Orbital Stabilization of Underactuated Nonlinear Systems.*

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*This presentation is based on [10] and [7]

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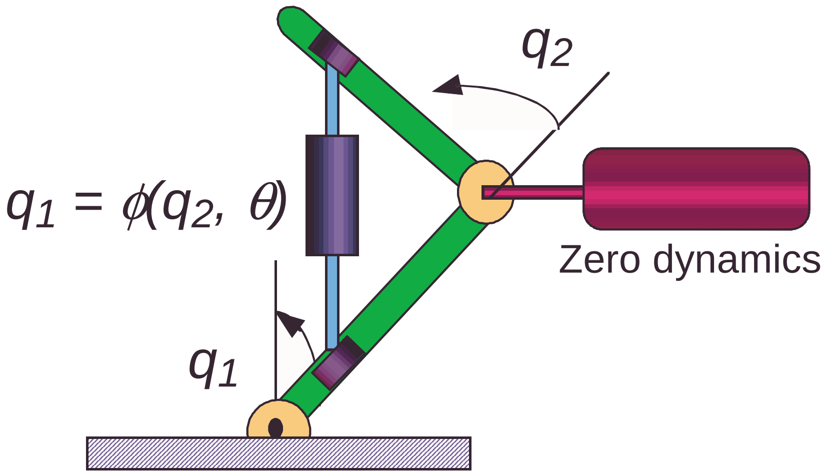
Previous Works

1. TRANSVERSAL LYAPUNOV FUNCTIONS. Hauser and Choo'94, [6].
2. ZERO DYNAMIC MATCHING, Grognard and Canudas'02, [5]; Canudas-Espiau-Urrea'02, [7]; Marconi-Isidory-Sarrani'02 [8].
3. HAMILTONIAN FORMALISM. Aracil-Gordillo-Acosta'02 [2], Vivas-Rubio'03 [4]
4. IN: Shiriaev and Canudas-de-Wit'03 [10]
   - Notion of Virtual Limit system
   - Linear time-varying controllability
   - Explicit use of integral forms for stabilization

Balancing: the problem



Create a sustained oscillation in the full robot coordinates at the single leg support configuration (high-dimensional inverted pendulum).

- Learn how to create stable orbits without impacts,
- A tool for transition strategy between different motions phases (i.e. run/walk),

Key steps

We consider underactuated Lagrangian systems

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = B(q)u$$

with a degree of underactuation equal to one.

- $n-1$ *virtual constraint* $\varphi(q,p) = 0$, with p, being parameter vector.
- The *virtually constrained system* has the form

  $$\alpha(\theta,p)\ddot{\theta} + \beta(\theta,p)\dot{\theta}^2 + \gamma(\theta,p) = 0$$

  which posses a conserved quantity $I = I(\dot{\theta}, \theta, \dot{\theta}(0), \theta(0), p)$
- Characterize the LTV controllability.
- Construct a (local) stabilizable control law.

Balancing vs Walking: What changes ?

CONCEPTUAL DIFFERENCES:

- Motion without impacts, then
- Cycles should be stabilized dynamically.
- "Passive" stable cycles do not exist.

TWO POSSIBLE METHODS:

- Matching a desired limit cycle exo-system,
- Use the trajectory controllability of the target cycle.

The matching solution

The parameter $p(t)$ may be time-varying i.e.

$$y = h_0(q) - h_d(\theta(q), p(t))$$

Find $h_d(\theta(q), p(t))$ and an adaptation law (dynamic feedback) for $\ddot{p}$, such that:

the zero dynamics

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \beta(x, h_d, p(t), \dot{p}(t), \ddot{p}(t)) \end{cases}$$

exhibits stable periodic behaviour, with stable solutions for $p(t)$.

- TARGET ORBIT (EXOSYSTEM) defines a target orbit $\Omega_d(x)$ defines a closed path in the plane.

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \beta_d(x) \end{cases}$$

- DYNAMIC MATCHING CONDITION. Solve for $\ddot{p}$, to satisfy

$$\beta(x, h_d, p, \dot{p}, \ddot{p}) = \beta_d(x)$$

while ensuring boundedness of the obtained solutions for $p(t)$.

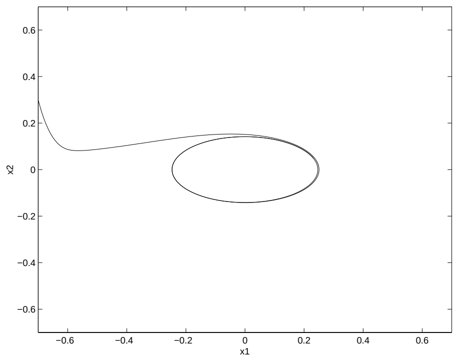
Internal Stability Issues

- IS THE INTERNAL DYNAMICS OF $p(t)$ STABLE ? average equation for $\bar{p}(t) = \frac{1}{T}\int p(\tau)d\tau$

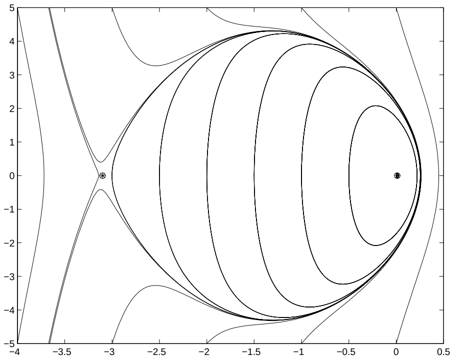
$$\ddot{\bar{p}} + \rho_1\dot{\bar{p}} + \rho_2\dot{\bar{p}}^2 + \rho_3 g(\bar{p}) = 0$$

