Appendix A – Formulation of the Linear Program (LP)

The equation

$$V^*\left(\breve{x}\right)=\min_{\breve{u}\in\breve{U}(x)}\left(\sum_{\breve{w}\in\breve{\mathcal{C}}}\Pr\left(\breve{w}\middle|\breve{x}\right)\left(c\left(\breve{x},\breve{u},\breve{w}\right)+\gamma\cdot\left(V^*\right)^T\cdot M_{\mathcal{X},\breve{\mathcal{X}}}\left(f_{HEV,aug}\left(\breve{x},\breve{u},\breve{w}\right)\right)\right)\right),\forall\breve{x}\in\breve{\mathcal{X}}\,. \quad (1)$$

is solved by linear programming as discussed in [1]. To simplify the description of the problem, assume that the sets $\breve{\mathcal{X}}$ and $\breve{\mathcal{U}}$ do not have duplicate members. First, note that **(1)** consists of a set of equations, one for each element in $\breve{\mathcal{X}}$. Additionally, any solution to **(1)** must satisfy the set of inequalities in (2) for each discrete state and discrete action.

$$V^*\left(\breve{x}\right)\le\left(\sum_{\breve{w}\in\breve{\mathcal{T}}}\Pr\left(\breve{w}\middle|\breve{x}\right)\left(c\left(\breve{x},\breve{u},\breve{w}\right)+\left(V^*\right)^T\cdot M_{\mathcal{X},\breve{\mathcal{X}}}\left(f_{HEV,aug}\left(\breve{x},\breve{u},\breve{w}\right)\right)\right)\right),\forall\breve{x}\in\breve{\mathcal{X}},\breve{u}\in\breve{U}\left(x\right) \quad (2)$$

Let equation (3) define the vector of all $V^*\left(\breve{x}\right)$. In this equation, the subscript on $\breve{x}$ is used to denote uniquely each element in $\breve{\mathcal{X}}$. $N_{\breve{\mathcal{X}}}$ is the total number of elements in that set. Assign to each element in $\breve{\mathcal{X}}$ a number from 1 to $N_{\breve{\mathcal{X}}}$. Let the function $n\left(\breve{x}\right)$ return this number. Let e_i be the natural basis vector for the i^{th} dimension in $\mathbb{R}^{N_{\breve{\mathcal{X}}}}$. Define the vector, V^*, to be the vector of value functions evaluated at each element in the discrete state space $\breve{\mathcal{X}}$.

$$V^*=\begin{bmatrix}V^*\left(\breve{x}_1\right) & V^*\left(\breve{x}_2\right) & \cdots & V^*\left(\breve{x}_N\right)\end{bmatrix}^T \quad (3)$$

Combining these ideas, the system of inequalities in (4) can be stated.

$$\left(e_{n(\breve{x})}-\sum_{\breve{w}\in\breve{\mathcal{T}}}\Pr\left(\breve{w}\middle|\breve{x}\right)\cdot M_{\mathcal{X},\breve{\mathcal{X}}}\left(f_{HEV,aug}\left(\breve{x},\breve{u},\breve{w}\right)\right)\right)^T\cdot V^*\le\left(\sum_{\breve{w}\in\breve{\mathcal{T}}}\Pr\left(\breve{w}\middle|\breve{x}\right)c\left(\breve{x},\breve{u},\breve{w}\right)\right),\forall\breve{x}\in\breve{\mathcal{X}}\breve{u}\in\breve{\mathcal{U}}\left(\breve{x}\right) \quad (4)$$

For each instance of this inequality, the left hand side of the inequality is a vector product and the right hand side is a scalar constant. The first vector is constant and the second vector is the unknown value function.

When all of the defined equations are combined, a matrix inequality is obtained as shown in (5).

$$A\cdot V^*\le b \quad (5)$$

In this matrix inequality, each row of A is formed from the first vector in the left hand side of (4) and the rows of b are formed from the scalars on the right hand side. The object for the LP can be formed to satisfy different criteria as discussed in **[2**]. For this work, the objective proposed by Bertsekas in **[1**] is used: The LP is formed by maximizing the sum of V^* subject to (5). Solution of that LP finds the solution to **(1)**.

References

[1] Bertsekas D. *Dynamic Programming and Optimal Control Vol I & II*. Athena Scientific: Belmont, Massachusetts, 1995.

[2] D. P. d. Farias and B. V. Roy, "The Linear Programming Approach to Approximate Dynamic Programming," *Operations Research*, vol. 51, pp. 850-865, Nov-Dec 2003.

Appendix B – The Dynamic Model

This HEV model used in this study is based on the model from [3], [4] and [5]. The model is repeated here for convenience and clarity on the modeling assumptions used in this work. The model consists of several distinct subsystems. Each is presented in turn. The combined system of equations obtained results in the system model. Figure 1 and Figure 2 show a block diagram of the system and a signal flow diagram of the system respectively.

The Lumped Kinematic Model.

All of the dynamics associated with the kinematics are lumped in one state variable. The dynamics associated with the engine are ignored. The vehicle speed is modeled as a first order ordinary differential equation (ODE). The vehicle model is a single wheel model that omits tire slip and weight transfer. The model assumes operation on level terrain. This ODE defines the acceleration of the vehicle as a function of the net torque applied at the wheels, $T_{whl,net}$, net force applied to the body, F_{veh}, and the effective vehicle mass, which is a function of the transmission gear, G. This relationship is shown in (6).

$$\dot{v}_{veh} = \frac{\left(T_{whl,net} / r_{whl} \right) + F_{veh}}{M_{veh,eff}\left(G\right)} \tag{6}$$

The net force applied to the vehicle is a function of the vehicle speed and consists of three components: the aerodynamic losses, the rolling losses and the bearing losses. These relationships are shown in (7) through (10).

$$F_{veh} = F_{aero} + F_{rolling} + F_{bearing} \tag{7}$$

$$F_{aero} = -0.5 \cdot A_{frontal} \cdot \rho_{air} \cdot C_D \cdot v_{veh}^2 \tag{8}$$

$$F_{rolling} = \begin{cases} F_{rolling,0} & \text{,if } v_{veh} \neq 0 \\ 0 & \text{,otherwise} \end{cases} \tag{9}$$

$$F_{bearing} = \frac{R_{bearing}}{\left(r_{whl}\right)^2} \cdot v_{veh} \tag{10}$$

The constants used in this portion of the model are summarized in Table 1.

Table 1 - Constants in Lumped Kinematic Model

Constant	Value
$M_{veh,eff}(1)$	8016
$M_{veh,eff}(2)$	7810
$M_{veh,eff}(3)$	7720
$M_{veh,eff}(4)$	7691
$A_{frontal}$	5.2026 [m^2]
ρ_{air}	1
C_D	0.8
$F_{rolling,0}$	268.4013 [N]
r_{whl}	0.3995 [m]
$R_{bearing}$	3 [N*m*s/rad]

The Battery Model.

