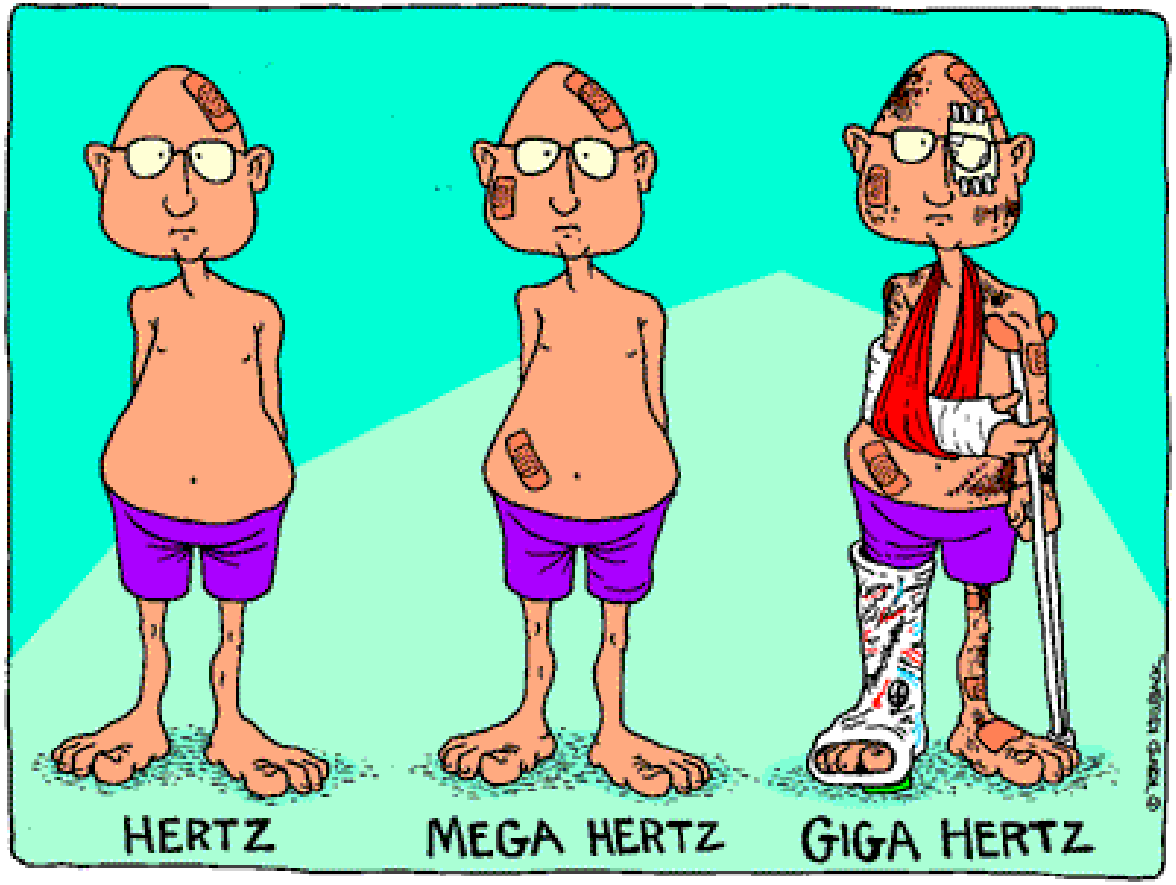

STEADY-STATE AC ANALYSIS I

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Winter, 2010

Material Order

- Ulaby & Maharbiz Chapters 7, 8, 9
- Lecture: Somewhat Different Order than Text Chapters
- Today: Big Picture (Part of Chp 9)
  - Finding AC Steady State Response
  - Relationship to Transient Response/Differential Equations
- Following Lectures Will Return to ~Text Book Time-Line

AC Circuit Analysis

- Objective: Find the Steady-State Response of a Linear Circuit to a Fixed AC Signal
  - $V_{in}(t)=V_0\cos(\omega t)$
- The Steady-State Response is the *Forced Solution* of the Differential Equation for the Circuit
- Why Do This?
  - Through Fourier Series, this gives us the response to ANY periodic signal
  - Directly Useful for All Analog Circuit Design, Understanding Clock Skew on Digital Lines, etc.