- INTEGRAL FORM

$$I = f((\bar{p}, \dot{\bar{p}}, \bar{p}(0), \dot{\bar{p}}(0)) = 0$$

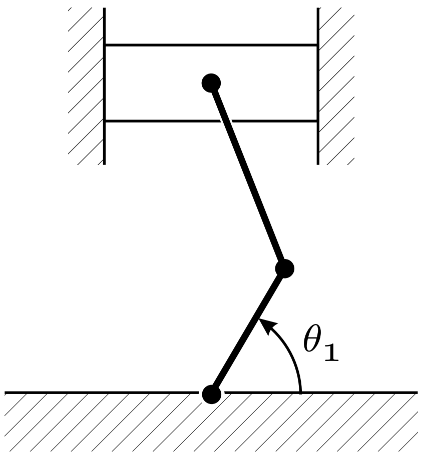


Evolution of $(x, \dot{x})$



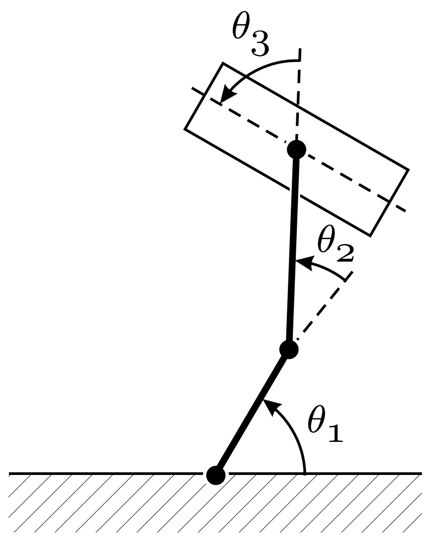
Evolution of $(\bar{p}, \dot{\bar{p}})$

II. Virtual constrains

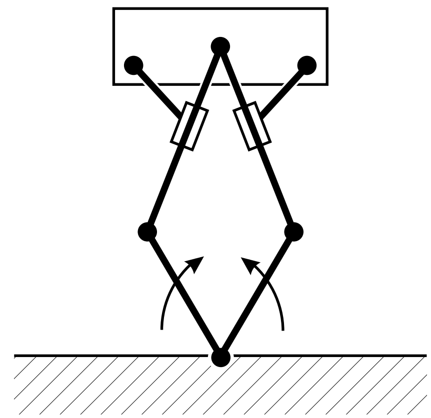


1-DOF Naturally constrained system

$y \equiv 0$



3-DOF Free system



Feedback-constrained 1-DOF system

$y \rightarrow 0$

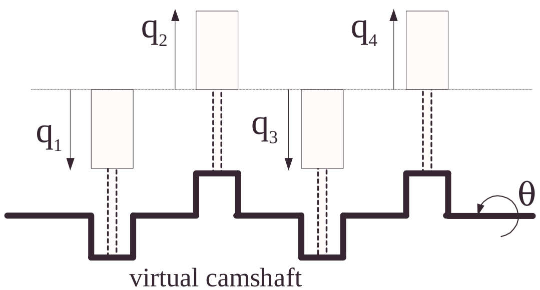
Implicit form

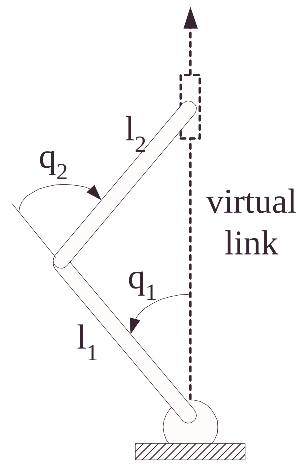
$$
\begin{aligned}
y_1 &= L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) = 0 \\
y_2 &= \theta_1 + \theta_2 + \theta_3 - \pi = 0.
\end{aligned}
$$

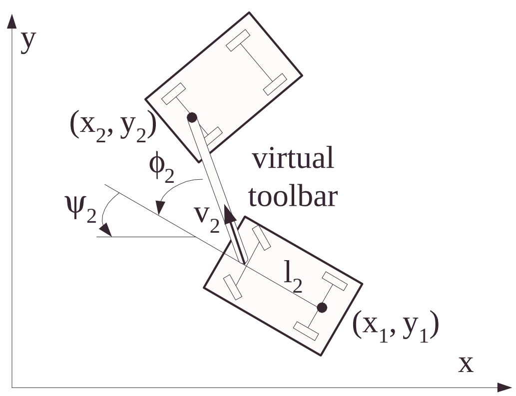
Explicit form

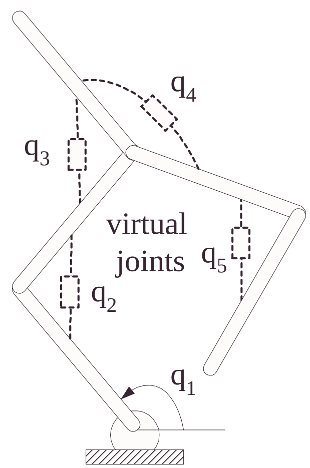
$$
\begin{aligned}
y_1 &= \theta_2 - (\pi - \theta_1 - \arccos(\tfrac{L_1}{L_2}\cos(\theta_1))) = 0 \\
y_2 &= \theta_3 - \arccos(\tfrac{L_1}{L_2}\cos(\theta_1)) = 0.
\end{aligned}
$$