The battery pack is a model of an 18 Ahr, 25 module PbA battery pack with a nominal open circuit voltage of 316 volts. The battery pack is modeled as an ideal voltage source connected to a parallel circuit with an ideal diode and resistor in each branch. In one of these branches, the diode allows current to flow into the pack when charging. In the other branch, the diode permits current to flow out of the pack while discharging. The voltage source and the resistors have values that are function of the battery SOC as shown in Figure 4 and Figure 5. The evolution of the battery SOC is described using an ordinary ODE. The rate of change of the charge in the battery is

$$\dot{q}_{batt} = \frac{i_{batt}}{Q_{\max,batt}}. \tag{11}$$

The current of the battery is

$$i_{batt} = \frac{P_{batt}}{V_{pack}}, \tag{12}$$

and the pack voltage is

$$V_{pack} = V_{OC}\left(q_{batt}\right) + i_{batt} \cdot R_{batt}\left(q_{batt}, i_{batt}\right). \tag{13}$$

The open circuit voltage is the function shown in Figure 4. The resistance is the function shown in Figure 5, where the charging resistance is used if i_{batt} is greater than 0, otherwise, the discharge resistance is used.

The battery use is constrained by SOC and voltage. These constraints on operation of the battery are summarized in (14) through (17).

$$q_{batt} \geq q_{batt,\min} \tag{14}$$

$$q_{batt} \leq q_{batt,\max} \tag{15}$$

$$V_{pack} \leq V_{pack,\max} \tag{16}$$

$$V_{pack} \geq V_{pack,\min} \tag{17}$$

The constants used in this subsystem are shown in Table 2.

Table 2 – Constants in Battery Pack & Machine Model

Constant	Value
$q_{batt,\min}$	0.36
$q_{batt,\max}$	0.74
$V_{pack,\max}$	412.5 [V]
$V_{pack,\min}$	237.5 [V]
$Q_{batt,\max}$	64800 [Amp-sec]

The Electric Machine Model.

The electrical subsystem consists of the electrical machine and the battery pack. The electrical machine is a 50 kW machine. The maximum torque of this machine is shown in Figure 6. This maximum torque is assumed available on a steady state basis. The power conversion efficiency of the electric machine is shown in Figure 7.

The Engine Model.

The engine model is of a V6, intercooled turbo, diesel with 5.475 liters of displacement. The dynamics on the engine are ignored. The engine is modeled as an algebraic function that maps engine speed, ω_e, and engine torque, T_e, to instantaneous fuel rate, NOx generation and PM generation as

$$W_f = f_{eng,fuel}\left(\omega_e, T_e\right), \tag{18}$$

$$W_{NOx} = f_{eng,NOx}\left(\omega_e, T_e\right) \tag{19}$$

and

$$W_{PM} = f_{eng,PM}\left(\omega_e, T_e\right). \tag{20}$$

These functions are illustrated in Figure 11,Figure 9 and Figure 10. The engine torque is constrained to the maximum shown in Figure 8. The engine power is the product of engine torque and engine speed:

$$P_e = T_e \cdot \omega_e. \tag{21}$$

The engine is constrained to operate between a maximum and minimum speed for generation of power. These constraints are

$$\omega_e \leq \omega_{e,\max} \tag{22}$$

And

$$\omega_e \geq \omega_{e,\min}. \tag{23}$$

The input to this system is a power command. If the power command is nonzero, the engine is on, consumes power and produces shaft power. The engine speed is selected that best meets the power command while minimizing fuel consumption. When producing power, the engine speed always satisfies(22), (23) and Figure 8. If the power command is zero, the engine is off. The engine is not back driven by the vehicle. The energy required for and engine start is assumed to be neglible.

The constants used in the engine model are shown in Table 3.

Table 3 - Constants in Engine Model

Constant	Value
$\omega_{e,\min}$	750 [rpm]
$\omega_{e,\max}$	2550 [rpm]

The Transmission, Torque Coupler, Differential and Brakes Model.

The transmission, torque coupler and differential are modeled to include spin losses and gear efficiency. The transmission output torque is calculated as

$$T_{trans,out} = \begin{cases} r_{trans}(G)\cdot\eta_{trans}(G)\cdot\left(T_{trans,input} - R_{trans,1}(G)\cdot\omega_{trans,in} - R_{trans,2}(G)\cdot\left(\omega_{trans,in}\right)^2\right) & \text{,if } \omega_e \geq \omega_{trans,in} \\ T_{trans,backdrive} & \text{,otherwise} \end{cases} . \quad (24)$$

The output of the torque couple is calculated as

$$T_{TC,out} = \begin{cases} T_m \cdot r_{TC} \cdot \eta_{TC} & \text{,if } T_m \geq 0 \\ T_m \cdot {}^{r_{TC}}\!/\!_{\eta_{TC}} & \text{,otherwise} \end{cases} . \quad (25)$$

The input to the differential is the sum of the torque couple and the transmission calaculated as

$$T_{diff,in} = T_{TC,out} + T_{trans,out} . \quad (26)$$

The differential then applies this torque to the wheel

$$T_{whl} = T_{brake} + \begin{cases} \left(T_{diff,in} - R_{diff,0}\cdot\mathrm{sgn}\left(\omega_{diff,in}\right) - R_{diff,1}\cdot\omega_{diff,in}\right)\cdot r_{diff}\cdot\eta_{TC} & \text{,if } \left(T_{diff,in} - R_{diff,0}\cdot\mathrm{sgn}\left(\omega_{diff,in}\right) - R_{diff,1}\cdot\omega_{diff,in}\right) \geq 0 \\ \left(T_{diff,in} - R_{diff,0}\cdot\mathrm{sgn}\left(\omega_{diff,in}\right) - R_{diff,1}\cdot\omega_{diff,in}\right)\cdot {}^{r_{diff}}\!/\!_{\eta_{TC}} & \text{,otherwise} \end{cases} \quad (27)$$

$$\omega_{whl} = \frac{v_{veh}}{r_{whl}} \quad (28)$$

$$\omega_{diff,in} = \omega_{whl} \cdot r_{whl} \quad (29)$$

$$\omega_{trans,in} = \omega_{diff,in} \cdot r_{trans}(G) \quad (30)$$

Table 4 - Transmission Constants

Constant	Value
$r_{trans}(1)$	3.45
$R_{trans,1}(1)$	0.0192
$R_{trans,2}(1)$	1.361e-5
$\eta_{trans}(1)$	0.9893
$r_{trans}(2)$	2.24
$R_{trans,1}(2)$	0.015
$R_{trans,2}(2)$	5.719e-6
$\eta_{trans}(2)$	0.966
$r_{trans}(3)$	1.41

$R_{trans,1}(3)$	0.031
$R_{trans,2}(3)$	-3.189e-5
$\eta_{trans}(3)$	0.9957
$r_{trans}(4)$	1
$R_{trans,1}(4)$	0.0367
$R_{trans,2}(4)$	-4.177e-5
$\eta_{trans}(4)$	1.00
r_{diff}	3.21
$R_{diff,0}$	8.34
$R_{diff,1}$	0.04087
η_{diff}	0.96

The Torque Converter Model.

The torque converter used in this work is modeled as a function two input, two output function. The inputs are the engine speed and the transmission input shaft speed. The outputs are the torque applied to the engine and the torque applied to the transmission input shaft.

If the engine speed is less than the transmission input speed, then a one-way clutch opens and the engine and transmission are decoupled. Under this condition there is no torque on the engine shaft or the transmission input shaft:

$$T_{eng,shaft} = T_{trans,in} = 0. \quad (31)$$

If the engine speed is greater than the transmission input speed then the one way clutch closes and the torque convert couples the engine and the transmission. To determine the torque applied to the engine and transmission, the speed of the engine and the speed of the transmission is compared. Based on the ratio between these speeds, k is calculated using different formula. These conditions and formula are shown in Table 5.

