SIAM J. CONTROL AND OPTIMIZATION
Vol. 15, No. 5, August 1977

# OPTIMAL PERIODIC CONTROL: A GENERAL THEORY OF NECESSARY CONDITIONS*

ELMER G. GILBERT†

**Abstract.** Does time-dependent periodic control yield better process performance than optimal steady-state control? This paper examines exhaustively the role of first order necessary conditions in answering this question. For processes described by autonomous, ordinary differential equations, a very general optimal periodic control problem (OPC) is formulated. By considering control and state functions which are constant, a finite-dimensional optimal steady-state problem (OSS) is obtained from OPC. Three solution sets are introduced: $\mathscr{S}$(OSS)—the solutions of OSS, $\mathscr{S}$(OPC)—the solutions of OPC, $\mathscr{S}$(SSOPC)—the solutions of OPC which are constant. Necessary conditions for elements of each of these sets are derived; their solution sets are denoted, respectively, by $\mathscr{S}$(NCOSS), $\mathscr{S}$(NCOPC), and $\mathscr{S}$(NCSSOPC). The relationship between these six solutions sets is a central issue. Under various hypotheses certain pair-wise inclusions of the six sets are determined and it is shown that no others can be obtained. Tests which imply that time-dependent periodic control is better than optimal steady-state control (($\mathscr{S}$(SSOPC) $= \varnothing$, $\mathscr{S}$(OSS) $\neq \varnothing$), including those based on relaxed steady-state control, are investigated and limits to what tests exist are established. The results integrate and amplify results which have appeared in the literature. Examples provide insight which supports the theory.

**1. Introduction.** Since the 1967 paper by Horn and Lin [13] there has been an increasing interest in the mathematical theory of periodic processes. The motivations for this theory came initially from the optimization of chemical processes [3], but there are other areas of potential application such as vehicle cruise [10]. The essence of most applications is the optimization of a "continuing process," a process which is fixed in its characteristics and is expected to operate continuously over an indefinitely long period of time. The traditional approach to such problems is to minimize process cost by selecting constant controls subject to the constraint that the (dynamic) process is in static equilibrium. Although this "steady-state" approach is simple (time does not appear) and has intuitive appeal, it is not necessarily best. It may be possible to exploit the process dynamics and obtain even lower cost. Experiments with actual processes have shown that this can indeed be the case. The theory has helped to explain some of the mechanisms for such improvement and suggests situations where "time-dependent" control may improve performance. Much of the literature on periodic control has been reviewed by Bailey [3] and Guardabassi, Locatelli and Rinaldi [11].

The natural starting point for a theoretical investigation of continuing processes is the formulation of a dynamic optimization problem. It is clear from the preceding discussion that this optimal control problem should satisfy certain requirements: 1. the system dynamics and control constraints should not depend explicitly on time, 2. the system state and control functions should be defined on the time interval $(-\infty, +\infty)$, 3. a meaningful "optimal steady-state" problem,

* Received by the editors December 8, 1975, and in revised form August 30, 1976.

† Department of Aerospace Engineering, Program in Computer, Information and Control Engineering, University of Michigan, Ann Arbor, Michigan 48109. This research was completed while the author was on leave at the Department of Electrical Engineering, Johns Hopkins University, Baltimore, Maryland. It was supported in part by the United States Air Force, Air Force Office of Scientific Research, Air Force Systems Command, under Grants 73-2517 and 77-3158.

which does not involve time, should result when the system state and control functions are assumed to be constant. This is the attitude taken in this paper; everything is based on the optimal control problem (OPC) which is stated in § 2. The structure of this problem is chosen so that requirements 1 and 3 are met directly. Requirement 2 is imposed indirectly by assuming that the system state and control functions are periodic. Although this is not absolutely essential it is consistent with the previous literature, is a practical constraint, and avoids certain mathematical difficulties. The problem OPC, which assumes the system dynamics are represented by ordinary differential equations, is quite general and includes most of the problems which have appeared to date as special cases.

Because of the special form of OPC there are three notions of optimality (solutions of OPC, solutions of OPC which are constant, solutions of the steady-state problem) and, correspondingly, three sets of necessary conditions. Hence many potential relationships exist between the necessary conditions and the various optima. The investigation of these relationships is the central theme of this paper. Apart from its intrinsic interest this investigation is valuable for a number of other reasons: it puts together in a larger, more consistent framework many of the scattered results in the literature; it produces stronger tests for optimality and properness (time-dependent control better than optimal-steady-state control); it establishes certain limits to what can be proved concerning these tests; it sheds new light on the role of relaxed steady-state controls.

The paper is organized as follows. Section 2 states the problem OPC and introduces notation for the three sets of solutions. In § 3 the necessary conditions are derived. The developments are restricted to the "first variation" and are, for the most part, applications of well established theory. Section 4 introduces notation for the sets of solutions of the necessary conditions and relates these sets to the three sets of optima. Section 5 presents a number of examples which show that it is not possible to obtain more set inclusions than those obtained in § 4. Tests for properness are considered in § 6 and it is shown that under certain reasonable conditions no other tests exist. Tests for optimality and relative optima are also discussed. Section 7 treats relaxed steady-state optima; one of the main consequences is an extension of the well known results of Bailey and Horn [1].

It is worth noting that the concept of a continuing process seems essential to much of what follows. While it is possible to pose optimal periodic control problems which do not satisfy requirements 1 and 3, the results concerning the comparison of time-dependent and steady-state optima are greatly weakened.

**2. Formulation of the problem.** In this section a problem of optimal periodic control is formulated which meets the general requirements of the previous section. It models a wide class of continuing processes and subsumes a meaningful steady-state problem. Solution sets related to the two optimization problems are defined and some simple facts concerning them are noted.

Before stating the optimal periodic control problem it is necessary to introduce the following notation and assumptions: $j$ and $k$ are nonnegative integers, $T \in R$ is positive, $U \subset R^m$ is an arbitrary set, $X \subset R^n$ and $Y \subset R^l$ are open sets, for $i = -j, \cdots, k$ the functions $g_i$: $Y \times X \to R$ are continuously differentiable, the functions $f: X \times U \to R^n$ and $\tilde{f}: X \times U \to R^l$ are continuous and for each $u \in U$ are continuously differentiable in $x$.

*Optimal periodic control problem* (OPC). Find $u(\cdot)$, $x(\cdot)$ and $\tau$ which minimize $J$ subject to

(2.1-1) $$J = g_0(y, x(0)),$$

(2.1-2) $$g_i(y, x(0)) \leqq 0, \qquad i = -j, \cdots, -1,$$

(2.1-3) $$g_i(y, x(0)) = 0, \qquad i = 1, \cdots, k,$$

(2.1-4) $$y = \frac{1}{\tau}\int_0^\tau \tilde{f}(x(t), u(t))\, dt \in Y,$$

(2.1-5) $$\dot{x}(t) = f(x(t), u(t)) \quad \text{almost all } t \in [0, T], \quad x(0) = x(\tau),$$

(2.1-6) $$u(\cdot) \in \mathscr{U} = \{u(\cdot): u(\cdot) \text{ measurable and essentially bounded on } [0, T],\ u(t) \in U \text{ for all } t \in [0, T]\},$$

(2.1-7) $$x(\cdot) \in \mathscr{X} = \{x(\cdot): x(\cdot) \text{ absolutely continuous on } [0, T],\ x(t) \in X \text{ for all } t \in [0, T]\},$$

(2.1-8) $$\tau \in (0, T].$$

Some general comments are in order. Equations (2.1-5) represent the dynamics of the process and the constraints that $x(\cdot)$ and $u(\cdot)$ are periodic on $(-\infty, +\infty)$ when appropriate extensions of their definitions are made: $x(t+\nu\tau) = x(t)$, $u(t+\nu\tau) = u(t)$, $t \in [0, \tau)$, $\nu =$ integer. The components of $\tilde{f}(x(t), u/t))$ are quantities of interest in the optimization problem, e.g., rates of process fuel consumption, material flow rates, overhead cost rates, value measures of process products. It is the *average* of these quantities $y$, as given by (2.1-4), which appear in the actual optimization of the process, i.e., the minimization of (2.1-1) subject to (2.1-2) and (2.1-3). The dependence of the $g_i$ on $x(0)$ allows consideration of factors relating to the "start-up" of each cycle of operation. It also allows constraints to be imposed on $x(0) = x(\tau)$. Note that $f$ and $\tilde{f}$ and the control constraint set $U$ do not depend on $t$ and the $g_i$ do not depend on $\tau$. This is essential if the requirements 1 and 3 of § 1 are to be satisfied. The bound (2.1-8) is consistent with the assumption of periodic operation. While $T = +\infty$ is not allowed, arbitrarily large $T$ is permitted. Thus the quasi-stationary approximation treated in the literature [3], [11] can be extended to OPC. This is not done here. The convention $j = 0$ is used to denote the absence of inequality constraints; similarly $k = 0$ denotes absence of equality constraints.

By appropriate changes in notation problem formulations considered previously in the literature become special cases of OPC. For example, the problem of Guardabassi, Locatelli and Rinaldi [11] requires $j = 0$ and $g_i$, $i = 0, \cdots, k$ equal to the components of $y$; the problem of Bailey and Horn [1] requires $j = k = 0$ and $g_0$ equal to a general function of $y$. The problem in [1] is somewhat more general than it may first appear because a simple substitution of variables allows it to include the case $j = 0$, $k > 0$ when the functions $g_i$, $i > 0$, are components of $y$ [2]. However, when restricted to the context of continuing systems, none of the previous formulations have the full generality of OPC.

The steady-state problem is obtained from OPC by adding the constraint that $x(\cdot)$ and $u(\cdot)$ are constant. As expected, this yields a finite-dimensional optimization problem which does not depend on $\tau$.

*Optimal steady-state problem* (OSS). Find $u$ and $x$ which minimize $J$ subject to

(2.2-1) $$J = g_0(y, x),$$

(2.2-2) $$g_i(y, x) \leqq 0, \qquad i = -j, \cdots, -1,$$

(2.2-3) $$g_i(y, x) = 0, \qquad i = 1, \cdots, k,$$

(2.2-4) $$y = \tilde{f}(x, u) \in Y,$$

(2.2-5) $$f(x, u) = 0,$$

(2.2-6) $$u \in U,$$

(2.2-7) $$x \in X.$$

It is of interest to compare the solutions of OPC with the solutions of OSS. This can be done conveniently by introducing the following solution sets, all of which are subsets of $\mathcal{U} \times \mathcal{X} \times (0, T]$:

(2.3) $$\mathcal{S}(\text{OPC} = \{(u(\cdot), x(\cdot), \tau): (u(\cdot), x(\cdot), \tau \text{ solves OPC}\},$$

(2.4) $$\mathcal{S}(\text{SS}) = \{(u(\cdot), x(\cdot), \tau): \text{(2.1-2)–(2,1-8) are satisfied and } u(\cdot) \text{ and } x(\cdot) \text{ are constant}\},$$

(2.5) $$\mathcal{S}(\text{SSOPC}) = \mathcal{S}(\text{OPC}) \cap \mathcal{S}(\text{SS}),$$

(2.6) $$\mathcal{S}(\text{OSS}) = \{(u(\cdot), x(\cdot), \tau): (u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\text{SS}) \text{ and } (u(0), x(0)) \text{ solves OSS}\}.$$

Of course, $\mathcal{S} = \varnothing$, the null set, is possible in any of the four cases. The particular circumstance $\mathcal{S}(\text{SSOPC}) = \varnothing$, $\mathcal{S}(\text{OSS}) \neq \varnothing$ implies that there exist time-dependent controls which do better than the best steady-state controls. If $\mathcal{S}(\text{SSOPC}) \neq \varnothing$ any $\psi \in \mathcal{S}(\text{SSOPC})$ is also in $\mathcal{S}(\text{OSS})$ since $\psi$ is optimum with respect to choices in $\mathcal{U} \times \mathcal{X} \times (0, T]$ and $\mathcal{S}(\text{SS}) \subset \mathcal{U} \times \mathcal{X} \times (0, T]$. Also, it is clear that all elements of $\mathcal{S}(\text{OSS})$ and $\mathcal{S}(\text{SSOPC})$ yield identical costs $J$. This leads to the following.

*Remark* 2.1. There are three mutually exclusive possibilities:

(i) $\mathcal{S}(\text{SSOPC}) = \mathcal{S}(\text{OSS}) \neq \varnothing$;

(ii) $\mathcal{S}(\text{SSOPC}) = \varnothing$, $\mathcal{S}(\text{OSS}) \neq \varnothing$;

(iii) $\mathcal{S}(\text{SSOPC}) = \mathcal{S}(\text{OSS}) = \varnothing$.

Possibility (iii) is not apt to occur since for well posed problems it is likely that $\mathcal{S}(\text{OSS}) \neq \varnothing$. Possibility (i) implies that OPC has a steady-state solution and consequently, there is no advantage (even though OPC may also have time-dependent solutions) in using time-dependent control. Possibility (ii) implies time-dependent control can do better than steady-state control (a statement which holds true even if $\mathcal{S}(\text{OPC}) = \varnothing$). Because of the importance of possibilities (i) and (ii) the following definitions are introduced.

DEFINITION 2.1. If $\mathcal{S}(\mathrm{SSOPC}) = \mathcal{S}(\mathrm{OSS}) \neq \varnothing$ the problem OPC is called *steady-state.*

DEFINITION 2.2. If $\mathcal{S}(\mathrm{SSOPC}) = \varnothing$, $\mathcal{S}(\mathrm{OSS}) \neq \varnothing$ the problem OPC is called *proper* (compare [5]).

The study of relative minima of OPC and OSS will prove to be of value, particularly in the case of steady-state minima.

DEFINITION 2.3. $(u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\mathrm{SS})$ is a *strong* {*weak*} *relative minimum* of OPC if there exists an $\varepsilon > 0$ such that for all $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$ which satisfy (2.1-2)–(2.1-8) and $\|\hat{x}(t) - x(0)\| < \varepsilon$ $\{\|\hat{x}(t) - x(0)\| < \varepsilon,\ \|\hat{u}(t) - u(0)\| < \varepsilon\}$, $t \in [0, T]$, it follows that $g_0(y, x(0)) \leqq g_0(\hat{y}, \hat{x}(0))$.

DEFINITION 2.4. $(u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\mathrm{SS})$ is a *strong* {*weak*} *relative minimum* of OSS if there exists an $\varepsilon > 0$ such that for all $(\hat{u}, \hat{x})$ which satisfy (2.2-2)–(2.2-7) and $\|\hat{x} - x(0)\| < \varepsilon$ $\{\|\hat{x} - x(0)\| < \varepsilon,\ \|\hat{u} - u(0)\| < \varepsilon\}$ it follows that $g_0(y, x(0)) \leqq g_0(\hat{y}, \hat{x})$.

In these definitions $\|\cdot\|$ denotes any norm on $\boldsymbol{R}^n$ or $\boldsymbol{R}^l$ and $\hat{y} = y$ for $u = \hat{u}$, $x = \hat{x}$, $\tau = \hat{\tau}$. Corresponding to each of the four types of relative minima, notations for the set of minima are adopted:

$$\mathcal{S}(\mathrm{SRMSSOPC}), \mathcal{S}(\mathrm{WRMSSOPC}), \mathcal{S}(\mathrm{SRMOSS}), \mathcal{S}(\mathrm{WRMOSS}).$$

For example,

$$\begin{aligned}(2.7)\quad \mathcal{S}(\mathrm{SRMSSOPC}) = \{(u(\cdot), x(\cdot), \tau) \colon & (u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\mathrm{SS}) \\ & \text{is a strong relative minimum of OPC}\}.\end{aligned}$$

Obviously, $\mathcal{S}(\mathrm{SSOPC}) \subset \mathcal{S}(\mathrm{SRMSSOPC}) \subset \mathcal{S}(\mathrm{WRMSSOPC})$ and $\mathcal{S}(\mathrm{OSS}) \subset \mathcal{S}(\mathrm{SRMOSS}) \subset \mathcal{S}(\mathrm{WRMOSS})$. By using the same reasoning which led to Remark 2.1 it is easy to see that $\mathcal{S}(\mathrm{SRMSSOPC}) \subset \mathcal{S}(\mathrm{SRMOSS})$. However, $\mathcal{S}(\mathrm{SRMSSOPC}) \neq \varnothing$ does not imply $\mathcal{S}(\mathrm{SRMSSOPC}) = \mathcal{S}(\mathrm{SRMOSS})$ because elements of $\mathcal{S}(\mathrm{SRMSSOPC})$ do not necessarily have the same cost as elements of $\mathcal{S}(\mathrm{SRMOSS})$. Similar reasoning applies to the case of weak relative minima. All of this is summarized in

*Remark* 2.2. The following conclusions are valid: $\mathcal{S}(\mathrm{SSOPC}) \subset \mathcal{S}(\mathrm{SRMSSOPC}) \subset \mathcal{S}(\mathrm{WRMSSOPC})$, $\mathcal{S}(\mathrm{OSS}) \subset \mathcal{S}(\mathrm{SRMOSS}) \subset \mathcal{S}(\mathrm{WRMOSS})$, $\mathcal{S}(\mathrm{SSOPC}) \subset \mathcal{S}(\mathrm{OSS})$, $\mathcal{S}(\mathrm{SRMSSOPC}) \subset \mathcal{S}(\mathrm{SRMOSS})$, $\mathcal{S}(\mathrm{WRMSSOPC}) \subset \mathcal{S}(\mathrm{WRMOSS})$.