Virtual constraints: more examples

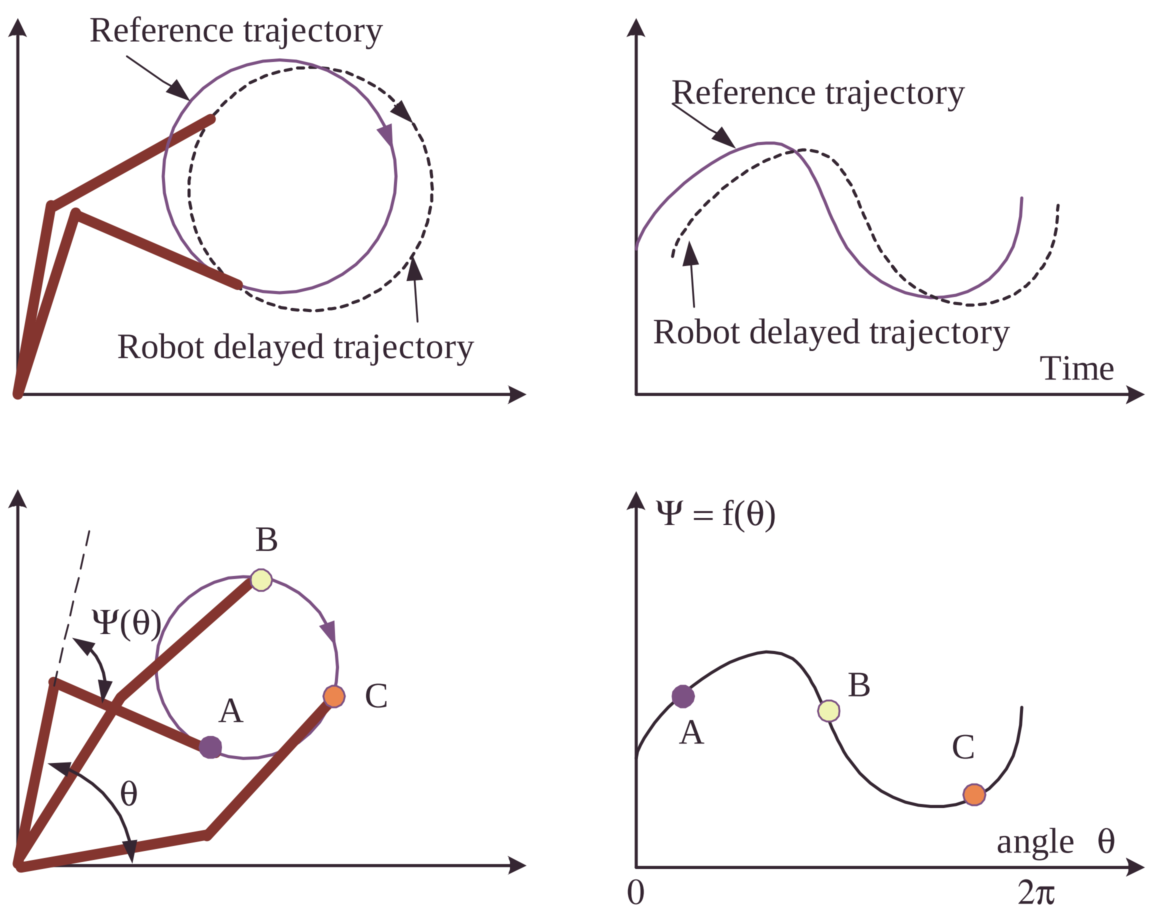








Time-invariance of periodic motions



Time invariance implies $\Psi = f(\theta)$

Multi-dimensional robots manipulators with open chain structure

Assuming that the link in contact with the ground is clamped, the equations of motion if a n-degree manipulator, with $(n-1)$ inputs is:

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Bu; \qquad \mathrm{rank}(B) = (n-1)$$

Let $\theta = q_k$ for some $k \in \{1, 2, \ldots, N\}$. Let the constraints be of the form

$$\varphi_i = q_i - \sum_{j=0}^{N} p_{(i,j)}\theta^j = 0, \quad \forall i = 1, \ldots, n, \quad i \neq k.$$

The zero dynamic

$$\alpha(\theta,p)\ddot{\theta} + \beta(\theta,p)\dot{\theta}^2 + \gamma(\theta,p) = 0$$

For the particular case where θ is chosen to be cyclic, then the above equation has the equivalent form

$$\begin{aligned} \dot{\theta} &= \tfrac{1}{I(\theta)}\sigma \\ \dot{\sigma} &= f(\theta) \end{aligned} \tag{1}$$

Else, In general (i.e. Pendubot) $f = f(\theta, \sigma)$ depends on σ as well. Implication of selection a cyclic coordinate is that the above equation can be integrated by eliminating the time;

$$\frac{d\sigma}{d\theta}\sigma = f(\theta) \cdot I(\theta) \quad \Longrightarrow \quad \frac{\sigma_1^2}{2} - \frac{\sigma_0^2}{2} = P_1 - P_0$$

where $P = \int_{\theta_0}^{\theta_1} f(s)I(s)ds$.

The function

$$\mathcal{L}_{VS} = K - P = \frac{\sigma^2}{2} - \int f(s)I(s)ds$$

that can be interpreted as the Lagrangian for the virtual system, is different to Lagrangian of the full mechanisms projected into the constraints.

Distinction between *physically* and *virtually* constrained systems

Consider a system of the general from:

$$\begin{aligned} \dot{x} &= f(x) + B(x)u, \\ y &= h(x). \end{aligned}$$

with $x \in R^n$, $y \in R^m$, and $u \in R^m$, $m < n$.

Lagrange reduction procedure:

$$\dot{x} = f(x) + B(x)u + \underbrace{J(x)\lambda}_{\text{extra inputs}}$$

where $J(x)^T = \frac{\partial h}{\partial x}(x)$. The reduced system has the form

$$\dot{x} = \left(I - J(x)J(x)^{\dagger}\right)[f(x) + B(x)u], \qquad J^{\dagger} = (J^T J)^{-1} J^T$$

Remark: the motion on this invariant manifold $(n - m)$ is still forced by the input u.

Systems subject to virtual constrains.

Use u in

$$\begin{aligned} \dot{x} &= f(x) + B(x)u, \\ y &= h(x). \end{aligned}$$

to zeroing the outputs y.

The resulting constrained dynamics is, in this case, given by

$$\dot{x} = \left(I - BB^T J (J^T BB^T J)^{-1} J^T\right) f(x)$$

Remark: inputs are not present in this autonomous equation which contains the zero dynamics.

Example

$$\begin{aligned} \dot{x}_1 &= x_2 + u, \\ \dot{x}_2 &= bu, \\ y &= x_1 + x_2. \end{aligned}$$

The two reduced systems are, with $x_1 = -x_2$:

$$\underbrace{\dot{x}_2 = -\frac{1}{2}\left(x_2 + (1-b)u\right)}_{\text{Lagrange red. syst.}} \qquad \underbrace{\dot{x}_2 = -\frac{b}{1+b}x_2}_{\text{Virtual limit system}}$$

- **Conclusion:** the limit system obtained from the use of the virtual constrains, will lead to different equations than the one obtained by standard reduction Lagrange method.
- The equation of the zero dynamics resulting from the virtual constrains is named the "*virtual limit system*".

