- \notin \mathcal{S} \\ x^+ &= \Delta(x^-) \quad\quad\quad\;\; x^- \in \mathcal{S}. \end{cases} \tag{1}$$

The elements of this model are as follows.

MH1) The continuous dynamics

$$\dot{x} = f(x) + g(x)u \tag{2}$$

is assumed to arise from a smooth (at least continuously differentiable) mechanical model of the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u, \tag{3}$$

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where u represents the set of actuator inputs[1]. The configuration coordinates of the robot are denoted by $q = (q_1; \cdots; q_N) \in \mathcal{Q}$, the state $x := (q; \dot{q})$ takes values in $\mathcal{X}$ and is assumed to be a simply connected open subset of $\mathbb{R}^{2N}$ corresponding to physically reasonable positions and velocities. The vector fields f and g are defined in the obvious manner.

MH2) The impact (or switching) surface $\mathcal{S}$ is an embedded smooth codimension-one submanifold of $\mathcal{X}$,

$$\mathcal{S} := \{x \in \check{\mathcal{X}} \mid H(x) = 0\}, \tag{4}$$

where $\check{\mathcal{X}}$ is an open subset of $\mathcal{X}$ and $H : \check{\mathcal{X}} \to \mathbb{R}$ is at least continuously differentiable and has constant rank on $\mathcal{S}$.

MH3) The impact (or reset) map $\Delta : \check{\mathcal{X}} \to \mathcal{X}$ is continuously differentiable[2]. When formulating the model (1), the impact map Δ may arise in several ways [34]. In [13], [31], [33] it models the outcome of the impact of two rigid bodies [16] representing the swing leg and the ground, whereas in [8], [9], [27], it represents the composition of a second continuous phase of the dynamics followed by an impact with the ground.

In simple terms, a solution of (1) is specified by the differential equation (2) until its state "impacts" the hyper surface $\mathcal{S}$ at some time t_I. At t_I, the impact map Δ compresses the impact event into an instantaneous moment of time, resulting in a discontinuity in the state trajectory. The impact model provides the new initial condition from which the differential equation evolves until the next impact with $\mathcal{S}$. In order to avoid the state having to take on two values at the "impact time" t_I, the impact event is, roughly speaking, described in terms of the values of the state "just prior to impact" at time "t_I^-", and "just after impact" at time "t_I^+". These values are represented by x^- and x^+, respectively. From this description, a formal definition of a solution is easily written down by piecing together appropriately initialized solutions of (2); see [35], [13], [14], [34].

The final assumption deals with the existence of a controller inducing stable periodic motion.

CH1) There exists a continuously differentiable state feedback controller $u_{\rm fb}(x)$ such that the closed-loop system

$$\Sigma_{\rm fb} : \begin{cases} \dot{x} &= f(x) + g(x)u_{\rm fb}(x) \quad x^- \notin \mathcal{S} \\ x^+ &= \Delta(x^-) \quad\quad\quad\quad\quad\; x^- \in \mathcal{S} \end{cases} \tag{5}$$

has an exponentially stable periodic orbit transversal[3] to $\mathcal{S}$. For later use, the periodic orbit is denoted $\mathcal{O} = \{x^*(t) \mid 0 \leq t < t^*\}$, where the (least) period is $t^* > 0$. It is further assumed that $\overline{\mathcal{O}} \cap \mathcal{S}$ is a singleton.

How to design such feedbacks is the topic of [23], [34], [1], [29], [30] as well as [13], [31], [33], [8], [9].

III. Two Observers and Elementary Properties

Two observer-based implementations of (5) are developed under the assumption that the configuration variables are measured,

$$y_{\rm M} = h_{\rm M}(x) := q,$$

the impact moments are detected, and Hypotheses MH1-MH3 and CH1 are satisfied. The structure of the observers is taken from [22], [11]. The method of analysis of the closed-loop system is based on [23].