**3. The necessary conditions.** Since explicit characterization of $\mathcal{S}(\mathrm{OPC})$, $\mathcal{S}(\mathrm{SSOPC})$ and $\mathcal{S}(\mathrm{OSS})$ is generally difficult or impossible, it is essential to consider necessary conditions for the elements of these sets. The necessary conditions for OPC will be obtained by applying some necessary conditions obtained by Neustadt (summarized in Appendix A). Similarly, known conditions for finite-dimensional optimization problems (summarized in Appendix B) are applied to OSS. An entirely separate derivation starting from the necessary conditions for OPC is required to obtain necessary conditions for elements of $\mathcal{S}(\mathrm{SSOPC})$. Relationships between the various necessary conditions and the solution sets introduced in the previous section are examined in § 4.

In what follows: let $f_x(x, u)$ and $\tilde{f}_x(x, u)$ denote respectively the Jacobian matrices of $f(x, u)$ and $\tilde{f}(x, u)$ with respect to $x$; for $i = -j, \cdots, k$, let $g_{iy}(y, x)$ and

$g_{ix}(y, x)$ denote respectively the Jacobian (row) matrices of $g_i(y, x)$ with respect to $y$ and $x$; let a prime denote the transpose of a (column) vector or matrix.

THEOREM 3.1 (necessary conditions for OPC). *Let*

$$H(x, u, p, \tilde{p}) = p'f(x, u) + \tilde{p}'\tilde{f}(x, u) \tag{3.1}$$

*where $p \in R^n$ and $\tilde{p} \in R^l$. Let $(u(\cdot), x(\cdot), \tau)$ solve OPC. Then there exist an absolutely continuous function $p(\cdot): [0, \tau] \to R^n$, $\tilde{p} \in R^l$ and real numbers $\alpha_{-j}, \cdots, \alpha_k$ such that the following conditions are satisfied*:

$$\max_{v \in U} H(x(t), v, p(t), \tilde{p}) = H(x(t), u(t), p(t), \tilde{p}) \qquad \textit{almost all } t \in [0, \tau], \tag{3.2-1}$$

$$\tilde{p}' = \sum_{i=-j}^{k} \alpha_i g_{iy}(y, x(0)), \tag{3.2-2}$$

$$\dot{p}'(t) = -p'(t) f_x(x(t), u(t)) - \tilde{p}'\tilde{f}_x(x(t), u(t)) \qquad \textit{almost all } t \in [0, \tau], \tag{3.2-3}$$

$$\begin{aligned} &p'(\tau) - p'(0) = \tau \sum_{i=-j}^{k} \alpha_i g_{ix}(y, x(0)), \\ &\alpha_i \leqq 0, \qquad i = -j, \cdots, 0, \\ &\alpha_i g_i(y, x(0)) = 0, \qquad i = -j, \cdots, -1, \\ &(\alpha_{-j}, \cdots, \alpha_k, p'(\tau)) \neq 0. \end{aligned} \tag{3.2-4}$$

*If $f(x(\cdot), u(\cdot))$ and $\tilde{f}(x(\cdot), u(\cdot))$ are continuous at $\tau$ the following additional condition is satisfied*:

$$\begin{aligned} &\tilde{p}'y \leqq H_M \quad \textit{if } \tau = T, \\ &\tilde{p}'y = H_M \quad \textit{if } \tau < T, \end{aligned} \tag{3.2-5}$$

*where*

$$H_M = \max_{v \in U} H(x(\tau), v, p(\tau), \tilde{p}). \tag{3.2-6}$$

*Proof.* With the following substitution OPC can be written as GOC of Appendix A: $\hat{n} = n + l$, $\mu = j + 1$, $\nu = k + n$, $\hat{X} = X \times R^l$, $\hat{x} = (x, \tilde{x})$, $\hat{f}(\hat{x}, \mu) = (f(x, u), \tilde{f}(x, u))$; for $i = -j, \cdots, k$, $\theta_i(\hat{x}^1, \hat{x}^2, \tau) = g_i(\tau^{-1}(\tilde{x}^2 - \tilde{x}^1), x^1)$; for $i = k+1, \cdots, k+n$, $\theta_i(\hat{x}^1, \hat{x}^2, \tau) = x^2_{i-k} - x^1_{i-k}$ where the subscripts denote the components of $x^2$ and $x^1$; $\theta_{-j-1}(\hat{x}^1, \hat{x}^2, \tau) = \tau - T$; $\hat{t}$ is any real number greater than $T$. By choosing $\hat{X}^1$ and $\hat{X}^2$ to be appropriate neighborhoods of $\hat{x}^1(0)$ and $\hat{x}^2(\tau)$ the constraint $y \in Y$ is assured. Using the conditions from Theorem A.1, letting $\hat{p} = (p, \tilde{p})$, and replacing $\alpha_i$ by $\tau\alpha_i$ gives conditions (3.2). To confirm the last line of (3.2-4), note that the last condition of (A.3-4) can be written $(\tilde{p}'y - H_M, \alpha_{-j}, \cdots, \alpha_k, p'(\tau)) \neq 0$. Since $(\alpha_{-j}, \cdots, \alpha_k, p'(\tau)) = 0$ implies $\tilde{p}'y - H_M = 0$, the last line of (3.2-4) must follow.

Before stating the necessary conditions for OSS it is necessary to introduce a procedure for obtaining "perturbations" in the constraint set $U$. This can be done in a variety of ways (see, e.g. [7], [17], [19]) without being very specific about the characterization of $U$. Here the presentation follows Canon, Cullum and Polak [7]. Let co $V$ = convex hull of $V$ and cl $V$ = closure of $V$.

DEFINITION 3.1. A convex cone $C(u, U) \subset R^m$, $u \in U$, is a *conical approximation* to $U$ at $u$ if for any collection $\{\delta u_1, \cdots, \delta u_s\}$ of vectors in $C(u, U)$ there exist an $\varepsilon > 0$ and a continuous function $\zeta$: co $\{u, u+\varepsilon\delta u_1, \cdots, u+\varepsilon\delta u_s\} \to U$, both dependent on $\{\delta u_1, \cdots, \delta u_s\}$, such that $\zeta(u+\delta u) = u+\delta u+o(\delta u)$ where $\|o(\delta u)\| \cdot \|\delta u\|^{-1} \to 0$ as $\delta u \to 0$.

When $U$ has simple characterizations so does $C(u, U)$. For example, suppose

$$U = \{u\colon h_i(u) \leqq 0,\ i = 1, \cdots, q\}, \tag{3.3}$$

where the $h_i$ are continuously differentiable on $R^m$ with Jacobian matrices $h_{iu}(u)$. Let $I(u) = \{i\colon h_i(u) = 0\}$. Then

$$\text{cl}\, C(u, U) = \{\delta u\colon h_{iu}(u)\delta u \leqq 0,\ i \in I(u)\} \tag{3.4}$$

if $U$ is convex or $\{h_{iu}(u)\}_{i\in I(u)}$ are linearly independent. For more details see [7].

Finally, the assumptions on $f$ and $\tilde{f}$ must be strengthened. When they exist, let $f_u(x, u)$ and $\tilde{f}_u(x, u)$ denote respectively the Jacobian matrices of $f(x, u)$ and $\tilde{f}(x, u)$ with respect to $u$.

*Assumption* A1. $f$ and $\tilde{f}$ are continuously differentiable on $X \times \hat{U}$ where $U \subset \hat{U}$ and $\hat{U} \subset R^m$ is an open set.

THEOREM 3.2 (necessary conditions for OSS). *Let $f$ and $\tilde{f}$ satisfy Assumption* A1 *and let $(u, x)$ solve OSS. Then there exist $p \in R^n$, $\tilde{p} \in R^l$ and real numbers $\alpha_{-j}, \cdots, \alpha_k$ such that the following conditions are satisfied for any $C(u, U)$ which is a conical approximation to $U$ at $u$:*

$$(p'f_u(x, u)+\tilde{p}'\tilde{f}_u(x, u))\delta u \leqq 0 \quad \textit{for all } \delta u \in \text{cl}\, C(u, U), \tag{3.5-1}$$

$$\tilde{p}' = \sum_{i=-j}^{k} \alpha_i g_{iy}(y, x), \tag{3.5-2}$$

$$-p'f_x(x, u)-\tilde{p}'\tilde{f}_x(x, u) = \sum_{i=-j}^{k} \alpha_i g_{ix}(y, x), \tag{3.5-3}$$

$$\begin{aligned} &\alpha_i \leqq 0, \qquad i = -j, \cdots, 0,\\ &\alpha_i g_i(y, x) = 0, \qquad i = -j, \cdots, -1,\\ &(\alpha_{-j}, \cdots, \alpha_k, p') \neq 0. \end{aligned} \tag{3.5-4}$$

*Proof.* With the following substitutions OSS can be written as FDO of Appendix B: $\hat{n} = n+l$, $\mu = j$, $\nu = k+n+l$, $\hat{X} = X \times Y$, $\hat{x} = (x, y)$; for $i = -j, \cdots, k$, $\theta_i(\hat{x}, u) = g_i(y, x)$; for $i = k+1, \cdots, k+n$, $\theta_i(\hat{x}, u) = f_{i-k}(x, u)$ where the subscripts denote components of $f(x, u)$; for $i = k+n+1, \cdots, k+n+l$, $\theta_i(\hat{x}, u) = \tilde{f}_{i-k-n}(x, u) - y_{i-k-n}$ where the subscripts denote components of $\tilde{f}(x, u)$ and $y$. Applying the conditions from Theorem B.1, letting $p' = (\alpha_{k+1}, \cdots, \alpha_{k+n})$ and $\tilde{p}' = (\alpha_{k+n+1}, \cdots, \alpha_{k+n+l})$, gives the conditions (3.5). The last line of (3.5-4) holds because $(\alpha_{-j}, \cdots, p') = 0$ and $\tilde{p} \neq 0$ is impossible.

By changing the hypotheses other necessary conditions for OSS may be obtained.

*Assumption* A2. The set

$$\hat{f}(x, U) = \{(f(x, u), \tilde{f}(x, u)) : u \in U\} \subset R^{n+l} \tag{3.6}$$

is convex for all $x \in X$.

THEOREM 3.3 (maximum principle for OSS). *Let $f$ and $\tilde{f}$ satisfy Assumption* A2. *Let $(u, x)$ solve OSS. Then there exist $p \in R^n$, $\tilde{p} \in R^l$ and real numbers $\alpha_{-j}, \cdots, \alpha_k$ such that conditions* (3.5-2), (3.5-3), (3.5-4) *and the following condition are satisfied*:

$$\max_{v \in U} H(x, v, p, \tilde{p}) = H(x, u, p, \tilde{p}). \tag{3.5-1}$$

*Proof.* Make the same notational assignments as in the proof of Theorem 3.2. Applying Theorem B.2 gives (3.5-1)′ instead of (3.5-1) while everything else remains the same as in the proof of Theorem 3.2.

*Remark* 3.1. By applying Theorem B.1 to the maximization problem (3.5-1)′, under Assumption A1, it can be seen that (3.5-1) is a necessary condition for (3.5-1)′. Thus the conditions obtained in Theorem 3.3 are stronger than those in Theorem 3.2.

THEOREM 3.4 (necessary conditions for SSOPC). *Let* $(u(\cdot), x(\cdot), \tau) \in \mathcal{S}$(SSOPC). *Then there exist $p \in R^n$, $\tilde{p} \in R^l$ and real numbers $\alpha_{-j}, \cdots, \alpha_k$ such that conditions* (3.5-1)′, (3.5-2), (3.5-3) *and* (3.5-4) *are satisfied for $u = u(0)$ and $x = x(0)$.*

*Proof.* Introduce the following notation: $\alpha' = (\alpha_{-j}, \cdots, \alpha_k)$, $g'(y, x) = (g_{-j}(y, x), \cdots, g_k(y, x))$, $g_y(y, x) =$ Jacobian matrix of $g(y, x)$ with respect to $y$, $g_x(y, x) =$ Jacobian matrix of $g(y, x)$ with respect to $x$. Since $u(\cdot)$ and $x(\cdot)$ are constant let $u(t) \equiv u^*$ and $x(t) \equiv x^*$ and define: $f_x^* = f_x(x^*, u^*)$, $\tilde{f}_x^* = \tilde{f}_x^*(x^*, u^*)$, $y^* = \tilde{f}(x^*, u^*)$, $g_y^* = g_y(y^*, x^*)$, $g_x^* = g_x(y^*, x^*)$. Clearly, $(u(\cdot), x(\cdot), \sigma) \in \mathcal{S}$(SSOPC) for all $\sigma \in (0, T]$. Thus, for each $\sigma$, $(u(\cdot), x(\cdot), \sigma)$ must satisfy the conditions of Theorem 3.1. For each $\sigma$ let $\alpha_i(\sigma)$, $i = -j, \cdots, k$, and $p(\sigma, \cdot)$ denote corresponding $\alpha_i$ and $p(\cdot)$ whose existence is guaranteed by Theorem 3.1. It is easy to show that (3.2-5) is satisfied automatically for $\tau = \sigma$ and imposes no conditions on $\alpha_i(\sigma)$ and $p(\sigma, \cdot)$. By introducing the sets

$$V^* = \{(\alpha, p) : \alpha_i \leqq 0,\ i = -j, \cdots, 0;\ \alpha_i g_i(y^*, x^*) = 0,\ i = -j, \cdots, -1\}, \tag{3.7}$$

$$C^* = \{(\alpha, p) : p'(f(x^*, v) - f(x^*, u^*)) + \alpha' g_y^*(\tilde{f}(x^*, v) - \tilde{f}(x^*, u^*)) \leqq 0 \quad \text{for all } v \in U\} \tag{3.8}$$

the conditions imposed by (3.2-1)–(3.2-4) on $\alpha(\sigma)$ and $p(\sigma, \cdot)$ can be written

$$(\alpha(\sigma), p(\sigma, \sigma)) \neq 0, \tag{3.9-1}$$

$$(\alpha(\sigma), p(\sigma, t)) \in V^* \cap C^* \quad \text{for all } t \in [0, \sigma], \tag{3.9-2}$$

$$\dot{p}'(\sigma, t) = -p'(\sigma, t) f_x^* - \alpha'(\sigma) g_y^* \tilde{f}_x^* \quad \text{for all } t \in [0, \sigma], \tag{3.9-3}$$

$$\sigma^{-1}(p'(\sigma, \sigma) - p'(\sigma, 0)) = \alpha'(\sigma) g_x^*. \tag{3.9-4}$$

These conditions must hold for all $\sigma \in (0, T]$; $\tilde{p}' = \alpha' g_y^*$ has been used to eliminate $\tilde{p}'$.

With the use of the variation of parameters formula condition (3.9-3) can be written

$$p'(\sigma, t) = p'(\sigma, 0)P(t) + \alpha'(\sigma)Q(t) \tag{3.10-1}$$

where the matrices $P(\cdot)$ and $Q(\cdot)$ are analytic on $[0, T]$ and satisfy the conditions: $P(0) =$ the identity matrix, $\dot{P}(0) = -f_x^*$, $Q(0) = 0$, $\dot{Q}(0) = -g_y^* \tilde{f}_x^*$. Note that if $\alpha(\sigma)$, $p(\sigma, \cdot)$ satisfy (3.9) then $\lambda\alpha(\sigma)$, $\lambda p(\sigma, \cdot)$ do also, where $\lambda$ is a positive real number. Thus $\alpha(\sigma)$, $p(\sigma, \cdot)$ can always be normalized so that (3.9-1) becomes $\|\alpha(\sigma)\| + \|p(\sigma, \sigma)\| = 1$. Because of (3.10-1) and the properties of $P(\cdot)$ and $Q(\cdot)$ there therefore exists a $\hat{T} \in (0, T]$ such that (3.9-2) yields

$$\alpha(\sigma), p(\sigma, t)) \in V^* \cap C^* \cap \{(\alpha, p): .5 \leqq \|\alpha\| + \|p\| \leqq 1.5\} \quad \text{for all } t, \sigma \in [0, \hat{T}]. \tag{3.10-2}$$

Finally, by using (3.10-1) and the properties of $P(\cdot)$ and $Q(\cdot)$ it is possible to write (3.9-4) as

$$-p'(\sigma, 0)f_x^* - \alpha'(\sigma)g_y^* \tilde{f}_x^* + \gamma(\sigma) = \alpha'(\sigma)g_x^* \quad \text{for all } \sigma \in [0, \hat{T}], \tag{3.10-3}$$

where $\gamma(\sigma) \to 0$ as $\sigma \to 0$.