Table 5 - k Calculation

Condition	Equation to solve for k
$\omega_{trans,input} \leq 0.993134 \cdot \omega_{eng}$	$k = \frac{\left(8.5 - 8.24 \cdot \left(\frac{\omega_{trans,in}}{\omega_{eng}}\right)\right)}{1 - 1.125381 \cdot \left(\frac{\omega_{trans,in}}{\omega_{eng}}\right) + 0.124636 \cdot \left(\frac{\omega_{trans,in}}{\omega_{eng}}\right)^2}$
$\omega_{trans,input} > 0.993134 \cdot \omega_{eng}$ and $\omega_{trans,input} \leq 1.07646 \cdot \omega_{eng}$	$k = 60 + 5508.01 \cdot \left(\frac{\omega_{trans,in}}{\omega_{eng}} - 0.993134\right)$
$\omega_{trans,input} > 1.07646 \cdot \omega_{eng}$	$k = \frac{5.58137}{1 - 1.10485 \cdot e^{\left(-0.102677 \cdot \left(\frac{\omega_{trans,in}}{\omega_{eng}}\right)\right)}}$

Based on the ratio between the transmission input shaft speed and the engine shaft speed, the torque ratio is determined. The conditions and equations used to calculate the torque ratio are shown in Table 6.

Table 6 - Torque Ratio Calculation

Condition	Equation to solve for T_{ratio}
$\omega_{trans,input} \leq 0.91 \cdot \omega_{eng}$	$T_{ratio} = 1.72 - 0.802198 \cdot \left(\frac{\omega_{trans,in}}{\omega_{eng}}\right)$
$\omega_{trans,input} > 0.91 \cdot \omega_{eng}$	$T_{ratio} = 0.99$

Using k the torque on the engine shaft is calculated as

$$T_{eng,shaft} = \left(\frac{\omega_{trans,in}}{k}\right)^2 \cdot \text{sgn}\left(1 - \left(\frac{\omega_{trans,in}}{\omega_{eng}}\right)\right). \tag{32}$$

Using the torque ratio, T_{ratio} and the engine torque, the transmission input shaft torque is calculated as

$$T_{trans,in} = T_{ratio} \cdot T_{eng,shaft} \, . \tag{33}$$

The Low Level Controls

At the low level there is a controller to select the transmission gear. This controller selects the gear that minimizes the same cost function used in the SP-SDP/SDP problem formulation. Additionally, there is a controller that interprets the power split ratio and the vehicle command. If the vehicle command is for positive power, this controller selects the engine speed that will induce enough torque to satisfy the power split ratio. If this ratio can not be satisfied, either additional engine power or electric machine power will be applied to insure that the vehicle command is satisfied. If the vehicle command is for negative power (braking), the engine is turned off, the electric machine is used for maximal regen and brakes are used to provide any additional power required.