A. Full-order observer

The full-order observer consists of a copy of the model (1) plus output injection, which may be designed as in [12] to be a constant matrix[4] depending on ϵ multiplying the output estimation error:

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x})u + L(\epsilon)\,(y_{\rm M} - \hat{y}_{\rm M}) \quad x^- \notin \mathcal{S} \\ \hat{y}_{\rm M} &= h_{\rm M}(\hat{x}) \\ \hat{x}^+ &= \Delta(\hat{x}^-) \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x^- \in \mathcal{S}, \end{cases} \tag{6}$$

where[5]

$$L(\epsilon) := \begin{bmatrix} \frac{\beta_0}{\epsilon} I \\ \frac{\beta_1}{\epsilon^2} I \end{bmatrix}, \quad \beta_0 = 1, \quad \beta_1 = 2. \tag{7}$$

Note that the switching condition is expressed in terms of the state of (1), consistent with the hypothesis that the impact instants of (1) are detected. When integrated with (5), we obtain the model

$$\Sigma_{cl} : \begin{cases} \dot{x} &= f(x) + g(x)u_{\rm fb}(\hat{x}) \quad x^- \notin \mathcal{S} \\ \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x})u_{\rm fb}(\hat{x}) \\ & \quad + L(\epsilon)\,(y_{\rm M} - \hat{y}_{\rm M}) \\ y_{\rm M} &= h_{\rm M}(x) \\ \hat{y}_{\rm M} &= h_{\rm M}(\hat{x}) \\ x^+ &= \Delta(x^-) \quad\quad\quad\quad\quad\; x^- \in \mathcal{S} \\ \hat{x}^+ &= \Delta(\hat{x}^-), \end{cases} \tag{8}$$

where, through abuse of notation, the switching condition is expressed as $x^- \in \mathcal{S}$ instead of $(x^-, \hat{x}^-) \in \mathcal{S} \times \mathcal{X}$.

Define the estimation error $e := x - \hat{x}$; then $\hat{x} = x - e$ and the model (8) becomes

$$\Sigma_{cl} : \begin{cases} \dot{x} &= f(x) + g(x)u_{\rm fb}(x - e) \quad x^- \notin \mathcal{S} \\ \dot{e} &= \hat{F}^\epsilon(x, e) \\ y_{\rm M} &= h_{\rm M}(x) \\ \hat{y}_{\rm M} &= h_{\rm M}(x - e) \\ x^+ &= \Delta(x^-) \quad\quad\quad\quad\quad\quad\; x^- \in \mathcal{S} \\ e^+ &= \hat{\Delta}(x^-, e^-), \end{cases} \tag{9}$$

where

$$\begin{aligned} \hat{F}^\epsilon(x, e) &:= f(x) - f(x - e) + (g(x) - g(x - e)) \\ &\quad\; \cdot u_{\rm fb}(x - e) - L(\epsilon)\,(y_{\rm M} - \hat{y}_{\rm M}) \end{aligned} \tag{10}$$

$$\hat{\Delta}(x^-, e^-) := \Delta(x^-) - \Delta(x^- - e^-). \tag{11}$$

[1] B is not required to have full rank.

[2] This is a stronger assumption than was made in [23], for example, where Δ was only assumed to be smooth on $\mathcal{S}$. The stronger assumption is needed for observer design because in general, $H(\hat{x}) \neq 0$ when $H(x) = 0$; see (8).

[3] See [23], [34] for definitions of these terms.

[4] The result of [12], though stated for scalar outputs, is easily extended to the case treated here.

[5] The more general gains of [2], [3] could also be considered, with $\beta_0 > 0$ and $\beta_1 > 0$.

Proposition 1: Assume Hypotheses MH1-MH3 and CH1. Then both $\hat{F}^\epsilon$ and $\hat{\Delta}$ are continuously differentiable and

$$\begin{aligned} \hat{F}^\epsilon(x,0) &\equiv 0 \quad (12)\\ \hat{\Delta}(x^-,0) &\equiv 0. \quad (13)\end{aligned}$$

Moreover, there exist $\delta > 0$, $\bar{\epsilon} > 0$ and a continuous function $K : (0,\bar{\epsilon}] \to \mathbb{R}$ such that $\lim_{\epsilon \searrow 0} K(\epsilon) = 0$ and for all initial conditions e_0 satisfying $||e_0|| < \delta$, the solution of the ordinary differential equation

$$\dot{e} = \hat{F}^\epsilon(x^*(t), e), \quad e(0) = e_0 \quad (14)$$

satisfies[6] $||e(t^*)|| \le K(\epsilon)||e_0||$.