Now let $(\sigma_q\}$ be a sequence in $[0, \hat{T}]$ such that $\sigma_q \to 0$. From (3.7) $V^*$ is closed and $C^*$ is closed because it is the dual cone [20] of the set $\{(\beta, \rho): \beta = g_y^*(\tilde{f}(x^*, v) - \tilde{f}(x^*, u^*)), \rho = f(x^*, v) - f(x^*, u^*), v \in U\}$. Thus the set on the right side of (3.10-2) is compact and there exists a subsequence of $\{\sigma_q\}$, $\{\sigma_{\hat{q}}\}$, such that $\sigma_{\hat{q}} \to 0$, $\alpha(\sigma_{\hat{q}}) \to \hat{\alpha}$ and $p(\sigma_{\hat{q}}, 0) \to \hat{p}$ where $(\hat{\alpha}, \hat{p}) \in V^* \cap C^*$ and $.5 \leqq \|\hat{\alpha}\| + \|\hat{p}\| \leqq 1.5$. This shows that $\hat{\alpha}$, $\hat{p}$ satisfy (3.5-1)$'$ and (3.5-4) and $\tilde{p}' = \hat{\sigma}' g_y^*$ satisfies (3.5-2). From (3.10-3) and $\gamma(\sigma_{\hat{q}}) \to 0$ it follows that $-\hat{p}' f_x^* - \hat{\alpha}' g_y^* \tilde{f}_x^* = \hat{\alpha} g_x^*$, which verifies (3.5-3).

*Remark* 3.2. The conditions in Theorems 3.3 and 3.4 are the same. Thus the reasoning used in Remark 3.1 shows that the conditions in Theorem 3.2 (with $u = u(0)$, $x = x(0)$) are necessary conditions for the elements of $\mathscr{S}$(SSOPC). However, since this (weaker) set of conditions arises from OSS it has no value in distinguishing the difference between "steady-state" and "time-dependent" control. Similar observations have been made in more restrictive circumstances by Horn and Lin [13].

*Remark* 3.3. It is not difficult to modify the preceding developments if $\tau$ is fixed ($\tau = T$). All the theorems are unchanged, except that condition (3.2-5) is eliminated from Theorem 3.1. The proofs are the same except: $\tau = T$ is treated as an equality constraint in the application of Theorem A.1 to the proof of Theorem 3.1, the elements of the sequence $\{\sigma_q\}$ in the proof of Theorem 3.4 are given by $\sigma_q = (q)^{-1}T$.

Several comments concerning Theorem 3.1 and its relation to previous results in the literature are in order. There are, of course, many necessary conditions which can be written. Theorem 3.1 represents a good compromise in getting strong necessary conditions with weak hypotheses. Previous derivations of necessary conditions [2], [8], [13] have required stronger assumptions, apply to more specialized problems, and have given the same or weaker conditions. It seems essential to follow a line of proof similar to that which has been taken above. The comprehensive approach taken by Bailey [2]·adapts the conditions from [18] by a change of variables. This approach applied to OPC would require the $g_i$, $i \neq 0$, to be twice differentiable (a hypothesis which for Bailey's problem is evident from equation (29) of [2]). Moreover, inequality constraints would be handled by the trick of Valentine which gives somewhat weaker necessary conditions ($\alpha_i \leqq 0$, $i = -j, \cdots, -1$, omitted from(3.2-4)). The requirement on the continuity of $f(x(\cdot), u(\cdot))$ and $\tilde{f}(x(\cdot), u(\cdot))$ which is needed for (3.2-5) is satisfied automatically when $u(\cdot)$ is piecewise continuous with a finite number of discontinuities. This accounts for the absence of the continuity requirement in the conditions obtained in [2]. Additional necessary conditions, e.g. the derivative condition on $H$ expressed by equation (17) of [14], require additional hypotheses which appear to be quite strong or difficult to verify generally. The necessary conditions obtained in [5], [12] are of considerable interest, but they involve consideration of the second variation and therefore go beyond the scope of this paper.

Consider what happens if OPC is modified by replacing $g_i(y, x)$ by $g_i(y, x, \tau)$ for $i = -j, \cdots, k$. The modified OPC is not a continuing process in the sense of § 1 because requirement 3 is not satisfied. All of the preceding definitions and results can be generalized to the modified OPC, except for Theorem 3.4. The proof of Theorem 3.4 fails because $(u(\cdot), x(\cdot), \tau^*) = \mathcal{S}(\text{SSOPC})$ no longer implies $(u(\cdot), x(\cdot), \sigma) \in \mathcal{S}(\text{SSOPC})$ for all $\sigma \in (0, T]$. Since much of what follows revolves about Theorem 3.4, this shows the importance of requirement 3. A similar observation applies to the relaxation of requirement 1.

**4. Relationships between the necessary conditions and the solution sets.** In order to simplify references to the necessary conditions and make clearer their relationship to the solution sets introduced in § 2 it is helpful to introduce the following definitions:

(4.1) $\mathcal{S}(\text{NCOPC}) = \{(u(\cdot), x(\cdot), \tau)$: equations (2.1-2)–(2.1-8) are satisfied and there exist $p(\cdot), \tilde{p}, \alpha_{-j}, \cdots, \alpha_k$ such that the conditions of Theorem 3.1 hold$\}$,

(4.2) $\mathcal{S}(\text{NCOSS}) = \{(u(\cdot), x(\cdot), \tau)$: $(u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\text{SS})$ and there exist $p, \tilde{p}, \alpha_{-j}, \cdots, \alpha_k$ such that the conditions of Theorem 3.2 hold with $u = u(0)$ and $x = x(0)\}$,

(4.3) $\mathcal{S}(\text{NCSSOPC}) = \{(u(\cdot), x(\cdot), \tau)$: $(u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\text{SS})$ and there exist $p, \tilde{p}, \alpha_{-j}, \cdots, \alpha_k$ such that the conditions of Theorem 3.4 hold$\}$.

The set $\mathcal{S}(\text{NCOSS})$ has been defined as a subset of $\mathcal{U} \times \mathcal{X} \times (0, T]$, even though Theorem 3.2 requires $(u, x) \in U \times X$. This is done as was the case with $\mathcal{S}(\text{OSS})$ to emphasize the fact that steady-state control is a special case of time-dependent control and to allow a direct comparison of all solution sets.

With the above definitions Theorems 3.1–3.4 can be paraphrased compactly by the following inclusions: $\mathcal{S}(\text{OPC}) \subset \mathcal{S}(\text{NCOPC})$; if A1 is satisfied $\mathcal{S}(\text{OSS}) \subset \mathcal{S}(\text{NCOSS})$; if A2 is satisfied $\mathcal{S}(\text{OSS}) \subset \mathcal{S}(\text{NCSSOPC})$;

$\mathcal{S}$(SSOPC) $\subset \mathcal{S}$(NCSSOPC). Furthermore, if A1 is satisfied it is clear from Remark 3.2 that $\mathcal{S}$(NCSSOPC) $\subset \mathcal{S}$(NCOSS).

Since Theorem 3.4 was obtained from Theorem 3.1 it is tempting to surmise that $\mathcal{S}$(NCSSOPC) $\subset \mathcal{S}$(NCOPC). The following example shows that this conclusion is not valid.

*Example* 4.1. $k=j=0$, $n=l=1$, $X=Y=R$, $U=[-1,1]\subset R$, $T=1$, $f=-x+u$, $\tilde{f}=x$, $g_0=y-\frac{3}{4}x-\frac{1}{8}x^2$. Application of the conditions in Theorem 3.4 shows that $\mathcal{S}$(NCSSOPC) is characterized by elements of the form: $u(t)\equiv x(t)\equiv 1$ or $-1$, $\tau\in(0,1]$. Now consider those elements of $\mathcal{S}$(NCOPC) which also belong to $\mathcal{S}$(SS). Application of the conditions in Theorem 3.1 is more difficult because $p(\cdot)$ is not necessarily constant. However, in this example it is not difficult to integrate (3.2-3) and verify that $\mathcal{S}$(NCOPC) $\cap\, \mathcal{S}$(SS) is characterized by elements of the form: $u(t)\equiv x(t)\equiv -1$, $\tau\in(0,1]$. The elements $u(t)\equiv x(t)\equiv 1$, $\tau\in(0,1]$ are excluded because condition (3.2-1) requires $p(t)\leqq 0$ on $[0,\tau]$ and this turns out to be impossible. Thus $\mathcal{S}$(NCSSOPC) $\not\subset \mathcal{S}$(NCOPC). Under the assumption which follows it is possible to prove $\mathcal{S}$(NCSSOPC) $\subset \mathcal{S}$(NCOPC).

*Assumption* A3. The functions $g_{-j}(y,x),\cdots,g_k(y,x)$ depend only on $y$.

THEOREM 4.1. *Let* A3 *be satisfied. Then* $\mathcal{S}$(NCSSOPC) $\subset \mathcal{S}$(NCOPC).

*Proof.* Suppose $(u(\cdot),x(\cdot),\tau)\in\mathcal{S}$(NCSSOPC) and let $u=u(0)$, $x=x(0)$. Then there exist $p\in R^n$, $\tilde{p}\in R^l$ and real numbers $\alpha_{-j},\cdots,\alpha_k$ which satisfy (3.5-1)′, (3.5-2)–(3.5-4). Because $g_{ix}(y,x)\equiv 0$, $i=-j,\cdots,k$, this implies $p(t)\equiv p$, $\tilde{p}$, $\alpha_{-j},\cdots,\alpha_k$ satisfy (3.2-1)–(3.2-4). Since $f(x(t),u(t))\equiv 0$ and $y=\tilde{f}(x(\tau),u(\tau))$ condition (3.2-5) is satisfied as an equality. Thus $(u(\cdot),x(\cdot),\tau)\in\mathcal{S}$(NCOPC).

*Remark* 4.1. For OPC problems which do not satisfy A3, Theorem 3.1 may (as Example 4.1 illustrates) offer a stronger test for $(u(\cdot),x(\cdot),\tau)\in\mathcal{S}$(SSOPC) than Theorem 3.4. This is not surprising because Theorem 3.4 is obtained from Theorem 3.1 by drawing certain conclusions as $\tau\to 0$. Unfortunately, the test may be much more difficult to apply because the (constant-coefficient, linear) differential equations (3.2-3) must be considered. For OPC problems which do satisfy A3 (this includes almost all the problems which have appeared in the literature on periodic control) Theorem 4.1 shows that Theorem 3.4 provides at least as strong a test as Theorem 3.1.

Now consider a variation of Example 4.1.

*Example* 4.2. Same as Example 4.1, except $T=2$. It is easy to show $\mathcal{S}$(NCSSOPC) is the same as in Example 4.1 and that $\mathcal{S}$(NCOPC) $\cap\, \mathcal{S}$(SS) is characterized by elements of the form: $u(t)\equiv x(t)\equiv -1$, $\tau\in(0,\tau^*]$. Here $\tau^*=1.5936\cdots$ is the positive root of $\tau=2(1-e^{-\tau})$. Elements of the form $u(t)\equiv x(t)\equiv -1$, $\tau\in(\tau^*,2]$ are excluded from $\mathcal{S}$(NCOPC) because (3.2-3) shows that it is impossible for $p(t)\geqq 0$ on $(0,\tau]$ if $\tau>\tau^*$ and $p(t)\geqq 0$ is required by (3.2-1). The characterization of $\mathcal{S}$(NCOPC) $\cap\, \mathcal{S}$(SS) leads to the following observation.

*Remark* 4.2. Let $(u(\cdot),x(\cdot),\hat{\tau})\in\mathcal{S}$(SSOPC). Since this implies $(u(\cdot),x(\cdot),\tau)\in\mathcal{S}$(SSOPC) for all $\tau\in(0,T]$ the conditions in Theorem 3.1 apply to $(u(\cdot),x(\cdot),\tau)$ for all $\tau\in(0,T]$. If Theorem 3.1 is to be exploited fully for testing $(u(\cdot),x(\cdot),\tau)\in\mathcal{S}$(SSOPC) all values of $\tau\in(0,T]$ must be considered. This is illustrated by Example 4.2. For $\tau\in(\tau^*,2]$ there are no elements of $\mathcal{S}$(SS) which satisfy the conditions of Theorem 3.1. Thus it may be concluded that $\mathcal{S}$(SSOPC) $=\varnothing$ For $\tau\in(0,\tau]$ it cannot be concluded from Theorem 3.1 that $\mathcal{S}$(SSOPC) $=\varnothing$.

Using the results of § 2 and this section it is now possible to summarize compactly what is known about the sets $\mathscr{S}(\text{OPC})$, $\mathscr{S}(\text{SSOPC})$, $\mathscr{S}(\text{OSS})$, $\mathscr{S}(\text{NCOPC})$, $\mathscr{S}(\text{NCSSOPC})$, and $\mathscr{S}(\text{NCOSS})$.

THEOREM 4.2. (i) $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{OSS})$, (ii) $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{OPC})$, (iii) $\mathscr{S}(\text{OPC}) \subset \mathscr{S}(\text{NCOPC})$, (iv) $\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{NCSSOPC})$, (v) *if* A1 *is satisfied* $\mathscr{S}(\text{OSS}) \subset \mathscr{S}(\text{NCOSS})$, (vi) *if* A1 *is satisfied* $\mathscr{S}(\text{NCSSOPC}) \subset \mathscr{S}(\text{NCOSS})$, (vii) *if* A3 *is satisfied* $\mathscr{S}(\text{NCSSOPC}) \subset \mathscr{S}(\text{NCOPC})$, (viii) *if* A2 *is satisfied* $\mathscr{S}(\text{OSS}) \subset \mathscr{S}(\text{NCSSOPC})$.

In reading the theorem it should be noted that assumptions A1 and A3 are satisfied in many applications of the theory. Assumption A2 is strong and, as will be seen later, has strong implications. Are there additional inclusions beyond those listed in the theorem? The answer is generally no, a conclusion which is made precise in the next section. The inclusions of Theorem 4.2 are summarized in Fig. 1.

$$
\begin{array}{ccc}
\mathscr{S}(\text{OPC}) & \subset & \mathscr{S}(\text{NCOPC}) \\
\cup & & \cup \;\; \text{A3} \\
\mathscr{S}(\text{SSOPC}) & \subset & \mathscr{S}(\text{NCSSOPC}) \\
\cap & & \cap \;\; \text{A1} \\
\mathscr{S}(\text{OSS}) & \underset{\text{A1}}{\subset} & \mathscr{S}(\text{NCOSS})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathscr{S}(\text{OPC}) & \subset & \mathscr{S}(\text{NCOPC}) \\
\cup & & \cup \;\; \text{A3} \\
\mathscr{S}(\text{SSOPC}) \subset \mathscr{S}(\text{OSS}) & \subset & \mathscr{S}(\text{NCSSOPC}) \\
 & & \cap \;\; \text{A1} \\
 & & \mathscr{S}(\text{NCOSS})
\end{array}
$$

Without A2 With A2

FIG. 1. *Summary of Theorem* 4.2. *See* (2.3), (2.4), (2.5), (4.1), (4.2) *and* (4.3) *for definitions of solution sets.*

The results of the previous section are also related to the solution sets of relative minima. For example, let $(u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SRMSSOPC})$. Then if $X$ is replaced by $X \cap \{\hat{x} : \|\hat{x} - x(0)\| < \varepsilon\}$, $\varepsilon > 0$ sufficiently small, $(u(\cdot), x(\cdot), \tau)$ is a regular minimum and the conditions of Theorem 3.4 apply without change to $(u(\cdot), x(\cdot), \tau)$. Thus $(u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{NCSSOPC})$. Similar arguments apply to weak relative minima but in the cases of Theorems 3.3 and 3.4 it is necessary to introduce a weak form of the maximum condition,

$$
\max_{v \in V, \|v-u\| < \varepsilon} H(x, v, p, \tilde{p}) = H(x, u, p, \tilde{p}), \tag{4.4}
$$

and define

$$
\begin{aligned}
\mathscr{S}(\text{WNCSSOPC}) = \{(u(\cdot), x(\cdot), \tau) :\ & (u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\text{SS}) \text{ and there} \\
& \text{exist } p, \tilde{p}, \alpha_{-j}, \cdots, \alpha_k \text{ such that conditions (4.7), (3.5-} \\
& \text{2), (3.5-3) and (3.5-4) hold with } u = u(0) \text{ and } x = x(0) \\
& \text{for some } \varepsilon > 0\}.
\end{aligned} \tag{4.5}
$$

In addition the following assumption, which is not necessarily stronger than A2, must be introduced.