Example: General RLC Problem

General Differential Equation

$$\frac{\partial^2 V_c}{\partial t^2}+\left(\frac{R_1+R_2}{L}\right)\frac{\partial V_c}{\partial t}+\left(\frac{1}{LC}\right)V_c=\left(\frac{1}{C}\right)\frac{\partial i_0}{\partial t}+\left(\frac{R_1}{LC}\right)i_0$$

$$\frac{\partial^2 V_c}{\partial t^2}+(2\alpha)\frac{\partial V_c}{\partial t}+\left(\omega_0^2\right)V_c=\left(\frac{1}{C}\right)\frac{\partial i_0}{\partial t}+\left(\frac{R_1}{LC}\right)i_0=f(t)$$

$$\alpha=\left(\frac{R_1+R_2}{2L}\right)$$

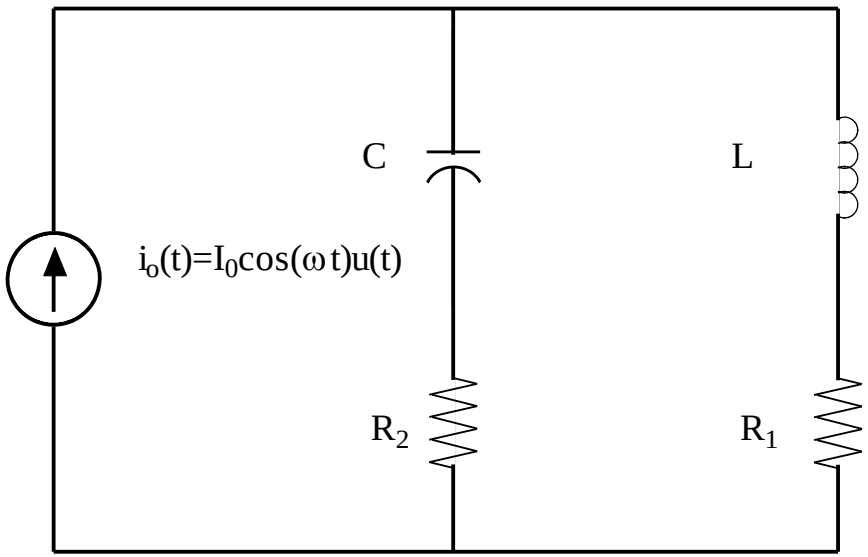
$$\omega_0=\left(\frac{1}{LC}\right)^{1/2}$$

$$Now\quad i_0=I_0\cos(\omega t)$$

$$f(t)=\left(-\frac{1}{C}\right)I_0\omega\sin(\omega t)+\left(\frac{R_1}{LC}\right)I_0\cos(\omega t)$$