Proof: Because the velocity components are obtained trivially by differentiating $y_{\rm M}$, the continuous portion of the system is observable. On any compact subset $\mathcal{K} \subset \mathcal{X}$ with $\overline{\mathcal{O}}$ contained in its interior, the Lipschitz conditions of [12, Thm. 3] are met. Following the Lyapunov argument of [12, Thm. 3] with

$$S_\infty(1/\epsilon) := \begin{bmatrix} \epsilon I & -\epsilon^2 I \\ -\epsilon^2 I & 2\epsilon^3 I \end{bmatrix},$$

one deduces that there exist $\delta > 0$ and $\bar{\epsilon} > 0$ such that for all $||e|| < \delta$ and $0 < \epsilon \le \bar{\epsilon}$,

$$\frac{d}{dt}\left(e' S_\infty(1/\epsilon) e\right)^{(1/2)} \le -\frac{1}{3\epsilon}\left(e' S_\infty(1/\epsilon) e\right)^{(1/2)},$$

which gives

$$||e(t)|| \le \sqrt{\frac{1+2\epsilon^2+(1+4\epsilon^4)^{(1/2)}}{1+2\epsilon^2-(1+4\epsilon^4)^{(1/2)}}}\exp(-\frac{t}{3\epsilon})||e_0||.$$

Evaluating at $t = t^*$ completes the proof. ■

It will be proved in Section IV that these properties imply that for $\epsilon > 0$ sufficiently small, the closed-loop system (8) has an exponentially stable periodic orbit.

B. Reduced-order observer

A reduced-order observer is now developed, based on [22]. Assuming the configuration variables q are available through $y_{\rm M} = q$, the equation governing the velocity states can be expressed with $\omega := \dot{q}$ and additional dummy states η as follows. Define η via

$$\omega = \eta + k y_{\rm M}, \quad (15)$$

where $k > 0$ is a scalar to be chosen. Differentiating the above equation along a solution of (3) yields

$$\begin{aligned} \dot{\eta} &= \dot{\omega} - k\dot{y}_m = D^{-1}(y_{\rm M})\{-C(y_{\rm M},\omega)\omega - G(y_{\rm M}) \\ &\quad + B(y_{\rm M})u\} - k\omega \\ &= D^{-1}(y_{\rm M})\{-C(y_{\rm M},(\eta+ky_{\rm M}))(\eta+ky_{\rm M}) - G(y_{\rm M}) \\ &\quad + B(y_{\rm M})u\} - k(\eta + k y_{\rm M}). \quad (16)\end{aligned}$$

The continuous portion of the reduced-order estimator is defined as

$$\begin{aligned} \dot{\hat{\eta}} &= D^{-1}(y_{\rm M})\{-C(y_{\rm M},(\hat{\eta}+ky_{\rm M}))(\hat{\eta}+ky_{\rm M}) - G(y_{\rm M}) \\ &\quad + B(y_{\rm M})u\} - k(\hat{\eta} + k y_{\rm M}) \quad (17)\\ \hat{\omega} &= \hat{\eta} + k y_{\rm M}.\end{aligned}$$

[6]Recall that t^* is the period of the orbit in Hypothesis CH1.