*Assumption* A4. There exists an $\bar{\varepsilon} > 0$ such that the set

$\hat{f}(x, U \cap \{u: \|u-v\| \leqq \varepsilon\})$ (see (3.6) for notation) is convex for all $x \in X$, $v \in U$, $\varepsilon \in (0, \bar{\varepsilon}]$.

The conclusions which follow along with the inclusions of Remark 2.2 are summarized as follows.

THEOREM 4.3. *The inclusions displayed in Fig.* 2 *are valid.*

Some applications of these inclusions are discussed in § 6.

$$
\begin{array}{ccccccc}
 & & \mathscr{S}(\text{NCSSOPC}) & \subset & \mathscr{S}(\text{WNCSSOPC}) & \underset{\text{A1}}{\subset} & \mathscr{S}(\text{NCOSS}) \\
 & & \cup & & \cup & & \\
\mathscr{S}(\text{SSOPC}) & \subset & \mathscr{S}(\text{SRMSSOPC}) & \subset & \mathscr{S}(\text{WRMSSOPC}) & & \\
\cap & & \cap & & \cap & & \\
\mathscr{S}(\text{OSS}) & \subset & \mathscr{S}(\text{SRMOSS}) & \subset & \mathscr{S}(\text{WRMOSS}) & \underset{\text{A1}}{\subset} & \mathscr{S}(\text{NCOSS}) \\
 & & \cap \text{A2} & & \cap \text{A4} & & \\
 & & \mathscr{S}(\text{NCSSOPC}) & \subset & \mathscr{S}(\text{WNCSSOPC}) & \underset{\text{A1}}{\subset} & \mathscr{S}(\text{NCOSS})
\end{array}
$$

FIG. 2. *Theorem* 4.3

**5. Some examples.** The examples of this section serve a number of purposes. First, they show that it is not possible to prove more inclusions than those which are contained in Theorem 4.2; this conclusion is formalized in Theorem 5.1 and extended somewhat in Theorem 5.2. Second, they delimit certain tests for optimality; this is discussed in the next section. Finally, they provide insight into the difficulties of applying and solving the various necessary conditions and into the wide variety of circumstances and phenomena which can occur in OPC problems.

*Example* 5.1. $k=j=0$, $n=l=1$, $X=Y=U=R$, $T>\sqrt{2}\pi$, $f=-x^2+u$, $\tilde{f}=-2x^2+u^2$, $g_0=y$. The assumption $T=\sqrt{2}\pi$ is sufficient to assure that the characterization of the solution sets is not changed by $T$. If $T<\sqrt{2}\pi$ one element of $\mathscr{S}$(NCOPC) disappears (d below) and everything else remains the same.

Omitting details, the conditions contained in Theorem 3.1 can be summarized as follows. From (3.2-2), $\tilde{p}=\alpha_0$. It is easy to show that for $\alpha_0=0$, (3.2) cannot have a solution and without loss of generality the case $\alpha_0<0$ can be treated as $\alpha_0=-1$. Condition (3.2-1) gives

$$u=\tfrac{1}{2}p. \tag{5.1}$$

The remaining conditions are (3.2-5) and

$$\dot{x}=-x^2+\tfrac{1}{2}p, \qquad \dot{p}=2xp-4x, \tag{5.2}$$

$$x(0)=x(\tau), \qquad p(0)=p(\tau), \qquad \tau \in (0, T]. \tag{5.3}$$

Figure 3 shows the $(x, p)$-phase plane for (5.2). Each characteristic curve corresponds to a fixed value of $H_M$ in the relation $H_M=-px^2+\frac{1}{4}p^2+2x^2$. The points labeled a, b and c are constant solutions of (5.2) and (5.3) and satisfy (3.2-5) with $\tilde{p}'y=H_M$ for all $\tau \in (0, T]$. The only other solutions of (5.2) and (5.3) are d, which has period $T$, and all the other solutions "inside" d (excluding c) which have periods $\tau \in (\sqrt{2}\pi, T)$. Calculation shows that for all these "time-dependent" solutions of (5.2) and (5.3), $\tilde{p}'y = -y < H_M$. Thus by (3.2-5) d is the only "time-dependent" solution of the conditions in Theorem 3.1. Because for each $(x(\cdot), p(\cdot), \tau)$ there is a corresponding $(u(\cdot), x(\cdot), \tau)$ the labels a, b, c and d can be used also to designate sets of elements in $\mathcal{U} \times \mathcal{X} \times (0, T]$. In particular, $\mathcal{S}$(NCOPC) "corresponds" to a, b, c and d, i.e., it is the union of elements designated a, b, c and d.

By using Theorems 3.2 and 3.4 it may be verified that both $\mathcal{S}$(NCOSS) and $\mathcal{S}$(NCSSOPC) correspond to a, b and c. Moreover, $\mathcal{S}$(OSS) corresponds to a and b and the cost associated with a and b is $J = -1$. Suppose there exist $u(\cdot), x(\cdot)$ and $\tau$ which satisfy (2.1-2)–(2.1-8) and give $J < -1$. This implies

$$(5.4) \qquad \begin{aligned} -1 > \frac{1}{\tau}\int_0^\tau (-2x^2+u^2)\,dt &= \frac{1}{\tau}\int_0^\tau (-2x^2+u^2-2\dot{x})\,dt \\ &= \frac{1}{\tau}\int_0^\tau (u^2-2u)\,dt \end{aligned}$$

which in turn implies

$$(5.5) \qquad 0 > \frac{1}{\tau}\int_0^\tau (u^2-2u+1)\,dt = \frac{1}{\tau}\int_0^\tau (u-1)^2\,dt.$$

This inequality is false and thus a and b are "contained" in $\mathcal{S}$(OPC). Any additional elements of $\mathcal{S}$(OPC) must be elements of $\mathcal{S}$(NCOPC). But c has cost $J = 0$ and it can be shown that d has cost $J > -1$. Thus $\mathcal{S}$(OPC) corresponds to a and b and $\mathcal{S}$(SSOPC) = $\mathcal{S}$(OPC).

The above results are summarized in the first line of Table 1. It is easy to show that $\mathcal{S}$(SRMOSS), $\mathcal{S}$(WRMOSS), $\mathcal{S}$(SRMSSOPC) and $\mathcal{S}$(WRMSSOPC) all correspond to a and b. The element d is a "time-dependent" strong relative minimum of OPC.

*Example* 5.2. $k = j = 0$, $n = l = 1$, $X = Y = R$, $U = [-2, 2] \subset R$, $T > 0$, $f = -x + u + 1$, $\tilde{f} = x(u+1)(u-1)^2$, $g_0 = y$. For $(u(\cdot), x(\cdot), \tau) \in \mathcal{S}$(SS), $x = u + 1$ and $y = (u+1)^2(u-1)^2$. Thus $\mathcal{S}$(OSS) corresponds to $u(t) \equiv 1$, $x(t) \equiv 2$, $\tau \in (0, T]$ (labeled a) and $u(t) \equiv -1$, $x(t) \equiv 0$, $\tau \in (0, T]$ (labeled b). Consideration of (3.5) shows that $\mathcal{S}$(NCOSS) corresponds to a, b and $u(t) \equiv 0$, $x(t) \equiv 1$, $\tau \in (0, T]$ (labeled c).

Theorem 3.1 leads to the characterization of $\mathcal{S}$(NCOPC). From (3.2-2), $\tilde{p} = \alpha_0$ and inspection of (3.2-3) and (3.2-4) shows that $\alpha_0 = 0$ is impossible. Thus without loss of generality assume $\alpha_0 = -1$. The maximization of

$$(5.6) \qquad H = p(-x+u+1) - x(u+1)(u-1)^2$$

with respect to $u \in U$ is complicated somewhat by the fact that the maximizing $u$ may be in the interior or in the boundary of $U$, depending on $x$ and $p$. Let $L_1$, $L_2$, $L_3$, $L_4$ be rays emanating from the origin of the $(x, p)$ plane which do not contain the origin and have, respectively, slopes: $1.2095\cdots$ (the root of $\frac{16}{27}+\frac{16}{27}\sqrt{1+\frac{3}{4}q}^{\,3}-\frac{1}{3}q=0$), 7, $2.5097\cdots$ (the root of $-\frac{16}{27}+\frac{16}{27}\sqrt{1+\frac{3}{4}q}^{\,3}+\frac{1}{3}q=0$), 15. Let $A_1$, $A_2$, $A_3$,

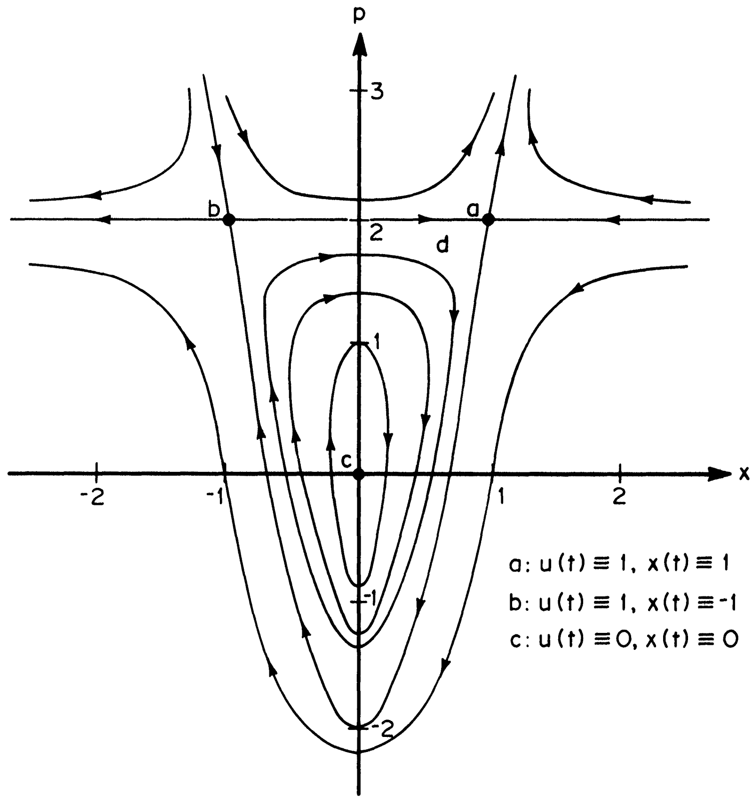


FIG. 3. *$(x, p)$-phase plane for Example* 5.1

TABLE 1
*Characterization of solution sets for examples*

| Example | $\mathcal{S}$(SSOPC) | $\mathcal{S}$(OSS) | $\mathcal{S}$(NCSSOPC) | $\mathcal{S}$(NCOSS) | $\mathcal{S}$(OPC) | $\mathcal{S}$(NCOPC) | $J$ OSS | $J$ OPC |
|---|---|---|---|---|---|---|---|---|
| 5.1, 5.7 | a, b | a, b | a, b, c | a, b, c | a, b | a, b, c, d | −1 | −1 |
| 5.2 | ∅ | a, b | b | a, b, c | ∅ | b | 0 | * |
| 5.3(i), 5.8(i) | ∅ | a | a | a | b | a, b, c, d | 0 | −1 |
| 5.3(ii), 5.8(ii) | a | a | a | a | a, d | a, b, c, d | −1 | −1 |
| 5.4 | a, b | a, b | a, b | a, b, c | a, b | a, b | 0 | 0 |
| 5.5 | ∅ | a, b | a, b, c | a, b, c | d | a, b, c, d (?) | 0 | <0 |
| 5.6 | ∅ | a, b | a, b | a, b, c | d | a, b, d | 0 | −1 |

* Minimum does not exist.

$A_4$ be the open sectors bounded by these rays (see Fig. 4). Then the maximizing $u$ is given by

$$
\begin{aligned}
u &= 2, & (x,p) &\in A_1 \cup L_2, \\
&= \tfrac{1}{3} + \tfrac{2}{3}\sqrt{1 + \tfrac{3}{4}\tfrac{p}{x}}, & (x,p) &\in A_2, \\
&= -2 \text{ or } 1.4651\cdots, & (x,p) &\in L_3, \\
&= -2, & (x,p) &\in A_3 \cup L_4, \\
&= \tfrac{1}{3} - \tfrac{2}{3}\sqrt{1 + \tfrac{3}{4}\tfrac{p}{x}}, & (x,p) &\in A_4, \\
&= 2 \text{ or } -.5873\cdots, & (x,p) &\in L_1, \\
&\in [-2, 2], & x &= p = 0.
\end{aligned}
\tag{5.7}
$$

Conditions (2.1-5) and (3.2-3) yield

$$\dot{x} = -x + u + 1, \qquad \dot{p} = p + (u+1)(u-1)^2, \tag{5.8}$$

$$x(0) = x(\tau), \qquad p(0) = p(\tau), \qquad \tau \in (0, T]. \tag{5.9}$$

With $u$ given by (5.7), equations (5.8) lead to the characteristic curves shown in the $(x, p)$-phase plane, Fig. 4. The point $x = p = 0$ corresponds to a constant solution if and only if $u(t) \equiv -1$. Points on the ray $L_1$ below $P_1$ correspond to a discontinuity in $u(t)$ ($u(t)$ at the discontinuity may be defined to be either 2 or $-.5873\cdots$). Above $P_1$ solutions of the system (5.7)–(5.8) cannot be continued across $L_1$ because from both $A_1$ and $A_4$ they lead into $L_1$. On $L_3$ solutions of (5.7)–(5.8) intersecting above $P_3$ or below $P_2$ can be continued across $L_3$ with a discontinuity in $u(t)$. On $L_3$ between $P_2$ and $P_3$ solutions lead away from $L_3$, going upward if $u(0) = 1.4651\cdots$ and downward if $u(0) = 2$. Thus the only solution of (5.7)–(5.8) which satisfies (5.9) is $u(t) \equiv -1$, $x(t) \equiv 0$, $p(t) \equiv 0$. This solution also satisfies (3.2-5) for all $\tau \in (0, T]$ and hence $\mathscr{S}$(NCOPC) corresponds to b in Table 1. It is also clear that $\mathscr{S}$(NCOPC) $= \mathscr{S}$(NCSSOPC).

The following argument shows that $\mathscr{S}$(SSOPC) $= \varnothing$. Suppose to the contrary. Then $\mathscr{S}$(SSOPC) $\subset \mathscr{S}$(NCSSOPC) implies that $\mathscr{S}$(SSOPC) corresponds to b in Table 1. But this contradicts $\mathscr{S}$(OSS) $= \mathscr{S}$(SSOPC) (Remark 2.1). Finally, $\mathscr{S}$(OPC) $= \varnothing$ because there are no "time-dependent" solutions of (5.7)–(5.8).

The above results are summarized in Table 1. Perhaps the most interesting conclusion is that $\mathscr{S}$(NCSSOPC) is a proper subset of $\mathscr{S}$(OSS). Clearly, $\mathscr{S}$(SRMOSS) $= \mathscr{S}$(WRMOSS) correspond to a and b. It is not difficult to show that with weak variations from $u(t) \equiv -1$, $J < 0$ can be obtained. To obtain $J < 0$ in the neighborhood of $x(t) \equiv 2$ it is necessary to use strong variations from $u(t) \equiv 1$. Thus $\mathscr{S}$(SRMSSOPC) $= \varnothing$ and $\mathscr{S}$(WRMSSOPC) corresponds to a.

*Example* 5.3. $k = 0$, $j = 1$, $n > 0$, $l = 2$, $x = R^n$, $Y = R^2$, $U = R$, $T > 0$, $f = Ax + \mathbf{b}u$, $\tilde{f}_1 = -\tfrac{1}{2}(\mathbf{c}'x)^2$, $\tilde{f}_2 = \tfrac{1}{2}u^2$, $g_0 = y_1$, $g_{-1} = y_2 - 1$. $A$ is a real $n \times n$ matrix and $\mathbf{b}, \mathbf{c} \in R^n$. This example is a special case of the problem considered in [6]. If $\mathbf{c}'x$ is interpreted as the output of the linear system $\dot{x} = Ax + \mathbf{b}u$, it corresponds to maximizing the average output power subject to a constraint on the average input power. Assume $A$ is stable (characteristic roots of $A$ have negative real parts), $(A, \mathbf{b})$ is controllable, $(\mathbf{c}', A)$ is observable [22] and let

$$G(s) = \mathbf{c}'(Is - A)^{-1}\mathbf{b} \tag{5.10}$$

denote the system transfer function.