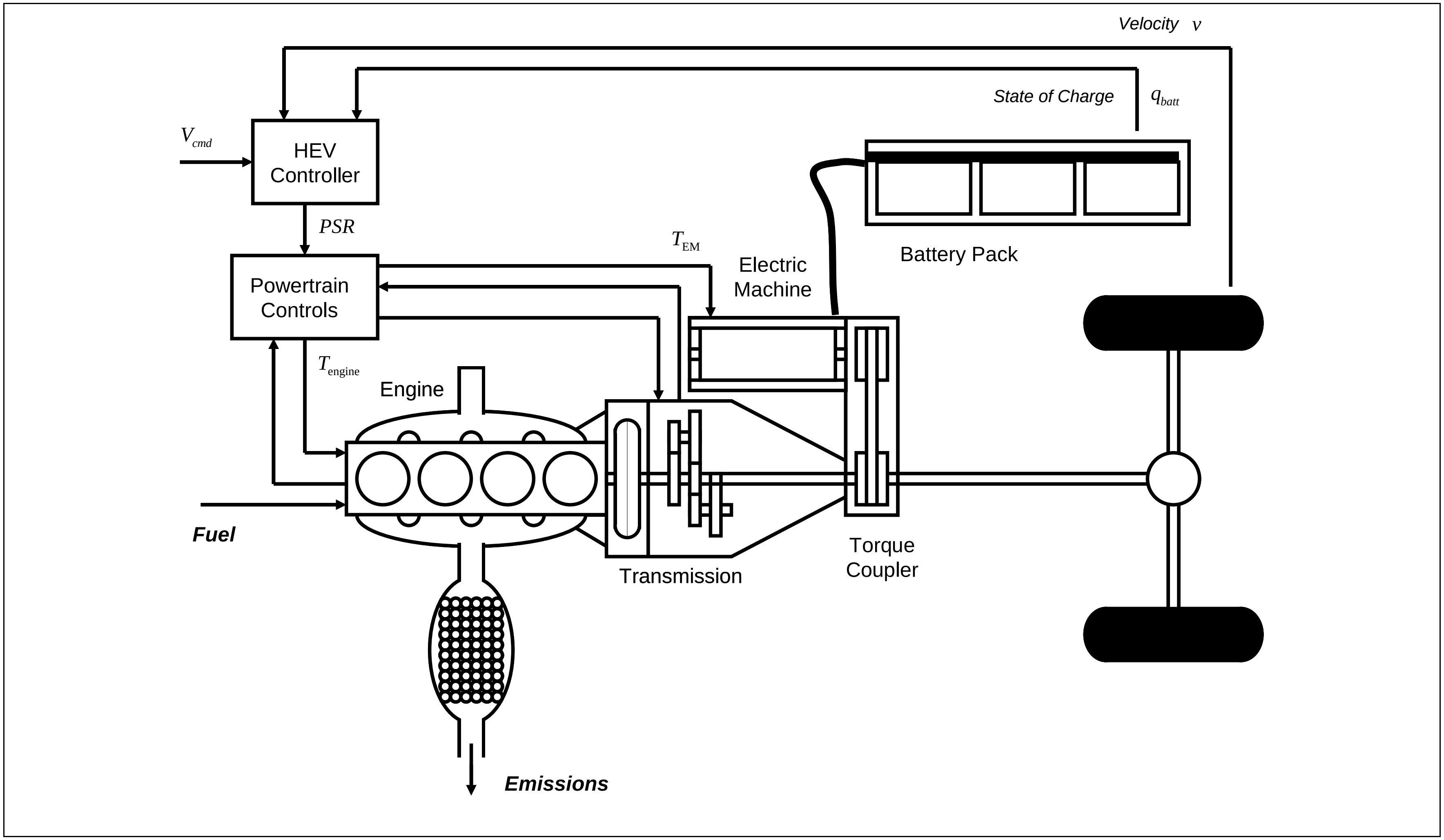

Figure 1 - The Environment, Driver and HEV as a System

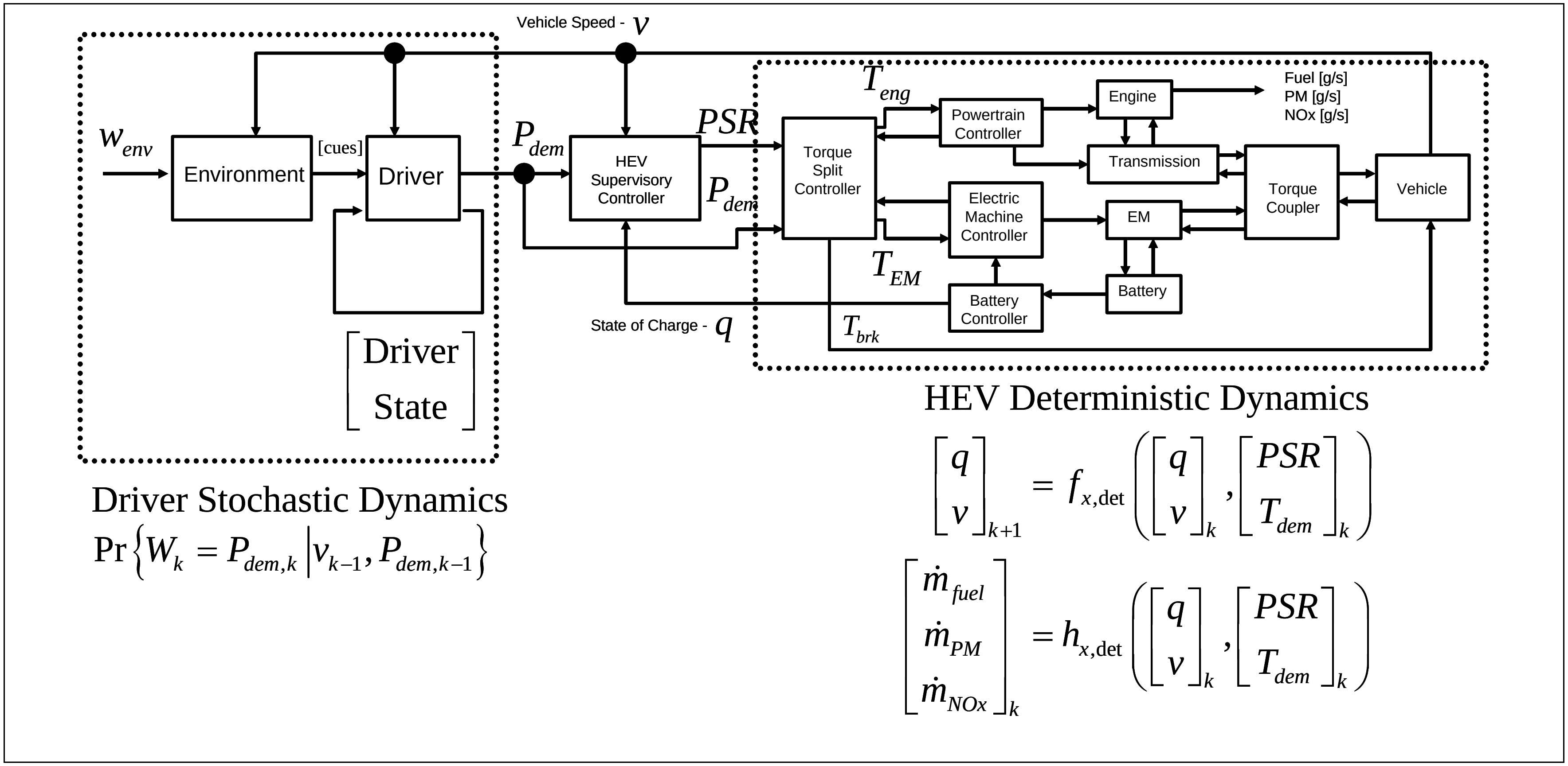

Figure 2 - A Signal Flow Diagram of the Environment, Driver and HEV as a System

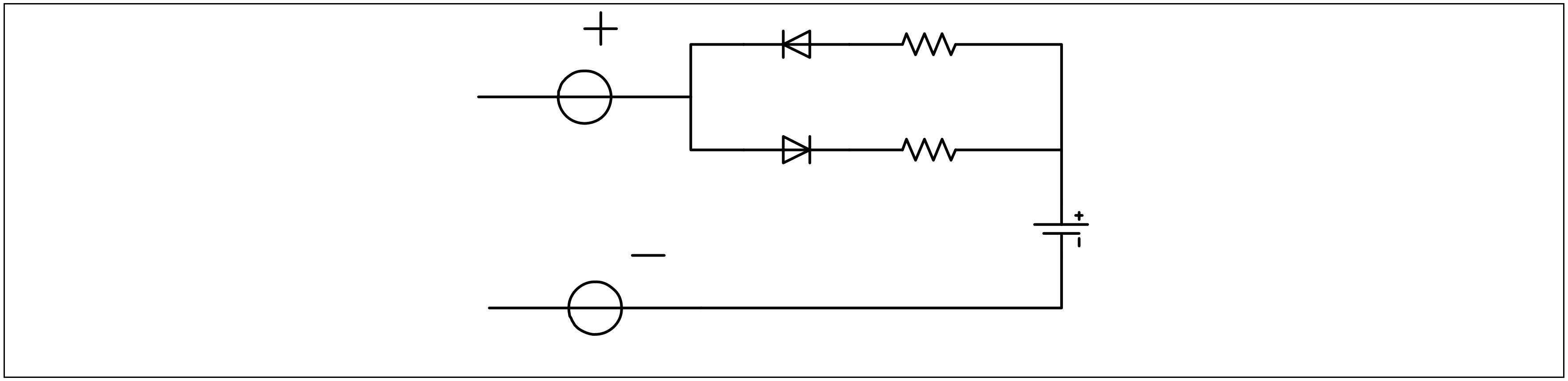