$$Trial\ Forced\ Solution\ of\ \ form\ of\ \ f(t),\ f'(t),\ f''(t)$$

$$v_{cf}(t)=A\cos(\omega t)+B\sin(\omega t)$$



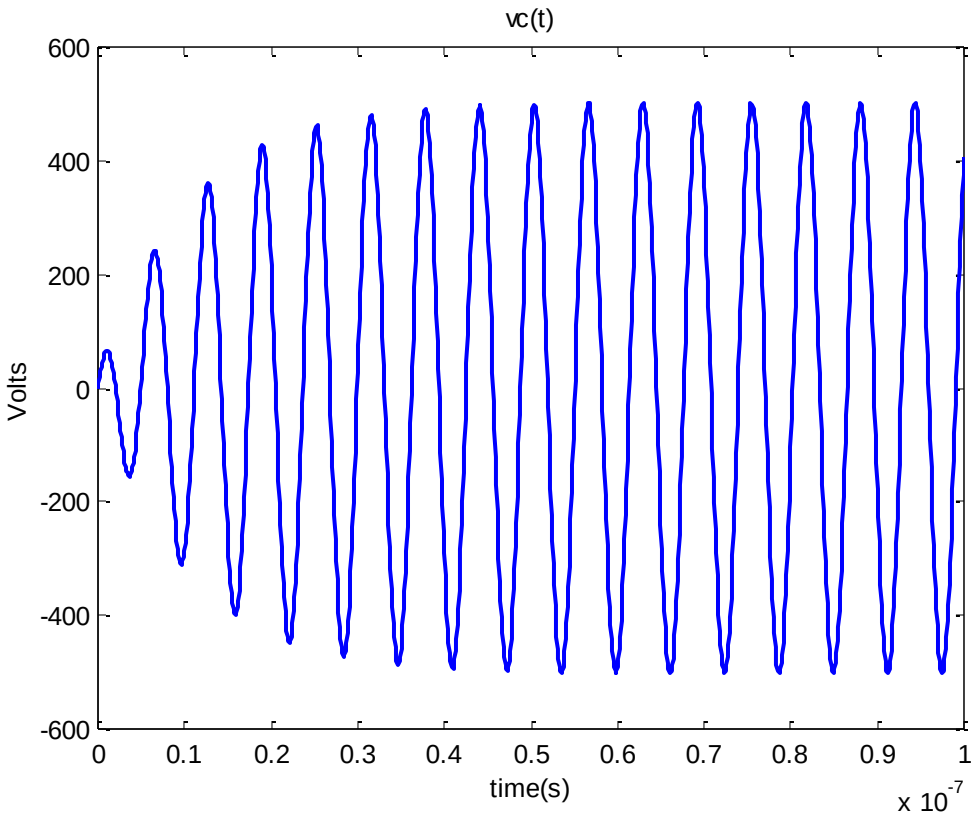
Later, we will use the following numerical values for this circuit:

$L=100nH\quad C=0.01nF=10pF$

$R_1=10\Omega\ \ R_2=10\Omega$

$I_0=1A\quad \omega=10^9\ s^{-1}$

Response



Later, we will find the response at the left for the numerical example on the previous page

For now, note the steady state period response for t>0.5s

Plug in to find A,B

General Differential Equation

$$v_{cf}(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\downarrow$$

$$\frac{\partial^2 V_c}{\partial t^2} + 2\alpha\frac{\partial V_c}{\partial t} + \omega_0^2 V_c = \left(-\frac{1}{C}\right) I_0 \omega \sin(\omega t) + \left(\frac{R_1}{LC}\right) I_0 \cos(\omega t)$$

$$\downarrow$$

$$\left\{\begin{array}{l} -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) \\ -2\alpha\,\omega A\sin(\omega t) + 2\alpha\,\omega B\cos(\omega t) \\ +\omega_0^2 B\sin(\omega t) + \omega_0^2 A\cos(\omega t) \end{array}\right\} = \left(-\frac{1}{C}\right) I_0 \omega \sin(\omega t) + \left(\frac{R_1}{LC}\right) I_0 \cos(\omega t)$$

$\cos(\omega t)$ *and* $\sin(\omega t)$ *must both match*

$$-\omega^2 A + 2\alpha\,\omega B + \omega_0^2 A = \left(\frac{R_1}{LC}\right) I_0$$

$$-\omega^2 B - 2\alpha\,\omega A + \omega_0^2 B = \left(-\frac{1}{C}\right) I_0 \omega$$

$$A\left(\omega_0^2 - \omega^2\right) + B\left(2\alpha\,\omega\right) = \left(\frac{R_1}{LC}\right) I_0$$

$$A\left(-2\alpha\,\omega\right) + B\left(\omega_0^2 - \omega^2\right) = \left(-\frac{1}{C}\right) I_0 \omega$$

$$A = \left[\frac{R_1\omega_0^2\left(\omega_0^2 - \omega^2\right) + \left(2\alpha\,\omega^2/C\right)}{\left(2\alpha\,\omega\right)^2 + \left(\omega_0^2 - \omega^2\right)^2}\right] I_0$$

Get B if you have the stomach for it.