Properties of Virtual Limit System

Theorem 1 *(Perram-Shriaev-Canudas'03)*

Given initial conditions $\left[\theta_0,\, \dot{\theta}_0\right]$*, if the solution* $\left[\theta(t,\theta_0),\, \dot{\theta}(t,\dot{\theta}_0)\right]$*, of the limit system*

$$\alpha(\theta,p)\ddot{\theta} + \beta(\theta,p)\dot{\theta}^2 + \gamma(\theta,p) = 0$$

exists for these initial conditions, then the function

$$I\left(\theta,\dot{\theta},\theta_0,\dot{\theta}_0\right) = \dot{\theta}^2 - \psi(\theta_0,\theta)\dot{\theta}_0^2 + \psi(\theta_0,\theta)\cdot\int_{\theta_0}^{\theta}\psi(s,\theta_0)\frac{2\gamma(s)}{\alpha(s)}\,ds$$

$$\psi(\theta_0,\theta_1) = \exp\left\{-2\int_{\theta_0}^{\theta_1}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}$$

preserves its value along this solution. This holds irrespective of the boundedness of the solution $\left[\theta(t,\theta_0),\, \dot{\theta}(t,\dot{\theta}_0)\right]$. ■

Proof (outline)

Introducing the new variable $Y = \dot{\theta}^2(t)$ One can then rewrite the virtual limit system in the equivalent form

$$\frac{d}{d\theta} Y + \frac{2\beta(\theta)}{\alpha(\theta)} Y + \frac{2\gamma(\theta)}{\alpha(\theta)} = 0$$

This is a linear equation with respect to function Y and with θ (instead of t) as independent variable. Its general solution has the following form:

$$Y(\theta) = \psi(\theta_0, \theta) \cdot Y(\theta_0) - \psi(\theta_0, \theta) \cdot \int_{\theta_0}^{\theta} \psi(s, \theta_0, s) \, \frac{2\gamma(s)}{\alpha(s)} \, ds$$

Introducing function I as

$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) \quad = \quad Y(\theta) - \psi(\theta_0, \theta) \cdot Y(\theta_0) + \psi(\theta_0, \theta) \cdot \int_{\theta_0}^{\theta} \psi(s, \theta_0, s) \, \frac{2\gamma(s)}{\alpha(s)} \, ds$$

results in the identity $I(\theta(t), \dot{\theta}(t)) = 0$.

Other useful relations of I

- I the virtual energy of the limit virtual system (conserved quantity).
- Consider now the forced virtual limit system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = u$$

then the following relation holds:

$$\frac{d}{dt} I = \dot{\theta} \left\{ \frac{2}{\alpha(\theta)} u - \frac{2\beta(\theta)}{\alpha(\theta)} I \right\}$$

- $I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = 0$ is the only invariant under $u = 0$. Thus it is not the first integral of system, but in number of examples this function could be readily rewritten in the form of the *first integral* $U(\theta, \dot{\theta})$ of the system, that is

$$f(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) \cdot I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = U(\theta, \dot{\theta}) - U(\theta_0, \dot{\theta}_0),$$

where $f(\theta, \dot{\theta})$ is some function. The 'modified' energy function $\frac{d}{dt} U = \dot{\theta} \cdot u$

Controller Design

Problem: Derive a family of feedback laws and conditions, that ensure exponential orbital stabilization of a particular class of periodic solution of the *virtual limit system*.

Main control design steps:

1. PARTIAL FEEDBACK LINEARIZATION,
2. CHOICE OF A PERIODIC SOLUTION FOR THE LIMIT VIRTUAL SYSTEM,
3. CONTROLLABILITY TEST OF THE AUXILIARY LPTV SYSTEM,
4. CONSTRUCTION OF THE FEEDBACK LAW.

1. Partial Feedback Linearization

The original nonlinear system is first transformed, via *partial feedback linearization,* to a form

$$\underbrace{\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta)}_{\text{virtual limit system}} = g_1(\theta, \dot{\theta}, y, \dot{y}) + g_2(\theta, \dot{\theta}, y, \dot{y}) \cdot v$$

$$\ddot{y} = v$$

Here $y \in R^{n-1}$, $\theta \in R^1$; $v \in R^{n-1}$. The g_i, are smooth functions

$$g_1(\theta, \dot{\theta}, 0, 0) + g_2(\theta, \dot{\theta}, 0, 0) \cdot v = 0, \quad v = 0_{(n-1)\times 1}.$$

In turn, due to the smoothness of $g_1 = g_y(\theta, \dot{\theta}, y, \dot{y}) \cdot y + g_{\dot{y}}(\theta, \dot{\theta}, y, \dot{y}) \cdot \dot{y}$.

This yields the Auxiliary System:

$$\dot{I} = \dot{\theta}\left\{ \frac{2}{\alpha(\theta)} \left[g_y \cdot y + g_{\dot{y}} \cdot \dot{y} + g_2 \cdot v\right] - \frac{2\beta(\theta)}{\alpha(\theta)}\, I \right\}$$

$$\ddot{y} = v$$

2. Choice of Periodic Solution

Select a set of parameter p such that the limit virtual system has a periodic solution

$$\left[\theta_\gamma(t), \dot\theta_\gamma(t)\right] = \left[\theta_\gamma(t+T), \dot\theta_\gamma(t+T)\right], \quad \forall\, t \tag{2}$$

of a period T. The problem is then: to determine a feedback controller that orbitally stabilize the following periodic solution, $\forall\, t$

$$\left[\,\theta_\gamma(t),\, \dot\theta_\gamma(t),\, y(t),\, \dot y(t)\,\right] = \left[\,\theta_\gamma(t+T),\, \dot\theta_\gamma(t+T),\, 0,\, 0\,\right] \tag{3}$$

of the nonlinear system

$$\begin{aligned} \underbrace{\alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta)}_{\text{virtual limit system}} &= g_1(\theta, \dot\theta, y, \dot y) + g_2(\theta, \dot\theta, y, \dot y)\cdot v \\ \ddot y &= v \end{aligned}$$

3. Controllability of Auxiliary LTVP system

Evaluation of input functions of the Auxiliary system along the solutions $\left[\theta_\gamma(t),\, \dot\theta_\gamma(t),\, 0,\, 0\right]$ gives

$$\begin{aligned} \dot I &= \frac{2\dot\theta_\gamma(t)}{\alpha\left(\theta_\gamma(t)\right)} \left\{ g_y(t)\cdot y + g_{\dot y}(t)\cdot \dot y + g_2(t)\cdot v - \beta\left(\theta_\gamma(t)\right)\cdot I \right\} \\ \ddot y &= v \end{aligned}$$

where

$$\begin{aligned} g_y(t) &= g_y\left(\theta_\gamma(t), \dot\theta_\gamma(t), 0, 0\right) \\ g_{\dot y}(t) &= g_{\dot y}\left(\theta_\gamma(t), \dot\theta_\gamma(t), 0, 0\right) \\ g_2(t) &= g_2\left(\theta_\gamma(t), \dot\theta_\gamma(t), 0, 0\right) \end{aligned}$$

with periodic coefficients.