Figure 3 - Battery Schematic

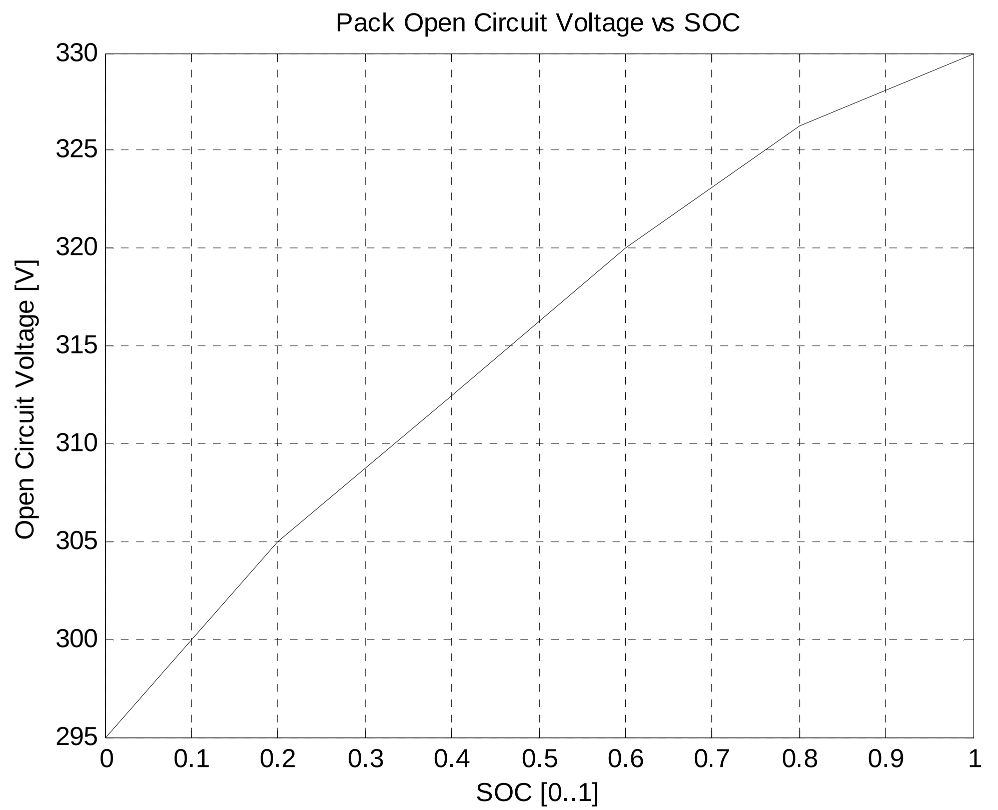

Figure 4 – Pack Voc vs SOC

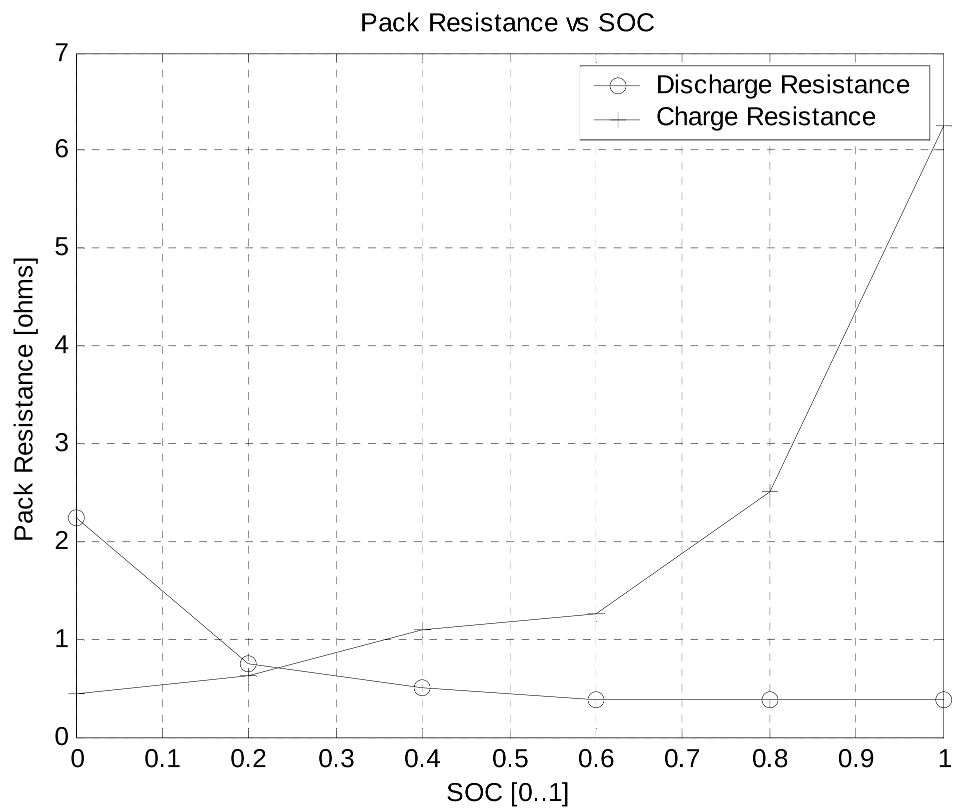

Figure 5 - Pack Resistance vs SOC

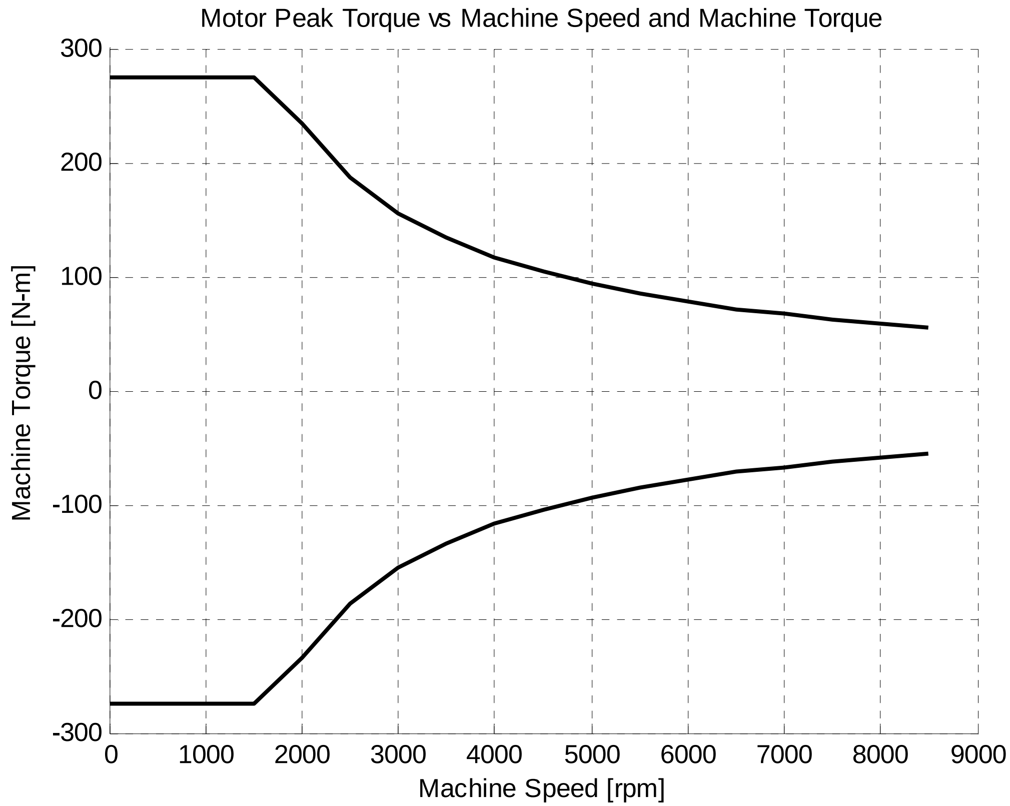

Figure 6 - Motor Max Torque

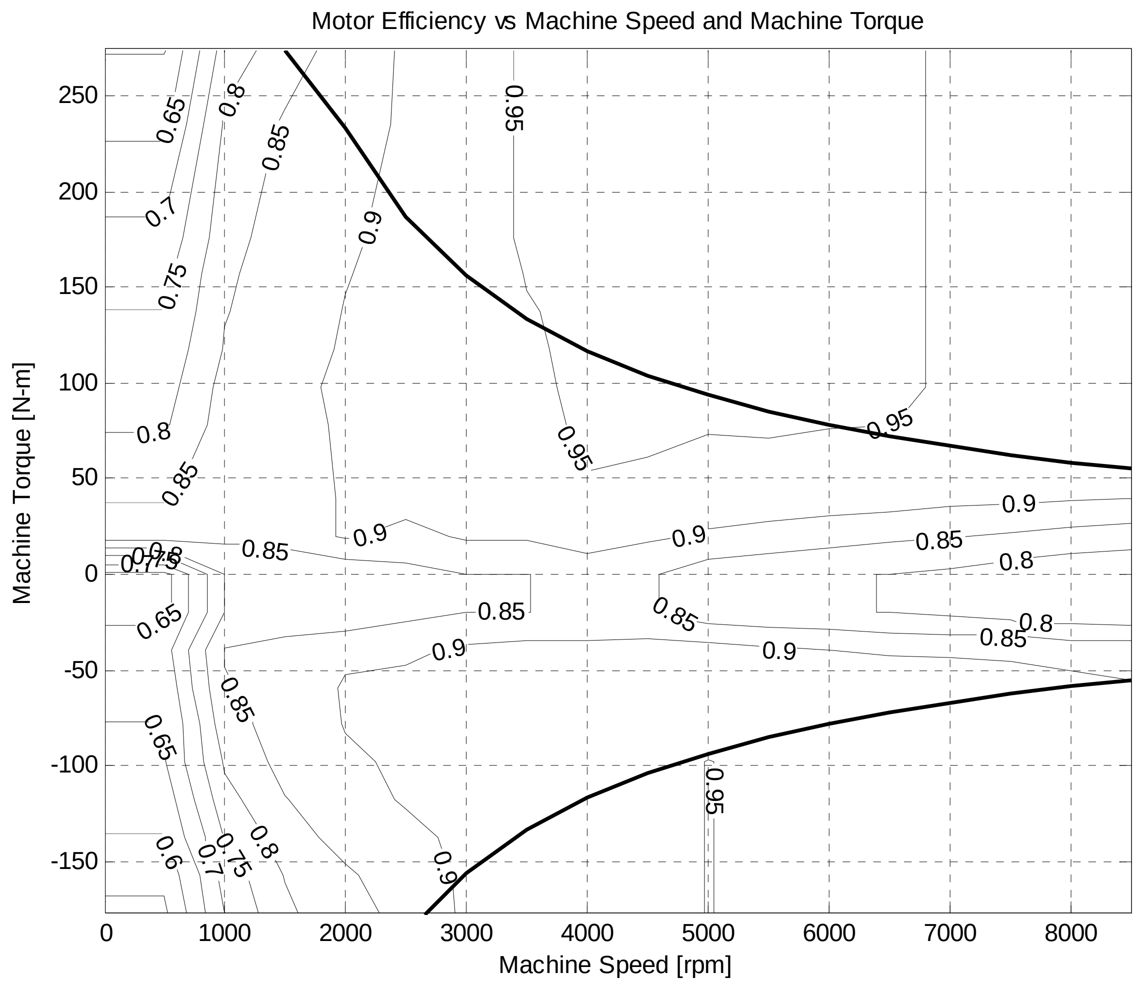