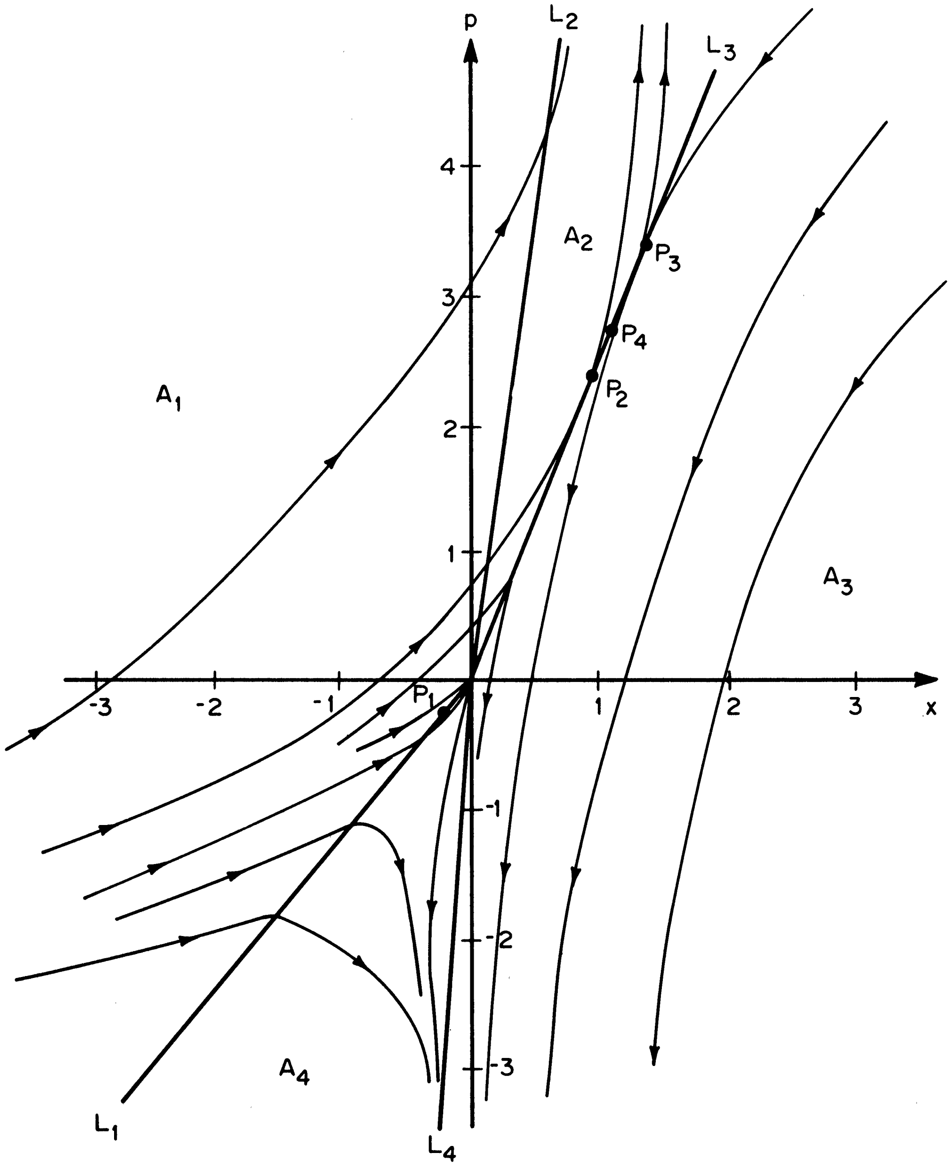


FIG. 4. $(x, p)$-*phase plane for Example* 5.2

Consider the characterization of $\mathcal{S}$(NCOPC). From (3.2-2), $\tilde{p}' = (\alpha_0, \alpha_{-1})$. First, assume $\alpha_{-1} \neq 0$. Conditions (3.2-1), (2.1-5) and (3.2-3) give

$$u = (\alpha_{-1})^{-1} p'\mathbf{b}, \tag{5.11}$$

$$\dot{x} = Ax + \mathbf{b}u, \qquad \dot{p}' = -p'A + (\alpha_0 \mathbf{c}'x)\mathbf{c}', \tag{5.12}$$

$$x(0) = x(\tau), \qquad p(0) = p(\tau), \qquad \tau \in (0, T]. \tag{5.13}$$

Since $A$ has no characteristic roots with zero real parts, $\alpha_0 = 0$ implies $p(t) \equiv 0$ and thus $u(t) \equiv 0$. But this gives $g_{-1}(y) < 0$ which contradicts (3.2-4). Since $\alpha_0 = 0$ is impossible, take $\alpha_0 = -1$. The system (5.11)–(5.12) is a linear, constant-coefficient, differential system of order $2n$ which has a periodic solution if and only if the characteristic equation has at least one root with real part zero. A simple calculation shows that there exist characteristic roots $\pm\mathbf{i}\omega$ ($\omega \in R, \omega \geqq 0$, $\mathbf{i} = \sqrt{1}$) if and only if

$$-\alpha_{-1} = G(\mathbf{i}\omega)G(-\mathbf{i}\omega) = |G(\mathbf{i}\omega)|^2, \tag{5.14}$$

an equation which always has a solution for $\alpha_{-1} < 0$ because controllability and observability imply $G(\mathbf{i}\omega) \not\equiv 0$. Since $y_2 = 1$ for $\alpha_{-1} < 0$, there must exist a $u(t)$ satisfying (5.11)–(5.13) of the form

$$u(t) = \begin{cases} 2\cos(\omega t + \theta), & \omega \geqq \dfrac{2\pi}{T}, \\ \pm\sqrt{2}, & \omega = 0, \end{cases} \tag{5.15}$$

where $\theta$ is arbitrary. The only remaining condition which must be satisfied if (3.2-5). A rather lengthy but straightforward computation shows that $H_M - \tilde{p}'y = -\omega(d/d\omega)|G(\mathbf{i}\omega)|^2$. Thus $\omega$ in (5.14) is a permissible value if and only if

$$\begin{aligned} &\omega = 0 \quad \text{or} \\ &\omega > \frac{2\pi}{T} \quad \text{and} \quad \frac{d}{d\omega}|G(\mathbf{i}\omega)|^2 = 0 \quad \text{or} \\ &\omega = \frac{2\pi}{T} \quad \text{and} \quad \frac{d}{d\omega}|G(\mathbf{i}\omega)| \leqq 0. \end{aligned} \tag{5.16}$$

Now consider $\alpha_{-1} = 0$. The possibility $\alpha_0 = 0$ is excluded because it implies $p(t) \equiv 0$ which violates (3.2-4). Thus take $\alpha_0 = -1$. Then it follows from (3.2-1) that $p'(t)\mathbf{b} \equiv 0$ on $[0, \tau]$ and $u$ is not determined by (3.2-1), i.e., $u$ is a singular control. The condition $p'(t)\mathbf{b} \equiv 0$ can be shown to imply: $u(t) = q\cos(\omega t + \theta)$ where $\theta$ is arbitrary, $0 \leqq q \leqq 2$, $\omega \geqq 2\pi/T$, $G(\mathbf{i}\omega) = 0$; or $u(t) \equiv q$ where $-\sqrt{2} \leqq q \leqq \sqrt{2}$, $G(0) = 0$. Thus for $\alpha_{-1} = 0$ the conditions on $\omega$ agree with (5.14) and (5.16) (observe that $(d/d\omega)|G(\mathbf{i}\omega)|^2 = 0$ for $\omega$ such that $G(\mathbf{i}\omega) = 0$). In the (relatively rare) circumstance that (5.14) and (5.16) permit multiple solutions ($\omega = \omega_1, \cdots, \omega_K$ satisfies (5.16) and $|G(\mathbf{i}\omega_i)|^2 = |G(\mathbf{i}\omega_1)|^2 = -\alpha_{-1}$, $i = 2, \cdots, K$) and $u(t) = \sum_{i=1}^{K} U_i \cos(\omega_i t + \theta_i)$ is periodic with period $\tau \leqq T$ then this $u(t)$ corresponds to a family of solutions of the necessary conditions for OPC provided the $U_i$ are chosen so that $g_{-1}(y) = 0$ (or $g_{-1}(y) \leqq 0$ if $\alpha_{-1} = G(\mathbf{i}\omega_1) = 0$).

Application of Theorem 3.4 shows that $\mathscr{S}$(NCSSOPC) may be obtained by specializing the above results to the case where $x(t)$ and $p(t)$ are constant. Thus for $G(0) \neq 0$: $u(t) \equiv \pm\sqrt{2}$, $x(t) \equiv \mp A^{-1}\mathbf{b}\sqrt{2}$, $\tau \in (0, T]$ corresponds to $\mathscr{S}$(NCSSOPC). For $G(0) = 0$: $\mathscr{S}$(NCSSOPC) corresponds to $u(t) \equiv q$, $x(t) \equiv -A^{-1}\mathbf{b}q$, $\tau \in (0, T]$, $q \in [-\sqrt{2}, \sqrt{2}]$. It is also clear from the form of $H$ and $U$ that $\mathscr{S}$(NCOSS) = $\mathscr{S}$(NCSSOPC).

Simple arguments (see [6]) show that $\mathcal{S}(\mathrm{OPC}) \neq \emptyset$. Since for elements in $\mathcal{S}(\mathrm{NCOPC})$, $g_0(y) = -|G(\mathbf{i}\omega)|^2$ it is clear that $\mathcal{S}(\mathrm{OPC})$ corresponds to those elements in $\mathcal{S}(\mathrm{NCOPC})$ with $\omega$ maximizing $|G(\mathbf{i}\omega)|^2$ on $\{0\} \cup [2\pi/T, +\infty)$. The maximum exists and is positive (because $|G(\mathbf{i}\omega)|^2 > 0$ for some $\omega$ and $|G(\mathbf{i}\omega)|^2 \to 0$ as $\omega \to +\infty$) and can occur only at a finite number of frequencies (because $|G(\mathbf{i}\omega)|^2$ is rational in $\omega^2$). $\mathcal{S}(\mathrm{SSOPC}) \neq \emptyset$ if and only if $|G(\mathbf{i}\omega)|^2 \leqq G^2(0)$ for all $\omega \geqq 2\pi/T$. $\mathcal{S}(\mathrm{OSS})$ corresponds to $u(t) \equiv \pm\sqrt{2}$, $x(t) \equiv \mp A^{-1}\mathbf{b}\sqrt{2}$, $\tau \in (0, T]$ if $G(0) \neq 0$ and to $u(t) \equiv q$, $x(t) = -A^{-1}\mathbf{b}q$, $\tau \in (0, T]$, $q \in [-\sqrt{2}, \sqrt{2}]$ if $G(0) = 0$.

Since the elements of the solution sets are characterized in terms of $|G(\mathbf{i}\omega)|^2$ it is easy to determine them even though $n$ may be large. Figure 5 gives two cases whose solution sets are summarized in Table 1. With the possible exception of d in Case (ii) it should be obvious what is meant by the designations of the solutions. For d, $u(t) = U_1 + U_2 \cos(\tilde{\omega}t + \theta)$, $\theta \in R$, $\frac{1}{2}U_1^2 + \frac{1}{4}U_2^2 = 1$, $\tau = (\tilde{\omega})^{-1}2\pi$. It is clear that $\mathcal{S}(\mathrm{NCOPC})$ may contain many more elements than $\mathcal{S}(\mathrm{OPC})$. Unfortunately, for most other OPC problems the suboptimal extremals are not so easily determined and rejected as they are in this example.

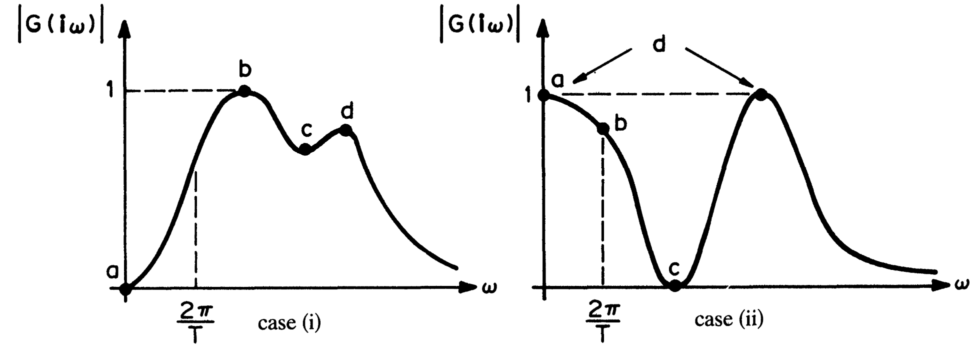


FIG. 5. *Designation of solution sets for Example* 5.3

*Example* 5.4. $k = j = 0$, $n = l = 1$, $X = Y = U = R$, $T > 0$, $f = -x + (u-1)^2(u+1)^2$, $\tilde{f} = x$, $g_0 = y$. Make the following designations: (a) $u(t) \equiv 1$, $x(t) \equiv 0$, $\tau \in (0, T]$; (b) $u(t) \equiv -1$, $x(t) \equiv 0$, $\tau \in (0, T]$; (c) $u(t) \equiv 0$, $x(t) \equiv 1$, $\tau \in (0, T]$. Then the characterizations for $\mathcal{S}(\mathrm{OSS})$, $\mathcal{S}(\mathrm{NCOSS})$ and $\mathcal{S}(\mathrm{NCSSOPC})$ given in Table 1 can be verified easily. Inspection of (3.2-3) shows that only the allowed solution for $p(t)$ is $p(t) \equiv \alpha_0$. This implies $\mathcal{S}(\mathrm{NCOPC}) = \mathcal{S}(\mathrm{NCSSOPC})$. From (2.1-5) and the form of $f$ it follows that $x(t) \geqq 0$ for all $t \in (0, T]$. This implies $J = y \geqq 0$ and $J = 0$ is only possible if $x(t) \equiv 0$. Thus $\mathcal{S}(\mathrm{OPC}) = \mathcal{S}(\mathrm{SSOPC}) = \mathcal{S}(\mathrm{OSS})$.

*Example* 5.5. $k = j = 0$, $n = 2$, $l = 1$, $X = R^2$, $Y = U = R$, $T = 2\pi$, $f_1 = x_2$, $f_2 = -x_1 - x_2 + u$, $\tilde{f} = (x_1 - 1)^2(x_1 + 1)^2 - (x_2)^2$, $g_o = y$. Make the following designations: (a) $u(t) \equiv x_1(t) \equiv 1$, $x_2(t) \equiv 0$, $\tau \in (0, T]$; (b) $u(t) \equiv x_1(t) \equiv -1$, $x_2(t) \equiv 0$, $\tau \in (0, T]$; (c) $u(t) \equiv x_1(t) \equiv x_2(t) \equiv 0$, $\tau \in (0, T]$. Then the characterizations for $\mathcal{S}(\mathrm{OSS})$, $\mathcal{S}(\mathrm{NCOSS})$ and $\mathcal{S}(\mathrm{NCSSOPC})$ given in Table 1 can be verified easily. Let $u(t) = 1 + A \cos \omega t$. Then $y$ may be computed easily from (2.1-4) and (2.1-5). For $\omega > \sqrt{2}$ and $A > 0$ sufficiently small the computation shows that $y < 0$. Since the optimal cost for OSS is $J = 0$ this proves that $\mathcal{S}(\mathrm{SSOPC}) = \emptyset$. From standard existence theorems it follows that $\mathscr{S}$(OPC) $\neq \emptyset$. Let the elements of $\mathscr{S}$(OPC) be designated by d. Since A3 is satisfied it is clear from Theorem 4.1 that a, b, c, d are "included" in $\mathscr{S}$(NCOPC). It is not known if there are additional elements in $\mathscr{S}$(NCOPC).

*Example* 5.6. $k=0, j=1, n=l=2, X=Y=R^2, U=\{u: u_3 \geqq u_2^2\} \subset R^3, T=3\pi$, $f_1=x_2$, $f_2=-x_1-x_2+u_2$, $\tilde{f}_1=(u_1-1)^2(u_1+1)^2-\frac{1}{2}(x_2)^2$, $\tilde{f}_2=\frac{1}{2}u_3$, $g_0=y_1$, $g_{-1}=y_2-1$. In each of the following designations assume that $x_1(t)\equiv u_2(t)\equiv q_2$, $u_3(t)\equiv q_3$, $x_2(t)\equiv 0$, $q_3 \in [0, 2]$, $q_3 \geqq q_2^2$, $\tau \in (0, T]$: (a) $u_1(t)\equiv 1$, (b) $u_1(t)\equiv -1$, (c) $u_1(t)\equiv 0$. The characterizations of $\mathscr{S}$(OSS), $\mathscr{S}$(NCOSS) and $\mathscr{S}$(NCSSOPC) are given in Table 1. To minimize $J$ in OPC it is necessary and sufficient to separately minimize the average of each of the two terms in $\tilde{f}_1(x(t), u(t))$. The first term is minimized by $u_1(t)\equiv \pm 1$ and the second term leads to a minimization problem of the type considered in Example 5.3, because at the minimum $u_3(t)=(u_2(t))^2$ (see also Example 5.8). This problem has a solution of the form: $u_2(t)=x_2(t)=2\cos(t+\theta)$, $u_3(t)=4\cos^2(t+\theta)$, $x_1(t)=2\sin(t+\theta)$ $\tau=2\pi$, $\theta \in R$. Let the set of all $(u(\cdot), x(\cdot), \tau)$ characterized in the above fashion be denoted by d. Then $\mathscr{S}$(OPC) corresponds to d. It can be shown that $\mathscr{S}$(NCOPC) corresponds to a, b and d.