Defining error states as $e = \eta - \hat{\eta}$ leads to the error dynamics

$$\begin{aligned} \dot{e} &= \dot{\omega} - \dot{\hat{\omega}} - ke \quad (18)\\ &= -D^{-1}(y_{\rm M})\{C(y_{\rm M},\omega)\omega - C(y_{\rm M},\hat{\omega})\hat{\omega}\} - ke,\end{aligned}$$

and it follows that $e(t) \to 0$ implies $\hat{\omega}(t) \to \omega(t)$. Assuming a common choice for C,

$$C_{ij}(q,\omega) = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial D_{ij}}{\partial q_k} + \frac{\partial D_{ik}}{\partial q_j} - \frac{\partial D_{kj}}{\partial q_i}\right)\omega_k, \quad (19)$$

yields [25]

$$\begin{aligned} C(q,\omega_1+\omega_2) &= C(q,\omega_1) + C(q,\omega_2) \\ C(q,\omega_1)\ \omega_2 &= C(q,\omega_2)\ \omega_1 \\ C(q,\alpha\omega) &= \alpha C(q,\omega) \quad \alpha \in \mathbb{R} \quad (20)\\ \dot{D}(q) - 2C(q,\omega) &= \text{skew symmetric.}\end{aligned}$$

Substituting these into (18) gives

$$\dot{e} = -D^{-1}(y_{\rm M})\{2C(y_{\rm M},\omega)e - C(y_{\rm M},e)e\} - ke. \quad (21)$$

Finally, passing directly to the closed-loop system with the observer written in error coordinates, we have

$$\Sigma_{cl}: \begin{cases} \begin{bmatrix} \dot{q} \\ \dot{\omega} \end{bmatrix} = f(q,\omega) + g(q,\omega)u & \begin{bmatrix} q^- \\ \omega^- \end{bmatrix} \notin \mathcal{S} \\ \dot{e} = \hat{F}^\epsilon(y_{\rm M},\omega,e) & \\ y_{\rm M} = q & \\ u = u_{\rm fb}(y_{\rm M},\omega - e) & \\ \begin{bmatrix} q^+ \\ \omega^+ \end{bmatrix} = \Delta(q^-,\omega^-) & \begin{bmatrix} q^- \\ \omega^- \end{bmatrix} \in \mathcal{S} \\ e^+ = \hat{\Delta}(q^-,\omega^-,e^-), & \end{cases} \quad (22)$$

where

$$\begin{aligned} \hat{F}^\epsilon(y_{\rm M},\omega,e) &:= -D^{-1}(y_{\rm M})\{2C(y_{\rm M},\omega)e - \\ &\qquad C(y_{\rm M},e)e\} - ke \quad (23)\\ \hat{\Delta}(q^-,\omega^-,e^-) &:= \Delta_{\dot{q}}(q^-,\omega^-) - \Delta_{\dot{q}}(q^-,\omega^- - e^-),\end{aligned}$$

and $\Delta_{\dot{q}}$ is the velocity component of the impact map, Δ.

Proposition 2: Assume Hypotheses MH1-MH3 and CH1. Then both $\hat{F}^\epsilon$ and $\hat{\Delta}$ are continuously differentiable and

$$\begin{aligned} \hat{F}^\epsilon(y_{\rm M},\omega,0) &\equiv 0 \quad (24)\\ \hat{\Delta}(q^-,\omega^-,0) &\equiv 0. \quad (25)\end{aligned}$$

Moreover, setting $k = \frac{1}{\epsilon}$ and letting $(y^*_{\rm M}(t),\omega^*(t)) = x^*(t)$, $0 \le t < t^*$, denote the periodic orbit of (5), there exist $\delta > 0$, $\bar{\epsilon} > 0$ and a continuous function $K : (0,\bar{\epsilon}] \to \mathbb{R}$ such that $\lim_{\epsilon \searrow 0} K(\epsilon) = 0$ and for all initial conditions e_0 satisfying $||e_0|| < \delta$, the solution of the ordinary differential equation

$$\dot{e} = \hat{F}^\epsilon(y^*_{\rm M}(t),\omega^*(t),e), \quad e(0) = e_0 \quad (26)$$

satisfies $||e(t^*)|| \le K(\epsilon)||e_0||$.