Its state representation with $\zeta = [\, I,\, y,\, \dot{y}\,]^T$ is

$$\dot{\zeta} = A(t)\zeta + b(t)v$$

with $A(t) = A(t+T)$, and $b(t) = b(t+T)$.

This system is controllable iff:

$$K = \int_0^T \left[X_0(t)\right]^{-1} b(t)b(t)^T \left[X_0(t)^T\right]^{-1} dt > 0$$

where the matrix function $X_0(t)$ is defined as

$$\frac{d}{dt}X_0 = A(t)X_0, \quad X_0(0) = I_{2n+1}.$$

Examples of possible feedbacks for the Auxiliary LTVP system

Let $\Gamma = \Gamma^T > 0$ $G = G^T > 0$, suppose that $\dot{\zeta} = A(t)\zeta + b(t)v$ is completely controllable, then $\exists\ R(t) = R(t)^T = R(t+T)$ that satisfies to the Riccati equation,

$$\dot{R}(t) + A(t)^T R(t) + R(t)A(t) + G = R(t)\, b(t)\Gamma^{-1} b(t)^T R(t),$$

and, the feedback controller

$$v \;=\; -\Gamma^{-1}\, b(t)^T R(t)\zeta$$

renders the linear periodic system exponentially stable. Moreover, along any solution $\left[I(t),\, y(t),\, \dot{y}(t)\right]$ of the closed loop system, the following holds

$$\frac{d}{dt}V(t) = -\zeta(t)^T G\zeta(t) - v(t)^T \Gamma v(t)$$

with $V(t) = \zeta(t)^T R(t)\zeta(t)$.

4. Constructive Procedure for Control Design

The previous LTV feedback controller

$$v = -\Gamma^{-1} b(t)^T R(t)\zeta$$

suggest the following control structure

$$v = -\Gamma^{-1} \underbrace{b(\theta, \dot{\theta}, y, \dot{y})^T}_{\neq b(t)} R(t)\zeta$$

with

$$b(\theta, \dot{\theta}, y, \dot{y})^T = \left[\frac{2\,\dot{\theta}\, g_2\left(\theta, \dot{\theta}, y, \dot{y}\right)}{\alpha\left(\theta\right)}, 0_{(n-1)\times(n-1)},\, I_{(n-1)} \right]^T$$

Question: Does this controller leads to any type of orbitally stability ?

Stability

Consider any underactuated system of the form $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = B(q)u$, with $B(q)$ of full rank. Assume that:

(*i*) A set of constraints (or outputs) $y_i = \phi_i(q,p) = 0 \quad \forall i = 1, 2, \ldots, (n-1)$, are defined, so that the resulting virtual limit system exhibes cycles;

(*ii*) Given the periodic solution $[\theta_\gamma(t), \dot{\theta}_\gamma(t)]$ of the virtual limit system with a period T, the corresponding auxiliary linear system $\dot{\zeta} = A(t)\zeta + b(t)v$ is completely controllable on $[0, T]$.

Then the control law $v = -\Gamma^{-1} b(\theta, \dot{\theta}, y, \dot{y})^T R(t)[I, y, \dot{y}]^T$ with

$$I = \dot{\theta}^2 - \psi(\theta_\gamma(0), \theta)\dot{\theta}_\gamma^2(0) + \psi(\theta_\gamma(0), \theta) \cdot \int_{\theta_\gamma(0)}^{\theta} \psi(s, \theta_\gamma(0), s)\frac{2\gamma(s)}{\alpha(s)}\, ds$$

makes the chosen solution of the closed loop system orbitally exponentially stable in a local sense. ■

Proof: outline

Taking $V = \zeta^T R(t)\zeta$, it can be shown that

$$\dot{V} = \zeta^T \left\{ -\underbrace{G}_{<0} - \underbrace{R(t)b(t)\Gamma^{-1}b(t)^T R(t)}_{<0} + \underbrace{\Delta(t)}_{perturbation} \right\} \zeta$$

Where the perturbation term Δ

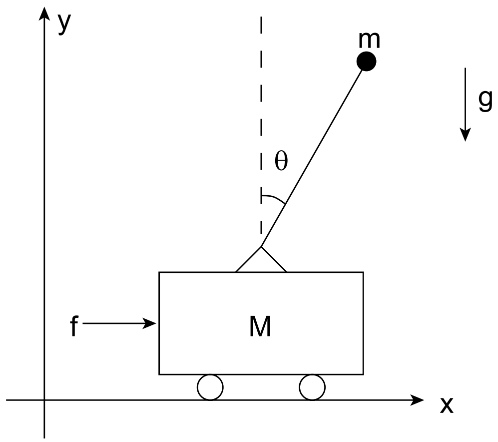
$$\Delta(t) = \left(\bar{A} - A(t)\right)^T R(t) + R(t)\left(\bar{A} - A(t)\right) R(t)\left(\bar{b} - b(t)\right)\Gamma^{-1}\left(\bar{b} + b(t)\right)^T R(t)$$

where $\bar{A} = A(\theta, \dot{\theta}, y, \dot{y})$, and $\bar{b} = b(\theta, \dot{\theta}, y, \dot{y})$.