*Example* 5.7. Same as Example 5.1, except for the following changes: $U=\{u: u_2 \geqq u_1^2\} \subset R^2$, $f=-x^2+u_1$, $\tilde{f}=-2x^2+u_2$. This example is essentially the same as Example 5.1. This can be seen by observing that in the characterization of all the solution sets it is required that $u_2(t)=(u_1(t))^2$. Thus the designations in Table 1 hold if: $u_1(t)=u(t)$, $u_2(t)=(u(t))^2$ where $u(t)$ is given as in Example 5.1; $x(t)$ is the same as $x(t)$ in Example 5.1.

*Example* 5.8. Same as Example 5.3, except for the following changes: $U=\{u: u_2 \geqq u_1^2\} \subset R^2$, $f=Ax+\mathbf{b}u_1$, $\tilde{f}=\frac{1}{2}u_2$. The modifications are similar to those used in Example 5.7. This leads to the designations shown in Table 1.

An immediate application of the examples is the following theorem.

THEOREM 5.1. *Let* A1 {A1 *and* A3} [A1, A2 *and* A3] *be satisfied. Then it is not possible to obtain additional inclusions beyond those which are implied by* (i)–(vi) {(i)–(vii)} [(i)–(viii)] *of Theorem* 4.2.

*Proof.* Of the 30 nontrivial, pair-wise inclusions involving $\mathscr{S}$(SSOPC), $\mathscr{S}$(OSS), $\mathscr{S}$(NCSSOPC), $\mathscr{S}$(NCOSS), $\mathscr{S}$(OPC), $\mathscr{S}$(NCOPC) which are possible (i)–(vi) {(i)–(vii)} [(i)–(viii)] of Theorem 4.2 imply that 8 {9} [11] are satisfied. Examples 4.1, 5.1, 5.2, 5.3(ii) {5.1, 5.2, 5.3(ii)} [5.5, 5.6] show that with A1 {A1 and A3} [A1, A2 and A3] satisfied the remaining 22 {21} [19] inclusions cannot hold generally.

Now consider the effect of stronger assumptions. Suppose as is the case in many practical problems that A1, A3 and $\mathscr{S}$(OSS) $\neq \emptyset$ are satisfied. Additional assumptions which are of interest are (i) OPC is proper ($\mathscr{S}$(SSOPC) $=\emptyset$), (ii) OPC is steady-state ($\mathscr{S}$(SSOPC) $\neq \emptyset$), (iii) OPC is proper and A2 is satisfied, (iv) OPC is steady-state and A2 is satisfied. For each of these cases Theorem 4.2 yields certain implications which are summarized in Fig. 6. It does not follow from Theorem 5.1 that these are the only implications concerning inclusion which can be drawn. However, the examples do show this. For instance, suppose that (ii) holds. Then Fig. 6 implies 13 nontrivial, pair-wise inclusions; Examples 5.1, 5.3(ii), 5.4 (which satisfy A1, A3, $\mathscr{S}$(OSS) $\neq \emptyset$, and (ii)) imply that the remaining

17 pair-wise inclusions cannot hold. All of the results are summarized in the following theorem.

THEOREM 5.2. *Let* A1, A3 *and* $\mathscr{S}(OSS) \neq \varnothing$ *be satisfied. Under the additional hypotheses* (i), (ii), (iii) *or* (*iv*) *the results of Fig.* 6 *are true. In each of the four cases it is not possible to prove additional inclusions exist.*

$$\begin{array}{ccccc} \mathscr{S}(\mathrm{OPC}) & \subset & \mathscr{S}(\mathrm{NCOPC}) \\ \cup & & \cup \\ \mathscr{S}(\mathrm{SSOPC}) = \mathscr{S}(\mathrm{OSS}) & \subset & \mathscr{S}(\mathrm{NCSSOPC}) \\ & & \cap \\ & & \mathscr{S}(\mathrm{NCOSS}) \end{array}$$

(ii) *and* (iv)

$$\begin{array}{ccc} \mathscr{S}(\mathrm{OPC}) & \subset & \mathscr{S}(\mathrm{NCOPC}) \\ & & \cup \\ & & \mathscr{S}(\mathrm{NCSSOPC}) \\ & & \cap \\ \mathscr{S}(\mathrm{OSS}) & \subset & \mathscr{S}(\mathrm{NCOSS}) \end{array}$$

(i)

$$\begin{array}{ccc} \mathscr{S}(\mathrm{OPC}) & \subset & \mathscr{S}(\mathrm{NCOPC}) \\ & & \cup \\ \mathscr{S}(\mathrm{OSS}) & \subset & \mathscr{S}(\mathrm{NCSSOPC}) \\ & & \cap \\ & & \mathscr{S}(\mathrm{NCOSS}) \end{array}$$

(iii)

FIG. 6. *Inclusions which are satisfied under* A1, A3, $\mathscr{S}(\mathrm{OSS}) \neq \varnothing$ *and*: (i) OPC *is proper,* (ii) OPC *is steady-state,* (iii) OPC *is proper and* A2, *or* (iv) OPC *is steady-state and* A2

**6. Tests for optimality.** If $\mathscr{S}$(OPC) and $\mathscr{S}$(OSS) are known it is possible to determine immediately whether or not time-dependent control improves performance and, if it does, the amount of the improvement. Since in most practical problems the solutions of OPC are not obtained easily, other paths must be pursued. One such path is suggested by Fig. 6. Under assumptions A1 and A3 it is clear that $\mathscr{S}(\mathrm{OSS}) \not\subset \mathscr{S}(\mathrm{NCSSOPC})$ implies that OPC is proper. Thus it can be determined that $\mathscr{S}(\mathrm{SSOPC}) = \varnothing$ without obtaining $\mathscr{S}$(OPC). This motivates the class of tests investigated in this section. Triples $\psi = (u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\mathrm{SS})$ are considered and it is supposed that it is possible to determine whether or not $\psi \in \mathscr{S}(A)$ for certain $A$. The principal concern is if OPC is proper or steady-state, but tests which may help in the search for solutions of OPC are examined too. The tests generalize (to OPC) and supplement tests which have appeared in the literature. An entirely new result is Theorem 6.1 which establishes limits to what can be tested in certain contexts.

To be complete the idea of Remark 4.2 is incorporated into the discussion. The condition given there corresponds to checking $\psi \in \mathscr{S}(\mathrm{NC'OPC})$ where

$$\mathscr{S}(\mathrm{NC'OPC}) = \{(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau}) \colon (\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau}) \in \mathscr{S}(\mathrm{SS}) \text{ and for all } \tau \in [0, T] \text{ there exist } p(\cdot), p, \alpha_{-j}, \cdots, \alpha_k \text{ such that (3.2-1)–(3.2-4) are satisfied for } u(t) \equiv \hat{u}(0), x(t) \equiv \hat{x}(0)\}. \tag{6.1}$$

By tracing the proof of Theorem 3.4, it is easy to see that $\mathscr{S}(\mathrm{NC'OPC}) \subset \mathscr{S}(\mathrm{NCSSOPC})$. Moreover, under A3, Theorem 4.1 states that $\mathscr{S}(\mathrm{NCSSOPC}) \subset \mathscr{S}(\mathrm{NCOPC})$; since $(u(\cdot), x(\cdot), \hat{\tau}) \in \mathscr{S}(\mathrm{NCSSOPC})$ implies $(u(\cdot), x(\cdot), \tau) \in \mathscr{S}(\mathrm{NCSSOPC})$ for all $\tau \in (0, T]$, this shows that $\psi \in \mathscr{S}(\mathrm{NCSSOPC})$ implies $\psi \in \mathscr{S}(\mathrm{NC'OPC})$. These facts and the content of Remark 4.2 are summarized in

*Remark* 6.1. $\mathscr{S}$(NC′OPC) satisfies the following inclusions: $\mathscr{S}(\mathrm{SSOPC}) \subset \mathscr{S}(\mathrm{NC'OPC}) \subset \mathscr{S}(\mathrm{NCSSOPC})$. If A3 is satisfied, $\mathscr{S}(\mathrm{NC'OPC}) = \mathscr{S}(\mathrm{NCSSOPC})$.

From this and the results of § 4, it is clear that the following tests are valid.

*Test* T1. The existence of $\psi$, $\psi \in \mathscr{S}(\mathrm{SS})$, $\psi \in \mathscr{S}(\mathrm{OPC})$, implies OPC is steady-state.

*Test* T2. The existence of $\psi$, $\psi \in \mathscr{S}(\mathrm{OSS})$, $\psi \notin \mathscr{S}(\mathrm{OPC})$, implies OPC is proper.

*Test* T3. The existence of $\psi$, $\psi \in \mathscr{S}(\mathrm{OSS})$, $\psi \notin \mathscr{S}(\mathrm{NC'OPC})$, implies OPC is proper.

*Test* T4. The existence of $\psi$, $\psi \in \mathscr{S}(\mathrm{OSS})$, $\psi \notin \mathscr{S}(\mathrm{NCSSOPC})$, implies OPC is proper.

Tests T1 and T2 arise directly from the definitions of proper and steady-state. Since T1 requires the determination of an element of $\mathscr{S}(\mathrm{SSOPC})$, it is the most difficult test to apply in practice. Usually, it involves inequalities which make use of particular structures in the problem data as in Example 5.1. Test T2 is easier to apply since it only requires exhibiting an admissible time-dependent triple $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$ which has lower cost than any element of $\mathscr{S}(\mathrm{OSS})$. See Example 5.5. General tests which implement T2 have been based on sinusoidal perturbations from an optimum steady-state solution [5], [12] and relaxed controls (see [1] and T8 of the next section). From Remark 6.1 it is seen that T2, T3 and T4 are successively weaker tests. Under A3 Remark 6.1 shows that T3 and T4 are equivalent; however, when A3 is not satisfied T3 may be a stronger test than T4 (Remark 4.2). Test T4 is stronger than tests of a similar type which have appeared previously [1], [13] in that it applies to a very general OPC problem and does not require $f_x(x(0), u(0))$, $(x(\cdot), u(\cdot), \tau) \in \mathscr{S}(\mathrm{OSS})$, to be nonsingular. The following theorem shows that T1, T2, T3 and T4 are not vacuous and that there exist no other tests in a reasonable class of tests.

THEOREM 6.1. *Suppose OPC satisfies no special assumptions* {A3} [A2] (A2 *and* A3). *Then tests* T1, T2, T3 *and* T4 {T1, T2 *and* $T3 = T4$} [$T1$, $T2$ *and* T3] (T1 *and* T2) *are not vacuous* (*always negative*) *or pairwise equivalent* (*one test positive always implies the other test positive*). *Let* $\psi \in \mathscr{S}(\mathrm{SS})$. *In the class of tests which employ an evaluation of* all *five conditions*, $\psi \in \mathscr{S}(A)$ *or* $\psi \notin \mathscr{S}(A)$ *for* $A =$ *OSS, NCSSOPC, NCOSS, OPC, NC'OPC, there exist no tests other than* T1, T2, T3 *and* T4 {T1, T2 *and* T3 = T4} [T1, T2 *and* T3] (T1 *and* T2) *which can show that OPC is proper or steady-state.*

*Proof.* First suppose that OPC satisfies no special assumption. Attach to $\psi \in \mathscr{S}(\mathrm{SS})$ the designation $h(\psi)$ where $h(\psi) = (h_{\mathrm{OSS}}, h_{\mathrm{NCSSOPC}}, h_{\mathrm{NCOSS}}, h_{\mathrm{OPC}}, h_{\mathrm{NC'OPC}})$ is a five digit binary number such that $h_A = 1$ if $\psi \in \mathscr{S}(A)$ and $h_A = 0$ if $\psi \notin \mathscr{S}(A)$. From Theorem 4.2 and Remark 6.1 it follows that 24 of the 32 possible values of $h(\psi)$ are excluded. The remaining eight with examples taken from Table 1 (where $\mathscr{S}(\mathrm{NC'OPC}) = \mathscr{S}(\mathrm{NCSSOPC})$) and § 4 are: $(1, 1, 1, 1, 1)$—Example 5.4 with $\psi \sim$ a or b; $(1, 0, 1, 0, 0)$—Example 5.2 with $\psi \sim$ a; $(1, 1, 1, 0, 0)$—Example 4.2 with $\psi \sim u(t) \equiv x(t) \equiv -1$; $(1, 1, 1, 0, 1)$—Example 5.5 with $\psi \sim$ a or b; $(0, 1, 1, 0, 1)$—Example 5.5 with $\psi \sim$ c and Example 5.7 with $\psi \sim$ c; $(0, 1, 1, 0, 0)$—Example 4.1 with $\psi \sim u(t) \equiv x(t) \equiv 1$ and Example 4.2 with $\psi \sim u(t) \approx x(t) \equiv 1$; $(0, 0, 1, 0, 0)$—Example 5.4 with $\psi \sim$ c and Example 5.6 with $\psi \sim$ c; $(0, 0, 0, 0, 0)$—Example 5.4 with $\psi \sim u(t) \equiv \frac{1}{2}$, $x(t) \equiv \frac{9}{16}$ and Example 5.6 with $\psi \sim u(t) \equiv x_1(t) \equiv \frac{1}{2}$, $x_2(t) \equiv 0$. The first result of the theorem follows because: $(1, 1, 1, 1, 1)$ implies T1 positive; $(1, 0, 1, 0, 0)$ implies T2, T3, T4 positive; $(1, 1, 1, 0, 0)$ implies T2, T3 positive; $(1, 1, 1, 0, 1)$ implies T2 positive. For each of the four remaining values of $h(\psi)$ there are examples of OPC which are both proper and steady-state. This is a consequence of Table 1, Example 4.2 being proper (see Remark 4.2) and Example 4.1 being steady-state (to show this requires an investigation of the solutions of (3.2) and an application of an existence theorem to OPC). Thus there are no additional tests for proper or steady-state. Now consider A3. Since $\mathcal{S}(\text{NC}'\text{OPC}) = \mathcal{S}(\text{NCSSOPC})$, $h(\psi) = (1, 1, 1, 0, 0)$ and $h(\psi) = (0, 1, 1, 0, 0)$ are impossible. The remaining examples apply as before. Under A2 Theorem 4.2 gives $\mathcal{S}(\text{OSS}) \subset \mathcal{S}(\text{NCSSOPC})$ and this eliminates $h(\psi) = (1, 0, 1, 0, 0)$. All of the above stated examples except Example 5.2 satisfy A2 and thus the results for A2 are obtained. When A2 and A3 both hold, the argument is essentially a combination of the previous two arguments.

The preceding results and Fig. 2 suggest how a search for solutions of OPC might proceed. Since the determination of elements of $\mathcal{S}(\text{NCOPC})$ requires the solution of the difficult two-point-boundary-value problem (3.2), it is worthwhile to see what can be learned by trying triples $\psi = (u(\cdot), x(\cdot), \tau) \in \mathcal{S}(\text{SS})$. If there is some reason to believe that OPC is proper, it is useful to have tests which indicate how to begin a search for time-dependent controls. For $\psi \in \mathcal{S}(\text{SS})$ conditions which may be checked (listed in order of increasing difficulty) include: $\psi \in \mathcal{S}(\text{NCOSS})$, the system (3.5); $\psi \in \mathcal{S}(\text{WNCSSOPC})$, for some $\varepsilon > 0$ the system (4.4), (3.5-2)–(3.5-4); $\psi \in \mathcal{S}(\text{NCSSOPC})$, the system (3.5-1)$'$, (3.5-2)–(3.5-4); $\psi \in \mathcal{S}(\text{NC}'\text{OPC})$, the system (3.2-1)–(3.2-4) for all $\tau \in [0, T]$. The test $\psi \in \mathcal{S}(\text{NCOSS})$ has little value, except perhaps to narrow the search. If elements $\psi \in \mathcal{S}(\text{OSS})$ are known, T3 and T4 may be applied. While there may be fewer elements $\psi$ that satisfy $\psi \in \mathcal{S}(\text{OSS})$, $\psi \notin \mathcal{S}(\text{WNCSSOPC})$ than T4, this test provides somewhat greater information than T4. In particular, reference to Fig. 2 shows $\psi \in \mathcal{S}(\text{WRMOSS})$ and $\psi \notin \mathcal{S}(\text{WRMSSOPC})$. This gives

*Test* T5. The existence of $\psi$, $\psi \in \mathcal{S}(\text{OSS})$, $\psi \notin \mathcal{S}(\text{WNCSSOPC})$, implies OPC is locally proper [5], i.e., OPS is proper *and* for all $\varepsilon > 0$ there exists a time-dependent admissible triple $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$ with $\|\hat{u}(t) - u(0)\|$, $\|\hat{x}(t) - x(0)\| < \varepsilon$ for all $t \in [0, T]$ which has lower cost than $\psi$.