Isn't there any easier way?

YES!

AC Steady State Analysis

- Real Sinusoidal Input
  - $V_{in}=V_0\cos(\omega t)$
- Assume Input is Complex

$$V_{in}(t)=V_0e^{j\omega t}=V_0\left[\cos(\omega t)+j\sin(\omega t)\right]$$

$$\mathrm{Re}\,al\ Input\ Signal$$

$$V_{in}(t)=\mathrm{Re}\left[V_{in}(t)\right]$$

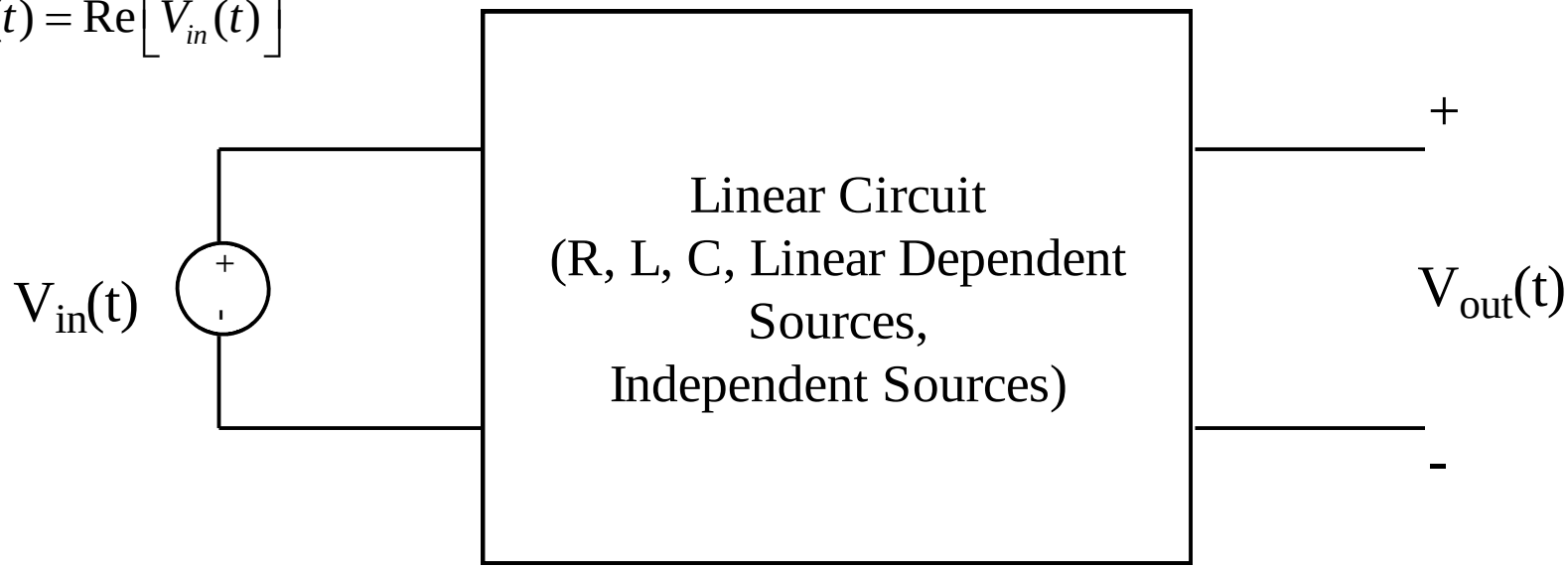
- By Superposition, ALL Circuit Quantities of Interest (voltages, currents) are given by taking the Real Part of the Response to the Complex Signal