Proof:

Let $V(e) = e' D(y^*_{\rm M}(t))e$. Then, after some algebra,

$$\dot{V}(e) = -\frac{2}{\epsilon}e'\left\{D(y^*_{\rm M}(t)) + \epsilon C(y^*_{\rm M}(t),\omega^*(t) - e)\right\}e. \quad (27)$$

Because $D(q)$ is everywhere continuous and positive definite, and the periodic orbit is bounded, there exist $0 < \mu_1 \leq \mu_2 < \infty$ such that, for all e and $0 \leq t < t^*$,

$$\mu_1 e'e \leq e' D(y^*_{\rm M}(t))e \leq \mu_2 e'e.$$

By continuity, there exist $\delta > 0$ and $\bar{\epsilon} > 0$ such that, for all $||e|| \leq \delta$, $0 < \epsilon \leq \bar{\epsilon}$, $0 \leq t < t^*$,

$$\frac{1}{2}\mu_1 e'e \leq e' \left\{D(y^*_{\rm M}(t)) + \epsilon C(y^*_{\rm M}(t), \omega^*(t) - e)\right\} e \leq 2\mu_2 e'e. \tag{28}$$

Therefore, (27) implies

$$V(e(t)) \leq \exp(-\frac{\mu_1}{\mu_2}\frac{t}{\epsilon})V(e_0), \tag{29}$$

and hence

$$||e(t)||^2 \leq \frac{\mu_2}{\mu_1}\exp(-\frac{\mu_1}{\mu_2}\frac{t}{\epsilon})||e_0||^2. \tag{30}$$

Evaluating at $t = t^*$ and taking the square root of both sides complete the proof. ■

IV. Main Convergence Result

An immediate corollary of the following theorem is that for $\epsilon > 0$ sufficiently small, if the full-state feedback used in (5) is implemented with either the full-order or the reduced-order observer of Sec. III, then exponential stability of the closed-loop system will be preserved. In particular, for $\epsilon > 0$ sufficiently small, the observer-based closed-loop systems (8), (9) and (22) will have an exponentially stable periodic orbit whenever (5) does.

Theorem 1: Consider a system with impulse effects that depends on a real parameter $\epsilon > 0$,

$$\Sigma^\epsilon_{\rm ext} : \begin{cases} \dot{x} = & F(x, e) \qquad x^- \notin \mathcal{S} \\ \dot{e} = & \hat{F}^\epsilon(x, e) \\ x^+ = & \Delta(x^-) \qquad x^- \in \mathcal{S} \\ e^+ = & \hat{\Delta}(x^-, e^-), \end{cases} \tag{31}$$

with state manifold $\mathcal{X} \times \mathcal{E}$, where $\mathcal{X}$ and $\mathcal{E}$ are open connected subsets $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ for some $n_1 > 0$ and $n_2 > 0$. Suppose that $\mathcal{S}$ is an embedded codimension-one C^1-submanifold of $\mathcal{X}$ and that for each value of $\epsilon > 0$, F, $\hat{F}^\epsilon$, Δ and $\hat{\Delta}$ are continuously differentiable. In addition, suppose that the following structural hypotheses are met:

CLH1) For all x and ϵ, $\hat{F}^\epsilon(x, 0) = 0$ and $\hat{\Delta}(x^-, 0) = 0$;

CLH2) The system with impulse effects

$$\Sigma : \begin{cases} \dot{x} = & F(x, 0) \qquad x^- \notin \mathcal{S} \\ x^+ = & \Delta(x^-) \qquad x^- \in \mathcal{S}, \end{cases} \tag{32}$$

has an exponentially stable period-one orbit transversal to $\mathcal{S}$. Let t^* denote the period and let $x^*(t)$ denote the periodic solution of (32).

CLH3) There exist $\delta > 0$, $\bar{\epsilon} > 0$ and a continuous function $K : (0, \bar{\epsilon}] \to \mathbb{R}$ such that $\lim_{\epsilon \searrow 0} K(\epsilon) =$ and for all initial conditions e_0 satisfying $||e_0|| < \delta$, the solution of the ordinary differential equation

$$\dot{e} = \hat{F}^\epsilon(x^*(t), e), \quad e(0) = e_0 \tag{33}$$

satisfies $||e(t^*)|| \leq K(\epsilon)||e_0||$.