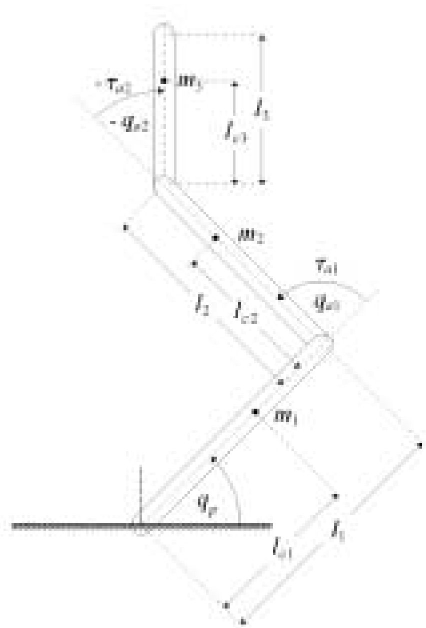
- $\lim_{\zeta \to 0} |\Delta(t)| = 0$
- $|\Delta(t)|$ can be made arbitrarily small in $[0, T]$
- It is possible to shown that

$$V(nT) \leq V(0) - \frac{\frac{1}{2}\min\left\{\lambda(G)\right\}}{\max\limits_{t\in[0,T]} \left\| R(t) \right\|} \int_0^{nT} V(\tau)d\tau.$$

Examples of systems controlled by this method



Generating Oscillations in 3-Links Pendulum



Virtual Constraints

$$y_1 = q_{a1} - a_1 \left(q_p - \varepsilon_1\right)$$

$$y_2 = q_{a2} - a_2 \left(q_p - \varepsilon_2\right)$$

$(y = 0)$ where a_1, a_2 and ε_1, ε_2 are the constant parameters to be determined later.

Partial Feedback Linearization

After partial linearization of the outputs y, we get

$$\begin{aligned} \alpha(q_p)\ddot{q}_p + \beta(q_p)\dot{q}_p^2 + \gamma(q_p) &= -c_{12}\,\dot{y}_1 - c_{13}\,\dot{y}_2 - m_{12}\cdot v_1 - m_{13}\cdot v_2 \\ \ddot{y}_1 &= v_1 \\ \ddot{y}_2 &= v_2 \end{aligned}$$

with,

$$\begin{aligned} \alpha(q_p) &= m_{11} + a_1\cdot m_{12} + a_2\cdot m_{13} \\ \beta(q_p) &= c_{11} + a_1\cdot c_{12} + a_2\cdot c_{13} \\ \gamma(q_p) &= g_1 \end{aligned}$$

Choice of Parameters of Virtual limit system

Conditions need to be fulfilled by the choice of a_1, a_2, ε_1 and ε_2

1. The limit system

$$\alpha(q_p)\ddot{q}_p + \beta(q_p)\dot{q}_p^2 + \gamma(q_p) = 0 \tag{4}$$

   has a stable equilibrium around $q_p = \pi/2$; i.e. search zero of $\gamma(q_p)$ in some vicinity of $\frac{\pi}{2}$;

2. Compute the linearization of (4) at the found equilibrium q_e

$$\ddot{z} + \omega_e z = 0, \tag{5}$$

   where $\omega_e = \frac{d}{dq_p}\left(\frac{\gamma(q_p)}{\alpha(q_p)}\right)\Big|_{q_p=q_e}$. If ω_e is positive, then the linear system (5) is a focus, and its solutions are periodic;

3. To verify that the nonlinear system (4) has periodic solutions in some neighborhood of the equilibrium q_e, check level sets of its integral I. If these sets are closed curves, then there exist periodic solutions.

Selected range of possible values:

$$0 \leq a_1 \leq 10, \quad 0 \leq a_2 \leq 10, \quad \varepsilon_1 = -\frac{\pi}{2} - 0.05, \quad \varepsilon_1 = \frac{\pi}{2} + 0.1$$

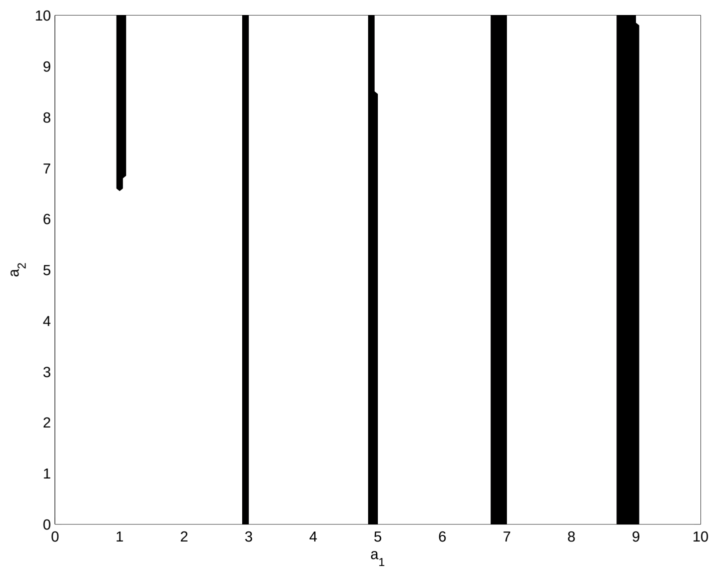


Figure 1: The areas in black correspond to those values of a_1 and a_2, where ω_e is positive and the equilibrium q_e of the virtual limit system (4) deviates from $\frac{\pi}{2}$ no more than 0.1 rad.

Choose any point (a_1, a_2) belonging to areas in black depicted at Figure 1 and verify that limit system has a periodic solutions.