Figure 7 - Motor Efficiency Map

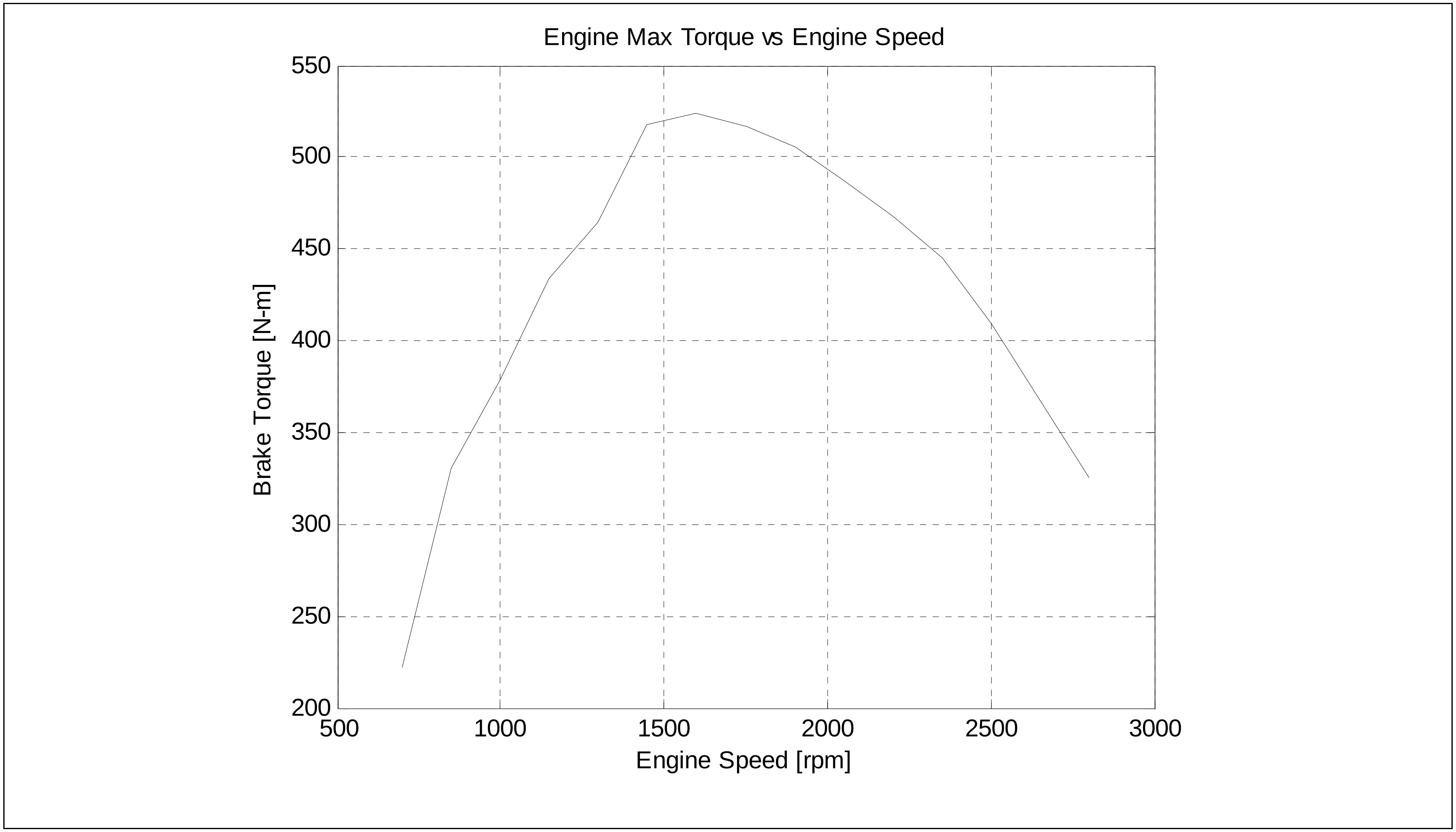

Figure 8 - Engine Max Torque

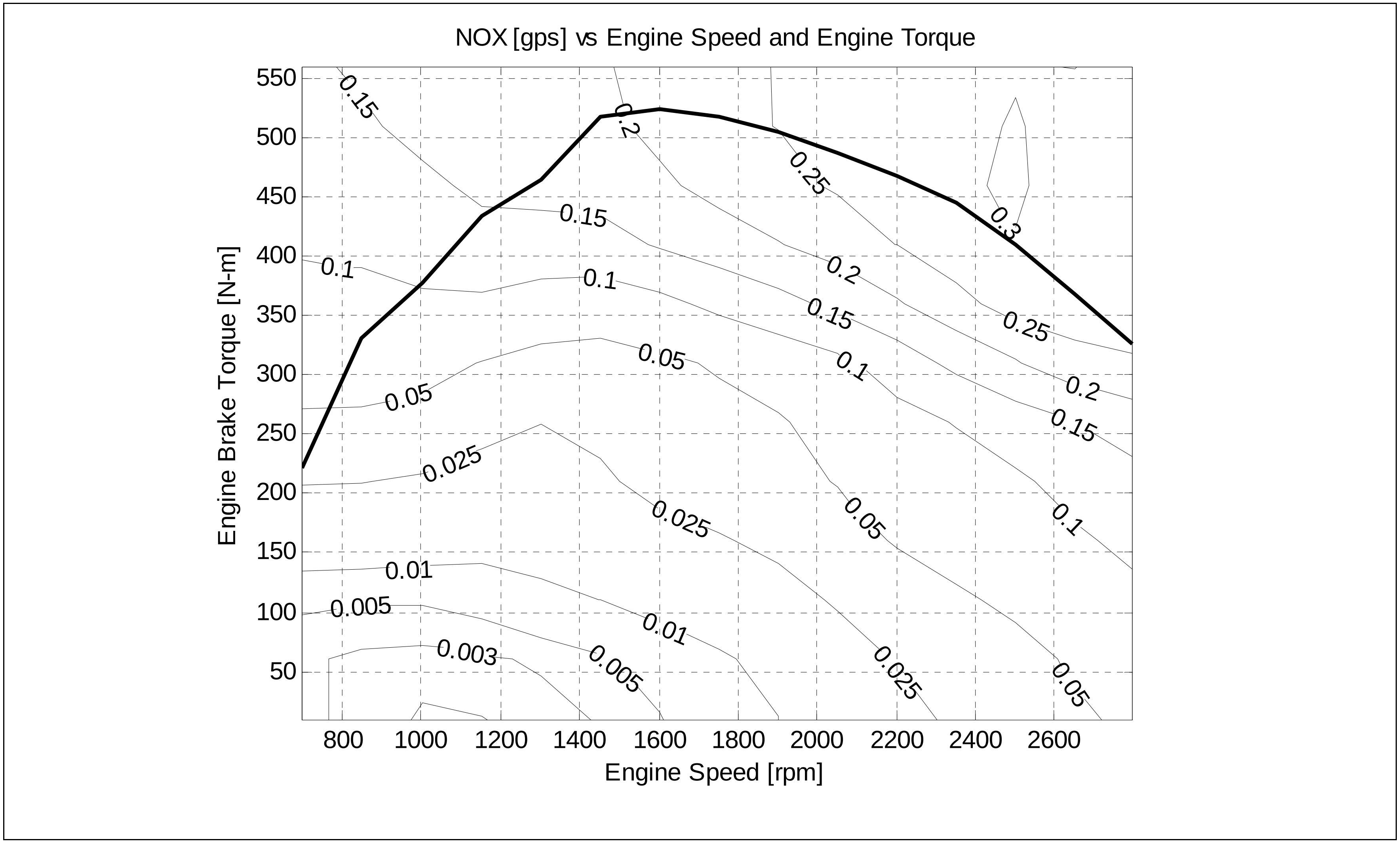

Figure 9 - Engine NOx Map

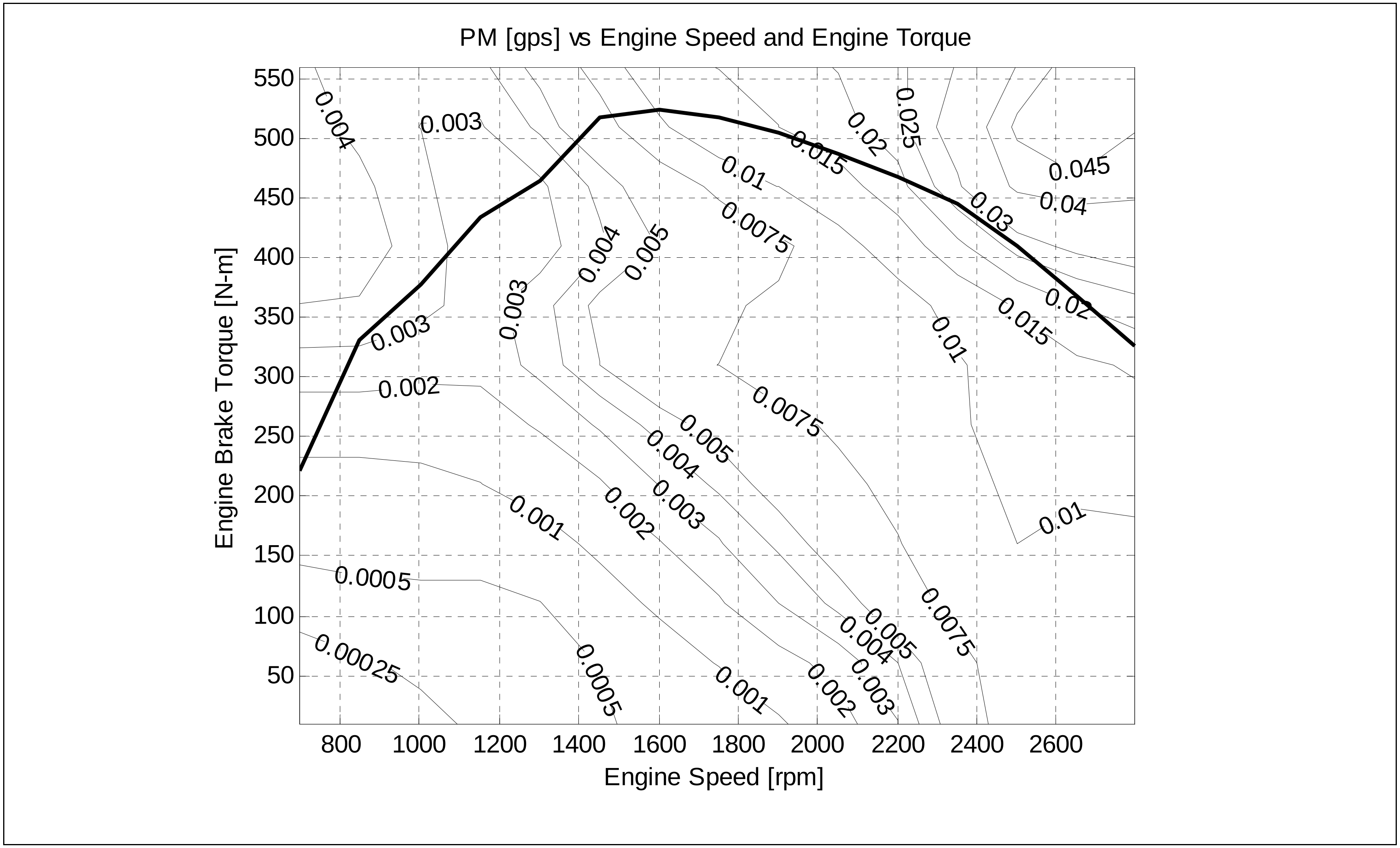

Figure 10 - Engine PM Map

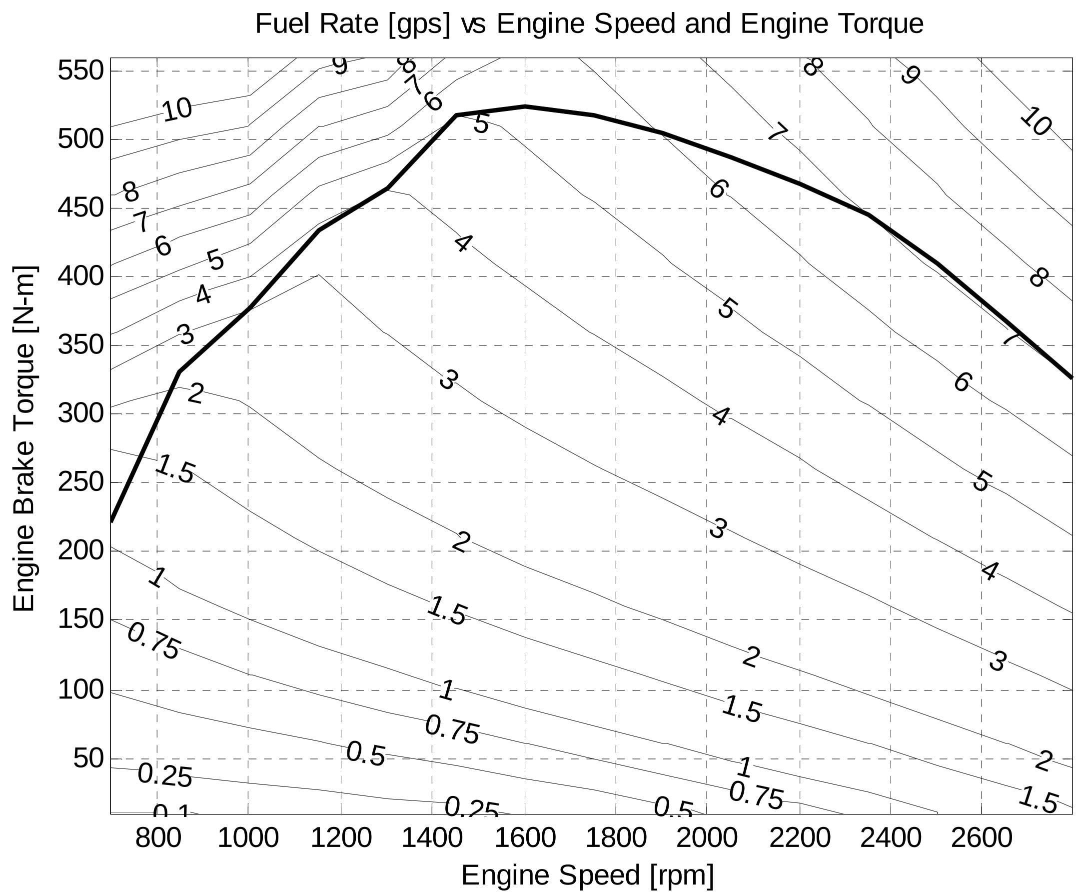