Then there exists $\epsilon^* > 0$ such that for all $0 < \epsilon \leq \epsilon^*$, $\mathcal{O}_{\rm ext} = \{(x^*(t), 0) \mid 0 \leq t \leq t^*\}$ is an exponentially stable periodic orbit of (31).

The proof is given in the appendix.

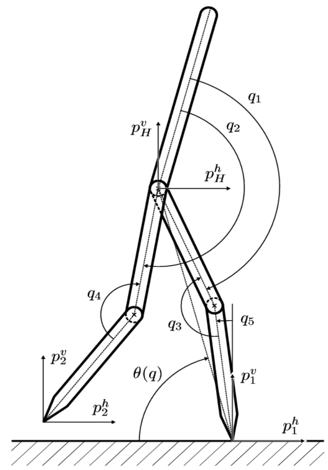

Fig. 2. Example 5-link robot. The parameter values are given in [31, Table 1].

V. Numerical Example

The reduced-order observer is illustrated on the underactuated, 5-link, planar, bipedal robot of [31, Sect. VII]; see Fig. 2. The system has five degrees of freedom in single support, and hence five velocity components to estimate. A state feedback was designed on the basis of virtual constraints as in [31], and then implemented with an input-output linearizing controller as in [23]. The system with state feedback has an exponentially stable periodic orbit, as depicted in Fig. 3. The state feedback was then implemented using the reduced-order observer, with a gain of $\epsilon = 0.3$. The asymptotic convergence of the velocity errors is shown in Fig. 4.

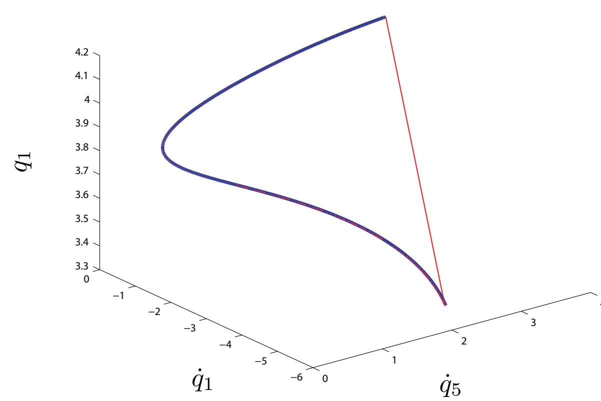

Fig. 3. Projection of a limit cycle of the 5-link robot using full-state feedback. The thin (red) line corresponds to the jump due to the impact map. The average walking rate is 1.04 ms^{-1}. The period is 0.51 s and the step length is 0.53 m. The generalized coordinates are in radians and radians per second.

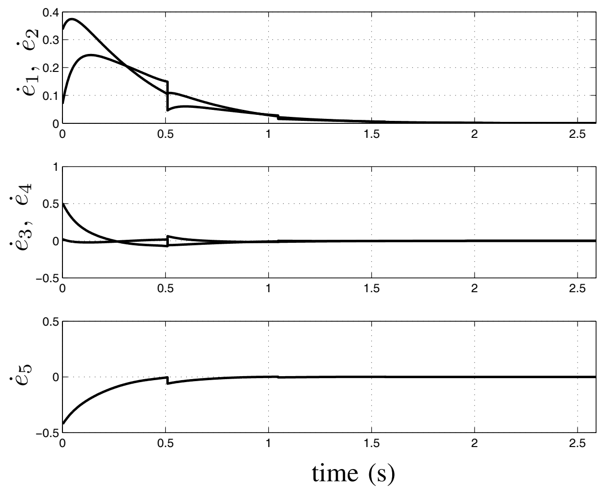

Fig. 4. Velocity errors (rad s^{-1}) of the 5-link robot with the reduced-order observer.