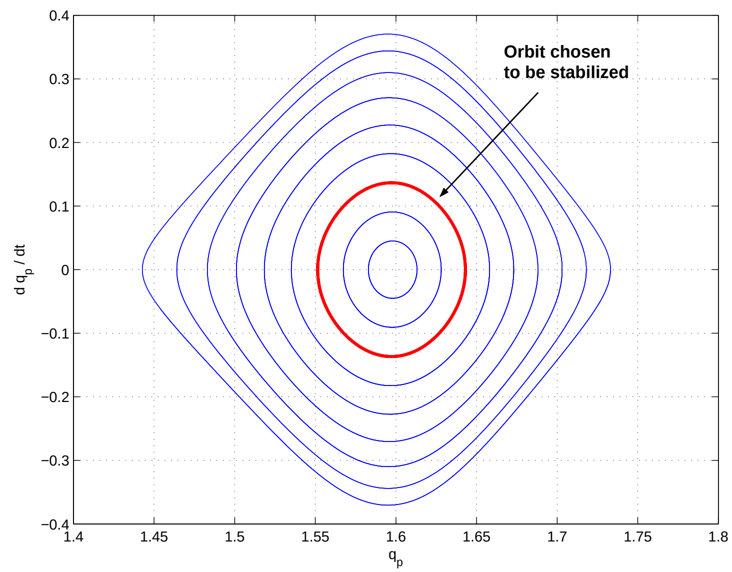
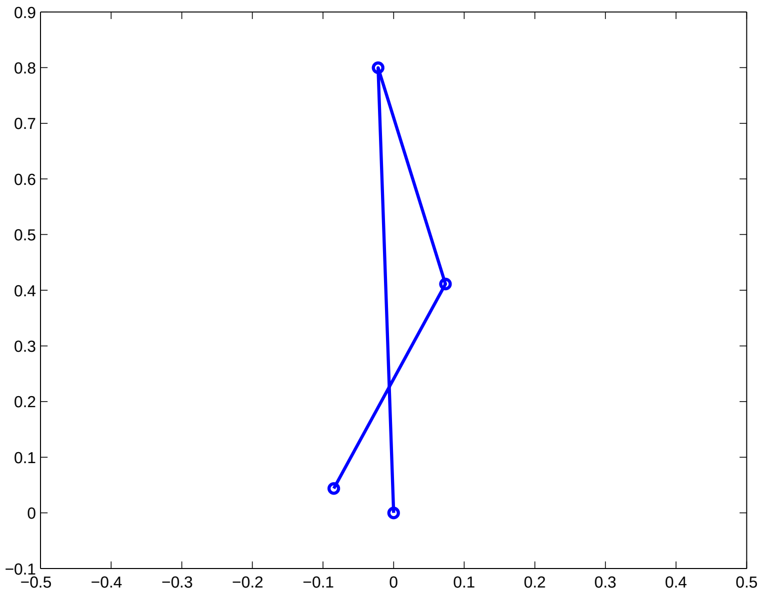


Figure 2: Level sets of the integral I when the constraints' parameters a_1, a_2, ε_1, ε_2 are chosen as in (1).

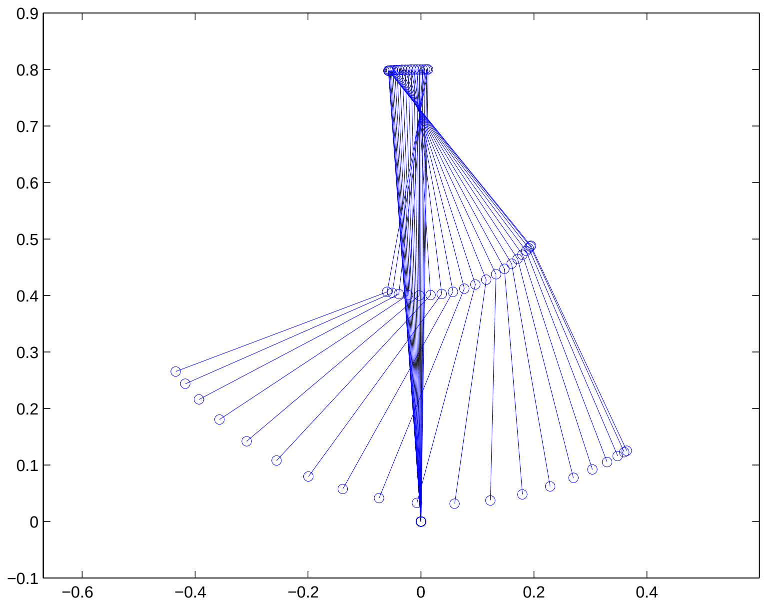
We have chosen the values as

$$a_1 = 8.85, \quad a_2 = 8.9, \quad \varepsilon_1 = -\frac{\pi}{2} - 0.05, \quad \varepsilon_1 = \frac{\pi}{2} + 0.1$$

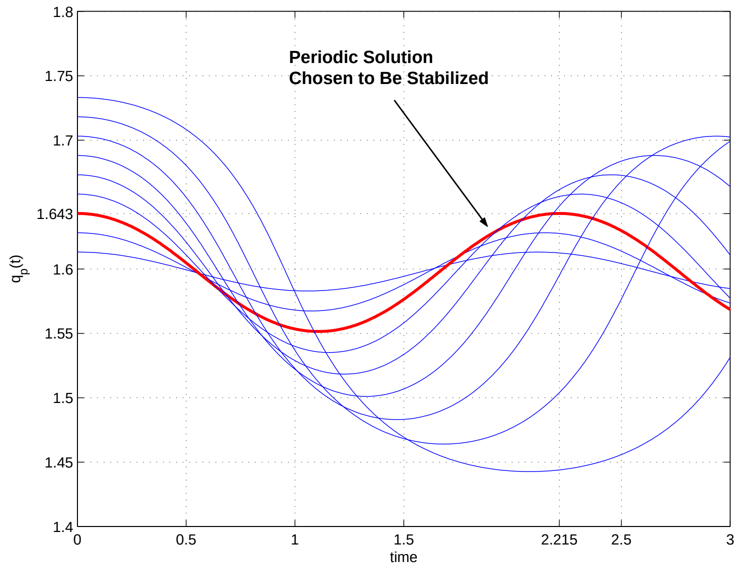
Evolution of the Rabbit restricted to 3 links



Equilibrium position (center)



