Continuous-Time Controllers for Stabilization of Periodic Orbits for Hybrid Systems: Application to an Underactuated 3D Bipedal Robot

Kaveh Akbari Hamed, Brian G. Buss, and J. W. Grizzle

I. Proof of Theorem 1

According to the invariance condition, $T(x_i^*,\xi)=T^*$ for all $\xi\in\Xi$. This fact together with (14) implies that the Jacobian of the Poincaré return map can be expressed as

$$\begin{aligned}\mathrm{D}_1 P(x_f^*,\xi) &= \mathrm{D}_1\,\varphi(T^*,x_i^*,\xi)\,\mathrm{D}_1\,T(x_i^*,\xi)\,\mathrm{D}\,\Delta(x_f^*)\\ &\quad+\mathrm{D}_2\,\varphi(T^*,x_i^*,\xi)\,\mathrm{D}\,\Delta(x_f^*).\end{aligned} \tag{33}$$

In our notation for a $\mathcal{C}^1$ function $h(z_1,\cdots,z_r)$,

$$\mathrm{D}_j\,h(z_1,\cdots,z_r):=\frac{\partial h}{\partial z_j}(z_1,\cdots,z_r),\quad j=1,\cdots,r.$$

Furthermore,

$$\begin{aligned}\mathrm{D}_1\,\varphi(T^*,x_i^*,\xi)&=\dot\varphi(T^*,x_i^*,\xi)\\ &=f^{\mathrm{cl}}\left(\varphi(T^*,x_i^*,\xi),\xi\right)\\ &=f^{\mathrm{cl}}(x_f^*,\xi)\\ &=f^{\mathrm{cl}}(x_f^*,\xi^*),\end{aligned} \tag{34}$$

in which in the last equality, we have made use of the invariance condition. $\mathrm{D}_2\,\varphi(T^*,x_i^*,\xi)$ can also be expressed as

$$\begin{aligned}\mathrm{D}_2\,\varphi(T^*,x_i^*,\xi)&=\frac{\partial\varphi}{\partial x}(T^*,x_i^*,\xi)\\ &=\Phi(T^*,x_i^*,\xi)\\ &=\Phi_f^*(\xi).\end{aligned} \tag{35}$$

From the switching and invariance conditions,

$$s(\varphi(T^*,x_i^*,\xi))=0,\quad\forall\xi\in\Xi$$

which together with the Implicit Function Theorem implies that

$$s(\varphi(T(x,\xi),x,\xi))=0 \tag{36}$$

for all x in an open neighborhood of x_i^* and all $\xi\in\Xi$. Differentiating (36) with respect to x around (x_i^*,ξ) results in

$$\begin{aligned}&\mathrm{D}\,s(x_f^*)\,\mathrm{D}_1\,\varphi(T^*,x_i^*,\xi)\,\mathrm{D}_1\,T(x_i^*,\xi)\\ &\qquad+\mathrm{D}\,s(x_f^*)\,\mathrm{D}_2\,\varphi(T^*,x_i^*,\xi)=0\end{aligned}$$

which in combination with (34), (35) and the transversality assumption results in

$$\mathrm{D}_1\,T(x_i^*,\xi)=-\frac{\frac{\partial s}{\partial x}(x_f^*)\,\Phi_f^*(\xi)}{\frac{\partial s}{\partial x}(x_f^*)\,f^{\mathrm{cl}}(x_f^*,\xi^*)}. \tag{37}$$

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K. Akbari Hamed. B. G. Buss, and J. W. Grizzle are with the Electrical Engineering and Computer Science Department of the University Michigan, Ann Arbor, MI, USA, {kavehah,bgbuss,grizzle}@umich.edu

Next, replacing (37) in (33) follows that

$$\frac{\partial P}{\partial x}(x_f^*,\xi)=\Pi(x_f^*,\xi^*)\,\Phi_f^*(\xi)\,\Upsilon(x_f^*). \tag{38}$$

In particular, from (38), the Jacobian of the Poincaré map, i.e., $\frac{\partial P}{\partial x}(x_f^*,\xi)$, depends on ξ only through the final state trajectory matrix $\Phi_f^*(\xi)$. One immediate consequence of (38) is that

$$\frac{\partial^2 P}{\partial\xi_i\partial x}(x_f^*,\xi^*)=\Pi(x_f^*,\xi^*)\,\frac{\partial\Phi_f^*}{\partial\xi_i}(\xi^*)\,\Upsilon(x_f^*)$$

for $i=1,\cdots,p$ which completes the proof.

II. Proof of Theorem 2, Part 1

We claim there exists a $B\in\mathbb{R}^{n\times np}$ matrix such that for all $\Delta\xi\in\mathbb{R}^p$,

$$\sum_{i=1}^{p}A_i\,\Delta\xi_i=B\,\left(I\otimes\Delta\xi\right). \tag{39}$$

To show this, let us partition the B matrix as follow

$$B=\begin{bmatrix}B_1 & B_2 & \cdots & B_n\end{bmatrix},$$

where $B_j\in\mathbb{R}^{n\times p}$ for $j=1,\cdots,n$. From the definition of the Kronecker product,

$$B\,\left(I\otimes\Delta\xi\right)=\begin{bmatrix}B_1 & \cdots & B_n\end{bmatrix}\begin{bmatrix}\Delta\xi & \cdots & 0\\ 0 & \cdots & 0\\ \vdots & \cdots & \vdots\\ 0 & \cdots & \Delta\xi\end{bmatrix}.$$

Hence, the j-th column of $B\,(I\otimes\Delta\xi)$ becomes $B_j\,\Delta\xi$ for $j=1,\cdots,n$. In order to satisfy (39), one can conclude that

$$B_j\,\Delta\xi=\sum_{i=1}^{p}A_i(:,j)\,\Delta\xi_i, \tag{40}$$

where $A_i(:,j)$ represents the j-th column of A_i. Next, differentiating (40) with respect to $\Delta\xi$ together with $\frac{\partial\Delta\xi_i}{\partial\Delta\xi}=e_i^\top, i=1,\cdots,p$ yields

$$B_j=\sum_{i=1}^{p}A_i(:,j)\,e_i^\top,\quad j=1,\cdots,n \tag{41}$$

which completes the proof.