VI. Conclusion

This communication has investigated the use of observer-based feedback control for the stabilization of periodic orbits arising in walking and running of a bipedal robot. It was assumed that the robot's configuration variables were measured and the moment of impact of the swing leg with the ground could be detected. It was then proved that if a continuously differentiable state variable feedback controller existed that induced an exponentially stable periodic orbit, then the same orbit could be exponentially stabilized with a continuously differentiable output feedback controller, based on an observer. The observer error had to be converging to zero sufficiently rapidly. The theoretical results were based on attractive invariant manifolds in systems with impulse effects. Simulation results supported the theoretical analysis.

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VII. Appendix: Proofs

The objective of this Appendix is to prove Theorem 1. This is accomplished as follows: Theorem 2 of [23] is extended to allow nonlinear terms in the transverse dynamics; the notation of [23, Thm. 2] is preserved so that only the required changes to the otherwise rather long proof need be given. The extended result is then applied to Theorem 1 in order to prove the stability of an observer-based feedback controller.

A. Extension of Theorem 2 of [23]

Consider a system with impulse effects that depends on a real parameter $\epsilon > 0$,

$$\Sigma^\epsilon : \begin{cases} \dot{x} = & f^\epsilon(x) \qquad x^- \notin \mathcal{S} \\ x^+ = & \Delta(x^-) \quad x^- \in \mathcal{S}, \end{cases} \tag{34}$$

and suppose that for each value of $\epsilon > 0$, hypotheses H1 hold:

H1.1) $\mathcal{X} \subset I\!R^n$ is open and connected,

H1.2) $f^\epsilon : \mathcal{X} \to \mathbb{R}^n$ is C^1,

H1.3) $H : \check{\mathcal{X}} \to I\!R$ is C^1, for $\check{\mathcal{X}}$ an open subset of $\mathcal{X}$,

H1.4) $\mathcal{S} := \{x \in \check{\mathcal{X}} \mid H(x) = 0\}$ is non-empty and $\forall x \in \mathcal{S}$, $\left.\frac{\partial H}{\partial x}\right|_x \neq 0$ (that is, $\mathcal{S}$ is C^1 and has co-dimension one),

H1.5) $\Delta : \mathcal{S} \to \mathcal{X}$ is C^1, and

H1.6) $\Delta(\mathcal{S}) \cap \mathcal{S} = \emptyset$.

In addition, suppose that the following structural hypotheses H2 are met:

H2.1) there exist global coordinates $x = (z, \eta)$ for $\mathcal{X} \subset I\!R^n$, $z \in I\!R^k$, and $\eta \in I\!R^{n-k}$, $1 < k < n$, in which f^ϵ has the form

$$f^\epsilon(x) := f^\epsilon(z, \eta) := \begin{bmatrix} f_{1:k}(z, \eta) \\ f^\epsilon_{k+1:n}(z, \eta) \end{bmatrix},$$

with $f_{1:k}(z, \eta)$ independent of ϵ and $f^\epsilon_{k+1:n}(z, 0) \equiv 0$;

H2.2) For $\mathcal{Z} := \{(z, \eta) \in \mathcal{X} \mid \eta = 0\}$, $\mathcal{S} \cap \mathcal{Z}$ is a $(k-1)$-dimensional, C^1-embedded submanifold of $\mathcal{Z}$, and

$$\Delta(\mathcal{S} \cap \mathcal{Z}) \subset \mathcal{Z}; \tag{35}$$

H2.3) (34) has a periodic orbit $\mathcal{O}$ that is contained in $\mathcal{Z}$, and hence the orbit is independent of ϵ;

H2.4) $x^* := \overline{\mathcal{O}} \cap \mathcal{Z}$ is a singleton;

H2.5) $L_{f^\epsilon} H(x^*) \neq 0$;

H2.6) there exists $\delta > 0$, $\bar{\epsilon} > 0$ and $K : (0, \bar{\epsilon}] \to [0, \infty)$ with $\lim_{\epsilon \searrow 0} K(\epsilon) = 0$, such that for all $||\eta_0|| \leq \delta$, the solution of the ordinary differential equation

$$\dot{\eta} = f^\epsilon_{k+1:n}(z^*(t), \eta), \quad \eta(0) = \eta_0 \tag{36}$$

satisfies

$$||\eta(t^*)|| \leq K(\epsilon)||\eta_0||, \tag{37}$$

where $(z^*(t), 0)$, $t \in [0, t^*)$ is the periodic orbit of Hypothesis H2.3 and t^* is its (least) period.