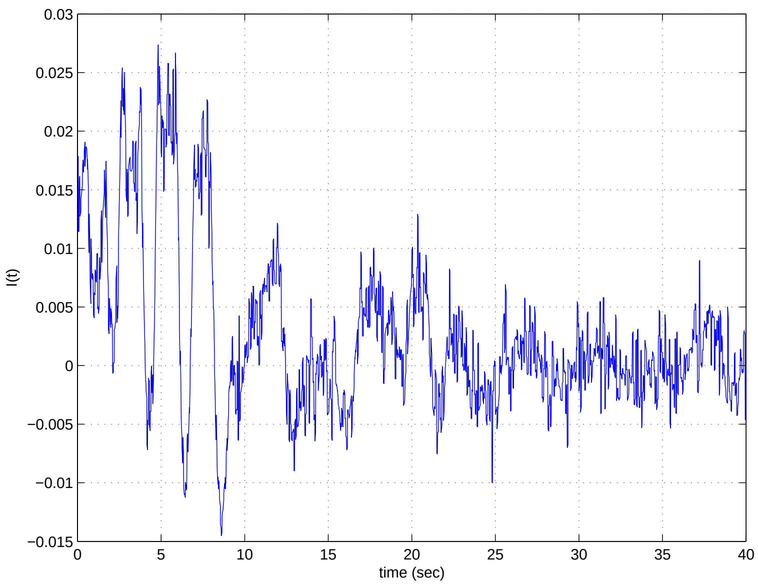
Motion over a period



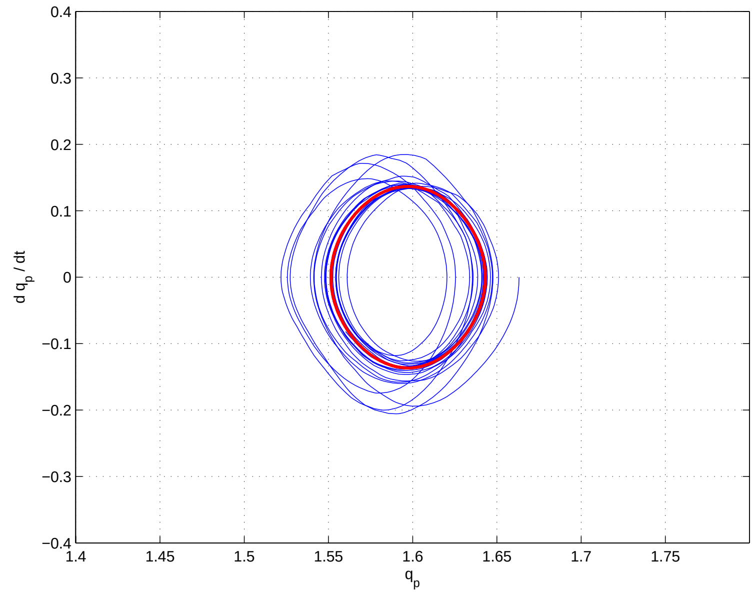
Verify the controllability of the auxiliar system along the chosen solution

$$\frac{d}{dt}\begin{bmatrix} I \\ y \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \kappa_1(t) & \kappa_2(t) & \kappa_3(t) \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}}_{A(t)} \begin{bmatrix} I \\ y \\ \dot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} \rho(t) \\ 0 \\ I \end{bmatrix}}_{b(t)} v,$$

Simulated closed-loop responses

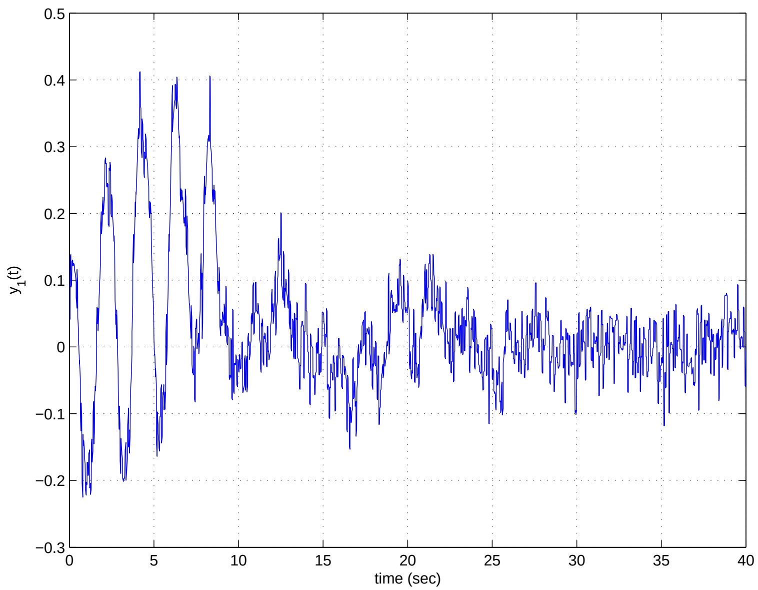


Behaviour of I

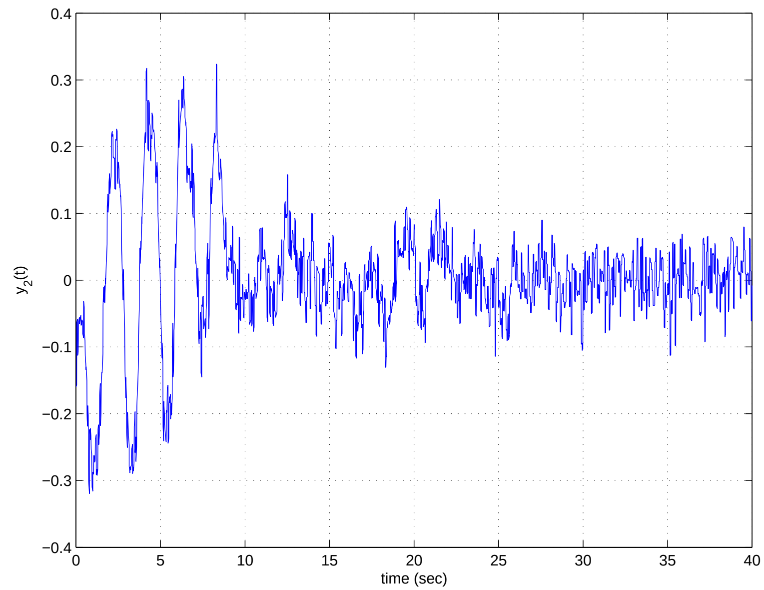


Motion of the under-actuated angle

Simulated closed-loop responses



Behaviour of y_1



Behaviour of y_2

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