Thus if T5 is positive the search for better time-dependent controls may begin with a guarantee of success in the neighborhood of $(u(0), x(0))$. If T4 is positive $\psi \in \mathcal{S}(\text{SRMOSS})$ and $\psi \notin \mathcal{S}(\text{SRMSSOPC})$. Thus there exist time-dependent admissible triples $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$ with $\hat{x}(t)$ in the neighborhood of $x(0)$ which reduce the cost, but large variations, $\hat{u}(t) - u(0)$, may be necessary. If $\psi \in \mathcal{S}(\text{OSS})$ and $\psi \in \mathcal{S}(\text{NC}'\text{OPC})$ ($\psi \in \mathcal{S}(\text{NCSSOPC})$ under A3), $\psi$ is a likely candidate for $\mathcal{S}(\text{SSOPC})$. Since $\psi \in \mathcal{S}(\text{NCOPC})$, Theorem 3.1 can reject $\psi$ only if other (time-dependent) solutions of (3.2) are found which have lower cost. However, since it is not known that $\psi \in \mathcal{S}(\text{WRMSSOPC})$, a search for better time-dependent controls might prove successful in the neighborhood of $(u(0), x(0))$.

If it is not possible to determine elements of $\mathcal{S}(\text{OSS})$ much less can be said. Figure 2 suggests several conditions for optimality including $\psi \in \mathcal{S}(\text{NCSSOPC}) \cap \mathcal{S}(\text{SRMOSS})$ and $\psi \in \mathcal{S}(\text{WNCSSOPC}) \cap \mathcal{S}(\text{WRMOSS})$. Checking $\psi \in \mathcal{S}(\text{SRMOSS})$ and $\psi \in \mathcal{S}(\text{WRMOSS})$ may be difficult. Since $\mathcal{S}(\text{NCSSOPC}) \subset \mathcal{S}(\text{WNCSSOPC}) \subset \mathcal{S}(\text{NCOSS})$ necessary conditions for elements of $\mathcal{S}(\text{SRMOSS})$ and $\mathcal{S}(\text{WRMOSS})$ are of value only if they are stronger than (3.5). Obvious candidates for such conditions are second order necessary conditions [9], [15]. Adjoining second order necessary conditions for OSS to the condition $\psi \in \mathcal{S}(\text{NCSSOPC})$ can produce a stronger necessary condition for elements of $\mathcal{S}$(SSOPC) than $\psi \in \mathcal{S}$(NCSSOPC). This happens in Example 5.1 where elements of $\mathcal{S}$(NCSSOPC) corresponding to c are eliminated.

Finally, it should be observed that the following simple tests, evident from Fig. 2, may be useful.

*Test* T6. The existence of $\psi$, $\psi \in \mathcal{S}$(SS), $\psi \notin \mathcal{S}$(NCSSOPC), implies that for all $\varepsilon > 0$ there exists an admissible triple $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$, possibly in $\mathcal{S}$(SS), with $\|x(t)-x(0)\| < \varepsilon$ for all $t \in [0, T]$ which has lower cost than $\psi$.

*Test* T7. The existence of $\psi$, $\psi \in \mathcal{S}$(SS), $\psi \notin \mathcal{S}$(WNCSSOPC), implies that for all $\varepsilon > 0$ there exists an admissible triple $(\hat{u}(\cdot), \hat{x}(\cdot), \hat{\tau})$, possibly in $\mathcal{S}$(SS), with $\|\hat{u}(t)-u(0)\|$, $\|\hat{x}(t)-x(0)\| < \varepsilon$ for all $t \in [0, T]$ which has lower cost than $\psi$.

*Remark* 6.2. The importance of the assumption A2 is clear. If A2 is satisfied T4 and T5 are vacuous. Moreover, $\psi \in \mathcal{S}$(SRMOSS) and $\psi \in \mathcal{S}$(OSS) are stronger necessary conditions for $\psi \in \mathcal{S}$(SSOPC) than $\psi \in \mathcal{S}$(NCSSOPC). Under A4 $\psi \in \mathcal{S}$(WRMOSS) is a stronger necessary condition than $\psi \in \mathcal{S}$(WNCSSOPC). Tests T1, T2, T3, T6 and T7 remain useful.

**7. Relaxed steady-state optima.** The replacement of an original optimal control problem by a relaxed optimal control problem is a well established technique in the application of existence theory [4], [21]. In the treatment of optimal periodic control problems it has been recognized [1], [3], [11], [13] that the replacement has an additional function. Steady-state analysis of the relaxed problem, which is relatively easy to carry out, may shed light on the dynamic behavior of the original problem. This path is pursued here; a principal objective is to extend the results of [1].

To introduce the relaxed problem let

$$w = (\rho^1, \cdots, \rho^{l+n+1}, \mu^1, \cdots, \mu^{l+n+1}) \in W \tag{7.1}$$

where

$$W = \left\{ w \colon \sum_{i=1}^{l+n+1} \rho^i = 1 \text{ and } \rho^i \geqq 0,\ \mu^i \in U \text{ for } i = 1, \cdots, l+n+1 \right\} \subset \boldsymbol{R}^{(l+n+1)(m+1)}. \tag{7.2}$$

Define $f^r \colon X \times W \to \boldsymbol{R}^n$ and $\tilde{f}^r \colon X \times W \to \boldsymbol{R}^l$ by

$$f^r(x, w) = \sum_{i=1}^{l+n+1} \rho^i f(x, \mu^i), \tag{7.3}$$

$$\tilde{f}^r(x, w) = \sum_{i=1}^{l+n+1} \rho^i \tilde{f}(x, \mu^i) \tag{7.4}$$

and let

$$y = \frac{1}{\tau} \int_0^\tau \tilde{f}^r(x(t), w(t))\, dt \in Y, \tag{2.1-4$'$}$$

$$\dot{x}(t) = f^r(x(t), w(t)) \quad \text{almost all } t \in [0, T], \quad x(0) = x(\tau), \tag{2.1-5$'$}$$

$$w(\cdot) \in \mathcal{W} = \{w(\cdot) \colon w(\cdot) \text{ measurable and essentially bounded on } [0, T]\ w(t) \in W \text{ for all } t \in [0, T]\}. \tag{2.1-6$'$}$$

The system (2.1-1), (2.1-2), (2.1-3), (2.1-4)$'$, (2.1-5)$'$, (2.1-6)$'$, (2.1-7), (2.1-8), which is denoted by (2.1)$^r$, constitutes the *relaxed* OPC problem. The same substitutions apply with obvious modifications elsewhere, e.g., in the statement of the *relaxed* OSS problem, (2.2)$^r$. Solution sets for the relaxed problem are defined as before and are denoted by $\mathcal{S}^r(\cdot)$. By the Carathéodory theorem [20], $\hat{f}^r(x, W) = \mathrm{co}\,\hat{f}(x, U)$. This result and an obvious modification lead to the following conclusions.

*Remark* 7.1. The relaxed OPC problem satisfies A2 and A4.

Suppose that OPC satisfies A2. Then $\hat{f}^r(x, W) = \hat{f}(x, U)$ and it is possible to show that for every solution of (2.1)$^r$ with cost $J$ there is a solution of (2.1) with cost $J$. Thus the relaxed problem has no interest when OPC satisfies A2.

DEFINITION 7.1. The sequence $\{(u^q(\cdot), x^q(\cdot), \tau)\}$ is an *approximate solution* of (2.1) with *period* $\tau$ and *cost* $J$ if: (i) for all $q>0$ $(u^q(\cdot), x^q(\cdot), \tau) \in \mathcal{U} \times \mathcal{X} \times (0, T]$, $\dot{x}^q(t) = f(x^q(t), u^q(t))$ for almost all $t \in [0, T]$ and $y^q = (1/\tau)\int_0^\tau \tilde{f}(x^q(t), u^q(t))\,dy \in Y$; (ii) for all $\varepsilon > 0$ there exists an integer $Q(\varepsilon)$ such that for $q > Q(\varepsilon)$

$$g_i(y^q, x^q(0)) \leqq \varepsilon \quad \text{for } i = -j, \cdots, -1, \quad |g_0(y^q, x^q(0)) - J| < \varepsilon,$$
$$|g_i(y^q, x^q(0))| < \varepsilon \quad \text{for } i = 1, \cdots, k \quad \text{and} \quad \|x^q(0) - x^q(\tau)\| < \varepsilon.$$

By suitably adapting well known results [4] the following theorem can be proved.

THEOREM 7.1. *Let* $(w(\cdot), x(\cdot), \tau)$ *satisfy* (2.1)$^r$. *Then there is an approximate solution of* (2.1) *with period* $\tau$ *and cost* $J$.

Since for every $(u(\cdot), x(\cdot), \tau)$ which satisfies (2.1) there exists a $w(\cdot)$ such that $(w(\cdot), x(\cdot), \tau)$ satisfies (2.1)$^r$, inf $J$ over (2.1)$^r$ is not greater than inf $J$ over (2.1). The system (2.1)$^r$ is of interest because it may have a solution whose cost is less than can be achieved in (2.1). In such a case the corresponding approximate solution of (2.1) has particular importance. These observations also apply when only steady-state solutions are considered. There may exist elements $\psi^r \in \mathcal{S}^r(\mathrm{SS})$ which have lower cost than the cost of any element $\psi \in \mathcal{S}(\mathrm{SS})$. Elements of $\mathcal{S}^r(\mathrm{SS})$ are relatively easy to determine and lead to approximate solutions of (2.1) which have a particularly simple form: $(w(\cdot), x(\cdot), \tau) \in \mathcal{S}^r(\mathrm{SS})$ implies $(u^q(\cdot), x^q(\cdot), \tau)$ can be constructed as a "chattering" solution [4] in which $x^q(\cdot)$ is approximately constant ($x^q(t) \to x(0)$ for all $t \in [0, T]$) and $u^q(t)$ takes on the value $\mu^i(0)$ on a subset of measure $\rho^i(0)T$. As suggested in § 6 this motivates an additional test for proper. Before stating the test it is necessary to extend the definition of proper to allow for approximate solutions.

DEFINITION 7.2. OPC is *approximately proper* if OSS has a minimum cost $J_{\mathrm{OSS}}$, and there exists an approximate solution of (2.1) with cost $J$ such that $J < J_{\mathrm{OSS}}$.

*Test* T8. Suppose there exist $\psi \in \mathcal{S}(\mathrm{OSS})$ and $\psi^r \in \mathcal{S}^r(\mathrm{SS})$ such that $\psi^r$ has lower cost than $\psi$. Then OPC is approximately proper.

The validity of the test is obvious from Theorem 7.1. It can be seen from Example 5.2 that the test is not vacuous (take $\rho^1(t) \equiv \frac{2}{3}$, $\rho^2(t) \equiv \frac{1}{3}$, $\mu^1(t) \equiv 1$, $\mu^2(t) \equiv -2$, $x(t) \equiv 1$ which gives $J = -3$). In fact, it is easy to find examples (in Example 5.2 replace $X = R$ by $X = \{x : x < 1.8\}$) where there exist no $\psi$ such that

T4 is positive and yet T8 is positive. The relationships between T8 and the tests T4, T5, T6 and T7 is clarified by the following theorem.

THEOREM 7.2. *Suppose* $(u(\cdot), x(\cdot), \tau) = \psi \in \mathcal{S}(\text{SS})$ *and* $\psi \notin \mathcal{S}(\text{NCSSOPC})$ $(\psi \in \mathcal{S}(\text{SS})$ *and* $\psi \notin \mathcal{S}(\text{WNCSSOPC}))$. *Then there exists* $(w(\cdot), x^r(\cdot), \tau) = \psi^r \in \mathcal{S}^r(\text{SS})$ *with lower cost than* $\psi$. *Furthermore, for any* $\varepsilon > 0$ *it is possible to choose* $\psi^r$ *so that* $\|x^r(0) - x(0)\| < \varepsilon$ $(\|x^r(0) - x(0)\| < \varepsilon$ *and* $\|\mu^i(0) - u(0)\| < \varepsilon$ *for* $i = 1, \cdots, l + n + 1)$.

*Proof.* Consider $w(\cdot)$ such that $\rho^1(t) \equiv 1$, $\mu^1(t) \equiv u(0)$. Then $\psi^r = (w(\cdot), x(\cdot), \tau) \in \mathcal{S}^r(\text{SS})$. Suppose $\psi^r \in \mathcal{S}^r(\text{NCSSOPC})$. Then there exist $p$, $\tilde{p}$, $\sigma_{-j}, \cdots, \alpha_k$ which satisfy the conditions of Theorem 3.4 with notation appropriately modified to account for the relaxed problem. Since $H^r(x, w, p, \tilde{p}) \geqq H^r(x, v, p, \tilde{p})$ for all $v \in W$ the same inequality holds for all $v = (1, 0, \cdots, \mu^1, 0, \cdots, 0)$ such that $\mu^1 \in U$. This implies that for the same $p$, $\tilde{p}$, $\alpha_{-j}, \cdots, \alpha_k$, $\psi$ satisfies the conditions of Theorem 3.4, i.e., $\psi \in \mathcal{S}(\text{NCSSOPC})$. This is a contradiction and hence $\psi^r \notin \mathcal{S}^r(\text{NCSSOPC})$. Now suppose $\psi^r \in \mathcal{S}^r(\text{OSS})$. Then because of Remark 7.1 and Theorem 4.2 $\psi^r \in \mathcal{S}^r(\text{NCSSOPC})$. Thus by contradiction $\psi^r \notin \mathcal{S}^r(\text{OSS})$ and there must exist an element of $\mathcal{S}^r(\text{SS})$ with lower cost than $\psi^r$. The argument still applies if $X$ is replaced by an arbitrarily small neighborhood of $x(0)$. Thus the part of the theorem corresponding to $\psi \notin \mathcal{S}(\text{NCSSOPC})$ is proved. For $\psi \notin \mathcal{S}(\text{WNCSSOPC})$ the argument is the same except $U$ is replaced by $U \cap \{\hat{u} : \|\hat{u} - u\| \leqq \varepsilon\}$ with $\varepsilon > 0$ sufficiently small and parts of Theorem 4.3 are used.

Applying Theorem 7.2 with $\psi \in \mathcal{S}(\text{OSS})$ shows that if T4 or T5 are positive there exists a $\psi^r \in \mathcal{S}^r(\text{SS})$ such that T8 is positive. Additionally, if OPC is proper then OPC is approximately proper. These facts and the comment before Theorem 7.2 are combined in the following conclusion.

*Remark* 7.2. T8 is a stronger test for OPC approximately proper than either T4 or T5.

To put this remark in perspective it should be observed that T8 has a weaker consequence than T4 or T5. Specifically, there are examples which show that "OPC is approximately proper" does not imply "OPC is proper."

*Example* 7.1. $j = 0$, $k = 2$, $n = 1$, $U = [-1, 1] \subset R$, $T > 0$, $f = -x + u$, $\tilde{f}_1 = x^2$, $\tilde{f}^2 = -u^2$, $g_0 = y_2$, $g_1 = y_1$. It is clear that (2.1) is satisfied if and only if $x(t) \equiv u(t) \equiv 0$ and $J = 0$. Thus OPC is steady-state. But $\rho^1(t) \equiv \frac{1}{2}$, $\rho^2(t) \equiv \frac{1}{2}$, $\mu^1(t) = 1$, $\mu^2(t) \equiv -1$, $x(t) \equiv 0$ satisfies $(2.1)^r$ with $J = -1$. Thus Theorem 7.2 implies OPC is approximately proper.

Similarly, Theorem 7.2 establishes a connection between relaxed steady-state solutions and tests T6 and T7. When T6 and T7 are positive there exists a $\psi^r \in \mathcal{S}^r(\text{SS})$ with lower cost than $\psi$. Moreover, $\psi^r$ can be chosen to that the "chattering" approximate solution of (2.1) corresponding to $\psi^r$ satisfies the same closeness requirements as do the regular solutions whose existence is guaranteed by T6 and T7. If it can be determined that $\psi \notin \mathcal{S}(\text{SRMOSS})$ (for T6) or $\psi \notin \mathcal{S}(\text{WRMOSS})$ (for T7) there is no need to resort to the relaxed problem and approximate solutions; it is clear that there are elements of $\mathcal{S}(\text{SS})$ which reduce the costs according to the requirements of T6 or T7. However, relaxed steady-state solutions may produce larger reductions in cost than the regular steady-state solutions.