Figure 11 - Engine Fuel Map

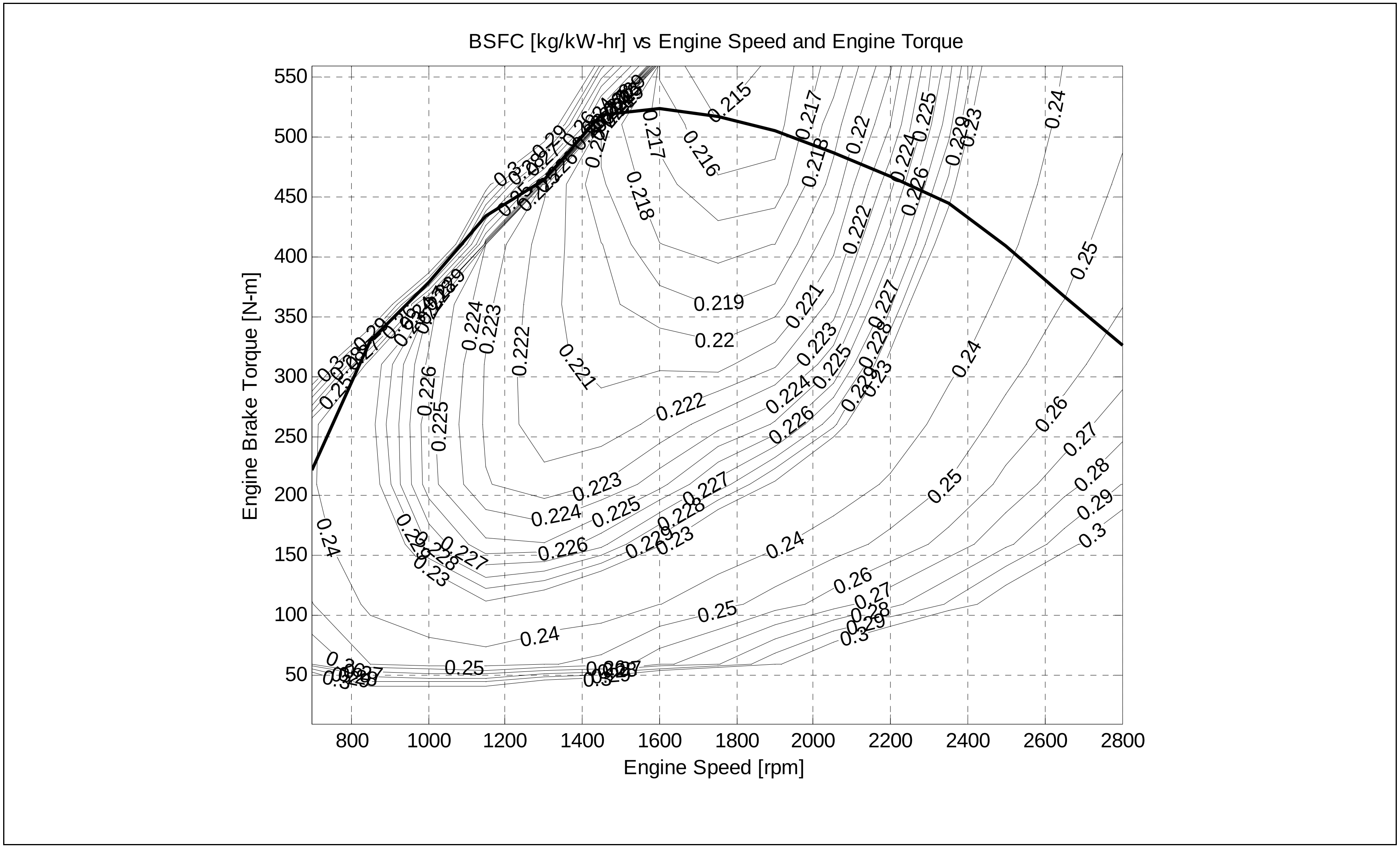

Figure 12 - Engine BSFC Map

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[3] Lin CC, Peng H, Grizzle JW, Kang JM, Power Management Strategy for a Parallel Hybrid Electric Truck. *IEEE Trans. on Control Systems Technology* Nov 2003; 11 (6): 839 – 849.

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[5] Lin CC. Modeling and Control Strategy Development for Hybrid Vehicles. Ph.D. Thesis. Department of Mechanical Engineering, University of Michigan, Ann Arbor, 2004.