Hypotheses H2.1 and H2.2 imply that the restriction of Σ^ϵ to the manifold $\mathcal{Z}$ is a well-defined system with impulse effects, called the *restriction dynamics*, $\Sigma_{\mathcal{Z}}$,

$$\Sigma_{\mathcal{Z}} : \begin{cases} \dot{z} = & f_{\mathcal{Z}}(z) \qquad z^- \notin \mathcal{S} \cap \mathcal{Z} \\ z^+ = & \Delta_{\mathcal{Z}}(z^-) \quad z^- \in \mathcal{S} \cap \mathcal{Z} \end{cases}, \tag{38}$$

where $f_{\mathcal{Z}}(z) := f^\epsilon(z, 0)_{|\mathcal{Z}} = f_{1:k}(z, 0)$, and $\Delta_{\mathcal{Z}} := \Delta(z, 0)_{|\mathcal{Z}}$. There is an obvious bijection between periodic orbits of (38) and periodic orbits of (34) that lie in $\mathcal{Z}$.

Theorem 2: Suppose that (34) satisfies Hypotheses H1 and H2. Then there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*]$, $\mathcal{O}$ is an exponentially stable periodic orbit of (34) lying in $\mathcal{Z}$ if, and only if, its restriction to $\mathcal{Z}$, denoted $\mathcal{O}_{\mathcal{Z}}$, is an exponentially stable periodic orbit of (38).

Remark 1: The result can be equivalently stated in terms of Poincaré maps; see [23].

B. Proof of the Extension of Theorem 2

The proof requires only a slight modification to the proof of Theorem 2 in [23, pp. 4202], where Hypothesis H2.6 was given as $f^{\epsilon}_{k+1:n}(\eta) = A(\epsilon)\eta$, and $\lim_{\epsilon \searrow 0} e^{A(\epsilon)} = 0$. The changes are:

a) On page 4203 of [23], remove part (iii) of Lemma 1.

b) On page 4203 of [23], in (24d) and (29), replace $e^{A(\epsilon)t^*}$ by

$$Q_{33}(\epsilon) := \frac{\partial}{\partial \eta_0} \tilde{\varphi}^{\epsilon}(t, \eta_0)\Big|_{t=t^*,\ \eta_0=0}, \qquad (39)$$

where $\tilde{\varphi}^{\epsilon}$ is the solution of (36).

c) Use Taylor's Theorem to show that the modified Hypothesis H2.6 implies that $\lim_{\epsilon \searrow 0} Q_{33}(\epsilon) = 0$, and hence M^{ϵ}_{22} in (29) of [23, Lemma 3] and [23, Sect. IV-C] goes to zero as ϵ goes to zero.

This concludes the required changes.

C. Proof of Theorem 1

The proof follows immediately from Theorem 2. The system Σ^{ϵ} of (34) is given by $\Sigma^{\epsilon}_{\text{ext}}$ in (31); it has state manifold $\mathcal{X}_{\text{ext}} := \mathcal{X} \times \mathcal{E}$ and impact surface $\mathcal{S}_{\text{ext}} := \mathcal{S} \times \mathcal{E}$. The smoothness Hypotheses H1.1 through H1.6 of Theorem 2 are contained in the first part of Theorem 1. The hypotheses H2.1 through H2.6 of Theorem 2 are now evaluated. Hypotheses H2.1 and H2.2 are immediate from (31) and Hypothesis CLH1. Hypotheses H2.3-H2.5 follow from CLH2. Hypothesis H2.6 is given by CLH3. The conclusion of Theorem 2 establishes that of Theorem 1.

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