The main practical value of Theorem 7.2 is that it provides a constructive approach for seeking controls which improve performance when any of the tests T4–T7 is positive. Bailey and Horn [1] make the same observation but with respect to T4 only. Their method of proof is more direct but requires $\psi \in \mathcal{S}(\text{OSS})$ *and* $f_x(x(0), u(0))$ nonsingular. The key to the proof presented here is part (viii) of Theorem 4.2 which is a direct consequence of Theorem 3.3.

Remark 7.2 makes it clear that the solution of the relaxed OSS problem deserves special attention. This is the conclusion of Bailey and Horn. Their sufficient condition I (equivalent to T8 under certain restrictions) is stronger than their sufficient condition II (equivalent to T4 under certain restrictions). Because of Remark 7.1, Remark 6.2 applies to the relaxed OPC problem. Hence, there is a hierarchy of necessary conditions which can be applied to the solution of the relaxed OSS problem: $\mathcal{S}^r(\text{OSS}) \subset \mathcal{S}^r(\text{SRMOSS}) \subset \mathcal{S}^r(\text{NCSSOPC}) \subset \mathcal{S}^r(\text{NCOSS})$. If it is not possible to obtain elements of $\mathcal{S}^r(\text{SRMOSS})$ it may be useful (see below) to combine second order necessary conditions for the relaxed OSS problem with $\psi^r \in \mathcal{S}^r(\text{NCSSOPC})$. Also notice that T4 and T5 are useless when applied to the relaxed OPC problem.

Example 5.2 illustrates some of the points which have been made in the preceding paragraphs. The solution of the relaxed OSS problem is given by $\psi^r$: $\rho^1 = .3896\cdots$, $\rho^2 = .6103, \cdots$, $\mu^1 = -2$, $\mu^2 = 1.5$, $x = 1.1363\cdots$, $J = -3.5511\cdots$. $\psi^r$ is also the (unique) solution of the relaxed OPC problem. This can be deduced from the application of Theorem 3.1 which yields an $(x, p)$-phase plane which is the same as Fig. 4 except: on $L_1$ there is a solution which moves from $P_1$ toward the origin, on $L_2$ there are solutions which move from $P_2$ and $P_3$ toward $P_4$, $P_4$ is an equilibrium solution. Thus $\mathcal{S}^r(\text{NCOPC})$ has two elements corresponding to $x(t) \equiv p(t) \equiv 0$, $J = 0$ and $x(t) \equiv 1.1363\cdots$, $p(t) \equiv 2.8518\cdots$, $J = -3.5511\cdots$. Because the relaxed OPC problem has a solution (an existence theorem can be applied) the second extremal must be optimal. OPC does not have a solution but all chattering solutions corresponding to $\psi^r$ satisfy (2.1-2)–(2.1-8) exactly and as $q \to \infty$ the cost approaches $-3.5511\cdots$. The elements of $\mathcal{S}(\text{OSS})$ labeled "a" are in $\mathcal{S}(\text{WNCCSSOPC})$ but not $\mathcal{S}(\text{NCSSOPC})$. Thus T4 and T6 are positive but T5 and T7 are negative. This is consistent with elements "a" in $\mathcal{S}(\text{WRMSSOPC})$ but not $\mathcal{S}(\text{SRMSSOPC})$. For element "b" T4–T7 are all negative but T8 is positive. $\mathcal{S}^r(\text{NCSSOPC})$ has two elements corresponding to $x = 0$ and $x = 1.1363\cdots$. The first element does not satisfy second order necessary conditions for the relaxed OSS problem.

Other examples illustrate that the relaxed OPC problem need not be steady-state. For instance, Example 5.8(i), which can be shown to be equivalent to the relaxed version of 5.3(i), is proper.

**Appendix A. Necessary conditions for a general optimal control problem.** Consider the following notation and assumptions: $\mu$ and $\nu$ are positive integers; $\hat{t} \in R$ is positive; $U \subset R^m$ is an arbitrary set; $\hat{X}, \hat{X}^1, \hat{X}^2 \subset R^{\hat{n}}$ are open sets; for $i = -\mu, \cdots, \nu$ the functions $\theta_i : \hat{X}^1 \times \hat{X}^2 \times (0, \hat{t}) \to R$ are continuously differentiable; the function $\hat{f}: \hat{X} \times U \to R^{\hat{n}}$ is continuous and for each $u \in U$ is continuously differentiable in $\hat{x}$. Let $\hat{f}_{\hat{x}}(\hat{x}, u)$ denote the Jacobian matrix of $\hat{f}(\hat{x}, u)$ with respect to $\hat{x}$; let $\theta_{i\hat{x}^1}(\hat{x}^1, \hat{x}^2, \tau)$, $\theta_{i\hat{x}^2}(\hat{x}^1, \hat{x}^2, \tau)$ and $\theta_{i\tau}(\hat{x}^1, \hat{x}^2, \tau)$ denote respectively the Jacobian matrices of $\theta_i(\hat{x}^1, \hat{x}^2, \tau)$ with respect to $\hat{x}^1$, $\hat{x}^2$ and $\tau$.

*General optimal control problem* (GOC). Find $u(\cdot)$, $\hat{x}(\cdot)$ and $\tau$ which minimize $J$ subject to

(A.1-1) $J = \theta_0(\hat{x}(0), \hat{x}(\tau), \tau)$,

(A.1-2) $\theta_i(\hat{x}(0), \hat{x}(\tau), \tau) \leqq 0, \qquad i = -\mu, \cdots, -1$,

(A.1-3) $\theta_i(\hat{x}(0), \hat{x}(\tau), \tau) = 0, \qquad i = 1, \cdots, \nu$,

(A.1-4) $\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), u(t))$ almost all $t \in [0, \hat{t}]$,

(A.1-5) $u(\cdot) \in \mathscr{U} = \{u(\cdot)$: measurable and essentially bounded on $[0, \hat{t}]$, $u(t) \in U$ for all $t \in [0, \hat{t}]\}$,

(A.1-6) $\hat{x}(\cdot) \in \hat{\mathscr{X}} = \{\hat{x}(\cdot)$: $\hat{x}(\cdot)$ absolutely continuous on $[0, \hat{t}]$, $\hat{x}(t) \in \hat{X}$ for all $t \in [0, \hat{t}]$, $\hat{x}(0) \in \hat{X}^1$, $\hat{x}(\tau) \in \hat{X}^2\}$,

(A.1-7) $\tau \in (0, \hat{t})$.

THEOREM A.1 (necessary conditions for GOC). *Let*

$$\hat{H}(\hat{x}, u, \hat{p}) = \hat{p}'\hat{f}(\hat{x}, u) \tag{A.2}$$

*where* $\hat{p} \in R^{\hat{n}}$. *Let* $(u(\cdot), \hat{x}(\cdot), \tau)$ *solve GOC. Then there exist an absolutely continuous function* $\hat{p}(\cdot)$: $[0, \tau] \to R^{\hat{n}}$ *and real numbers* $\alpha_{-\mu}, \cdots, \alpha_\nu$ *such that the following conditions are satisfied*:

(A.3-1) $\max_{v \in U} H(\hat{x}(t), v, \hat{p}(t)) = H(\hat{x}(t), \hat{u}(t), \hat{p}(t))$ *almost all* $t \in [0, \tau]$,

(A.3-2)
$$\hat{p}'(0) = -\sum_{i=-\mu}^{\nu} \alpha_i \theta_{i\hat{x}^1}(\hat{x}(0), \hat{x}(\tau), \tau),$$
$$\hat{p}'(\tau) = \sum_{i=-\mu}^{\nu} \alpha_i \theta_{i\hat{x}^2}(\hat{x}(0), \hat{x}(\tau), \tau),$$

(A.3-3)
$$\dot{\hat{p}}'(t) = -\hat{p}'(t)\hat{f}_{\hat{x}}(\hat{x}(t), u(t)) \quad \textit{almost all } t \in [0, \tau],$$
$$\alpha_i \leqq 0, \qquad i = -\mu, \cdots, 0,$$

(A.3-4)
$$\alpha_i \theta_i(\hat{x}(0), \hat{x}(\tau), \tau) = 0, \qquad i = -\mu, \cdots, -1,$$
$$(\alpha_{-\mu}, \cdots, \alpha_\nu) \neq 0.$$

*If* $\hat{f}(\hat{x}(t), u(t))$ *is continuous at* $t = \tau$ *the following additional condition is satisfied*:

$$\max_{v \in U} H(\hat{x}(\tau), v, \hat{p}(\tau)) = -\sum_{i=-\mu}^{\nu} \alpha_i \theta_{i\tau}(\hat{x}(0), \hat{x}(\tau), \tau). \tag{A.3-5}$$

*Proof.* With minor changes in notation the conditions are taken from § 7 of [16], assuming that: $\tau_1$ is fixed, $t_1 = \tau_1 = 0$, the $\theta_i$ do not depend on $\tau_3$ and $z(\tau_3)$. The regularity condition (7.3) of [16] is not required. This can be seen by changing the proof in [16] to follow the pattern used in [17].

**Appendix B. Necessary conditions for a finite-dimensional optimization problem.** Consider the following notation and assumptions: $\mu$ and $\nu$ are nonnegative integers, $\hat{U} \subset R^m$ and $\hat{X} \subset R^{\hat{n}}$ are open sets, $U \subset \hat{U}$ is an arbitrary set, for $i=-\mu, \cdots, \nu$ the functions $\theta_i: \hat{X} \times \hat{U} \to R$ are continuously differentiable. Let $\theta_{i\hat{x}}(\hat{x}, u)$ and $\theta_{iu}(\hat{x}, u)$ denote respectively the Jacobian matrices of $\theta_i(\hat{x}, u)$ with respect to $\hat{x}$ and $u$.

*Finite-dimensional optimization problem* (FDO). Find $u$ and $\hat{x}$ which minimize $J$ subject to

$$J=\theta_0(\hat{x}, u), \tag{B.1-1}$$

$$\theta_i(\hat{x}, u) \leqq 0, \qquad i=-\mu, \cdots, -1, \tag{B.1-2}$$

$$\theta_i(\hat{x}, u) = 0, \qquad i=1, \cdots, \nu, \tag{B.1-3}$$

$$u \in U, \tag{B.1-4}$$

$$\hat{x} \in \hat{X}. \tag{B.1-5}$$

THEOREM B.1 (necessary conditions for FDO). *Let $C(u, U)$ be a conical approximation to $U$ at $u \in U$. Let $(u, \hat{x})$ solve FDO. Then there exist real numbers $\alpha_{-\mu}, \cdots, \alpha_\nu$ such that the following conditions are satisfied*:

$$\sum_{i=-\mu}^{\nu} \alpha_i \theta_{iu}(\hat{x}, u)\delta u \leqq 0 \quad \text{for all } \delta u \in \operatorname{cl} C(u, U), \tag{B.2-1}$$

$$\sum_{i=-\mu}^{\nu} \alpha_i \theta_{i\hat{x}}(\hat{x}, u) = 0, \tag{B.2-2}$$

$$\begin{aligned} &\alpha_i \leqq 0, \qquad i=-\mu, \cdots, 0,\\ &\alpha_i\theta_i(\hat{x}, u)=0, \qquad i=-\mu, \cdots, -1,\\ &(\alpha_{-\mu}, \cdots, \alpha_\nu) \neq 0. \end{aligned} \tag{B.2-3}$$

*Proof.* Apply Theorem 2.3.12 of [7] letting: the equality constraint correspond to (B.1-3) and $\theta_i(\hat{x}, u)=v_{-i}$, $i=-\mu, \cdots, -1$; $z=(u, \hat{x}, v) \in R^{m+\hat{n}+\mu}$; $\Omega = U \times \hat{X} \times V$ where $V=\{v: v_i \leqq 0, i=1, \cdots, \mu\}$; $(\alpha_0, \alpha_1, \cdots, \alpha_\nu, \alpha_{-1}, \cdots, \alpha_{-\mu})$ correspond to $\psi$.

THEOREM B.2 (maximum principle for FDO). *Assume that for all $\hat{x} \in \hat{X}$ the set $\{(\theta_{-\mu}(\hat{x}, u), \cdots, \theta_\nu(\hat{x}, u)): u \in U\}$ is convex. Weaken the differentiability requirements on the $\theta_i$ to the following: for $i=-\mu, \cdots, \nu$ the functions $\theta_i$ are continuous and for each $u \in U$ continuously differentiable in $\hat{x}$. Then the conditions in Theorem* B.1 *apply with* (B.2-1) *replaced by*

$$\sum_{i=-\mu}^{\nu} \alpha_i \theta_i(\hat{x}, u) = \max_{v \in U} \sum_{i=-\mu}^{\nu} \alpha_i \theta_i(\hat{x}, v). \tag{B.2-1$'$}$$

*Proof.* See Theorem 4.6 of [19] and take note of the comment on p. 221. Alternatively, the approach taken in § 4.2 of [7] may be adapted.

**Acknowledgment.** The author expresses his thanks to George D. Ianculescu whose excitement in periodic control motivated his own interest in the subject.

## REFERENCES


[1] J. E. BAILEY AND F. J. M. HORN, *Comparisons between two sufficient conditions for improvement of an optimal steady-state process by periodic operation*, J. Optimization Theory Appl., 17 (1971), pp. 378–384.

[2] J. E. BAILEY, *Necessary conditions for optimality in a general class of non-linear mixed boundary value control problems*, Internat. J. Control, 16 (1972), pp. 311–320.

[3] ———, *Periodic operation of chemical reactors: a review*, Chem. Engrg. Comm., 1 (1973), pp. 111–124.

[4] L. D. BERKOVITZ, *Optimal Control Theory*, Springer-Verlag, New York, 1974.

[5] S. BITTANTI, G. FRONZA AND G. GUARDABASSI, *Periodic control: a frequency domain approach*, IEEE Trans. Automatic Control, AC-18 (1973), pp. 33–38.

[6] ———, *Periodic optimization of linear systems under control power constraints*, Automatica—J. IFAC, 9 (1973), pp. 269–271.

[7] M. CANON, C. CULLUM AND E. POLAK, *Theory of Optimal Control and Mathematical Programming*, McGraw-Hill, New York, 1970.

[8] K. S. CHANG, *Necessary and sufficient conditions for optimality*, Periodic Optimization, R. MARZOLLO, ed., vol. 1, Springer-Verlag, New York, 1972, pp. 183–217.

[9] A. FIACCO AND G. MCCORMICK, *Nonlinear Programming*, John Wiley, New York, 1968.

[10] E. G. GILBERT, *Vehicle cruise: improved fuel economy by periodic control*, Automatica—J. IFAC, 12 (1976), pp. 159–166.

[11] G. GUARDABASSI, A. LOCATELLI AND S. RINALDI, *Status of periodic optimization of dynamical systems*, J. Optimization Theory Appl., 14 (1974), pp. 1–20.

[12] G. GUARDABASSI AND N. SCHIAVONI, *Boundary optimal constant control versus periodic operation*, preprint Part 1C-IFAC, 6th Triennial World Congress, Internat. Fed. Automatic Control, Boston, 1975.

[13] F. J. M. HORN AND R. C. LIN, *Periodic processes: a variational approach*, Indust. and Eng. Chem. Proc. Design Dev., 6 (1967), pp. 21–30.

[14] M. MATSUBARA, Y. NICHIMURA AND N. TAKAHASHI, *Optimal periodic control of lumped parameter systems*, J. Optimization Theory Appl., 13 (1974), pp. 13–31.

[15] E. J. MESSERLI AND E. POLAK, *On second order necessary conditions for optimality*, this Journal, 7 (1969), pp. 272–291.

[16] L. W. NEUSTADT, *An abstract variational theory with applications to a broad class of optimization problems. II: Applications*, this Journal, 5 (1967), pp. 90–137.

[17] ———, *A general theory of extremals*, J. Comput. System Sci., 3 (1969), pp. 57–92.

[18] L. S. PONTRYAGIN, V. G. BOLTYANSKII, R. V. GAMKRELIDZE AND E. F. MISCHENKO, *The Mathematical Theory of Optimal Processes*, John Wiley, New York, 1962.

[19] B. N. PSHENICHNYI, *Necessary Conditions for an Extremum*, M. Dekker, New York, 1971.

[20] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[21] J. WARGA, *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.

[22] W. A. WOLOVICH, *Linear Multivariable Systems*, Springer-Verlag, New York, 1974.