Concept

$$V_{in}(t) = V_0 e^{j\omega t} = V_0\left[\cos(\omega t) + j\sin(\omega t)\right]$$

$$\text{Re}\,al\ Input\ Signal$$

$$V_{in}(t) = \text{Re}\left[V_{in}(t)\right]$$



$$V_{out}(t) = circuit\ response\ to\ V_0 e^{j\omega t} = V_0\left[\cos(\omega t) + j\sin(\omega t)\right]$$

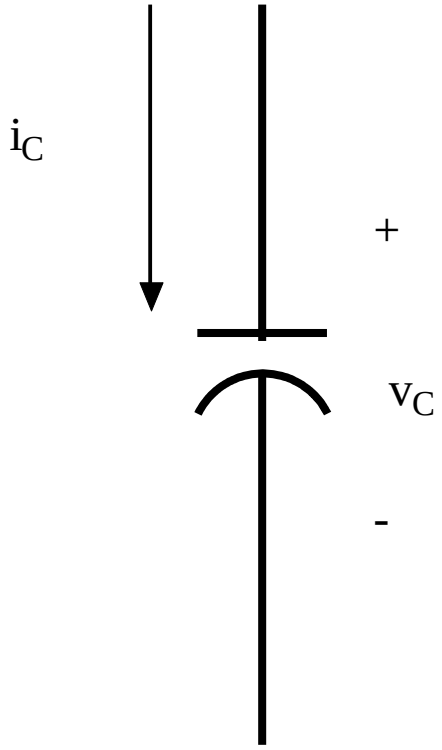
$$\text{Re}\,al\ Output\ Signal$$

$$V_{out}(t) = \text{Re}\left[V_{out}(t)\right]$$

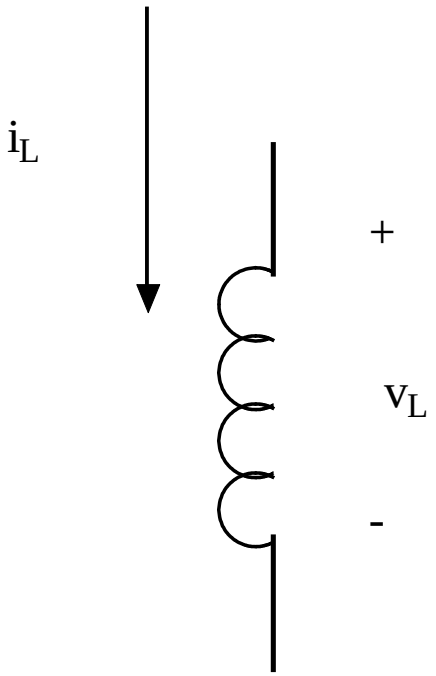
AC Steady State Analysis Outline

- Treat Sinusoidal Signals as Complex Exponentials
- Analyze Circuit Response Using Nodal, Mesh, *Whatever* Using Complex Impedances
- Find Desired Responses in Complex Form
- Take the Real Part to Get Physical Result
- *Wonderful Thing Is That This Approach Is Easier*

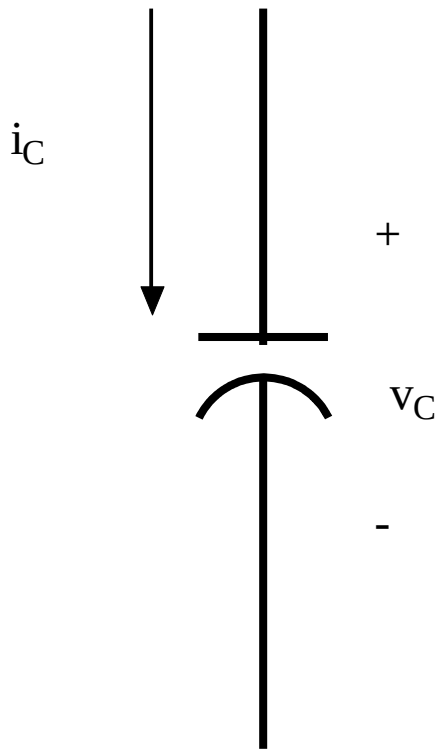
Capacitor in jω-Domain



Inductor in jω -Domain



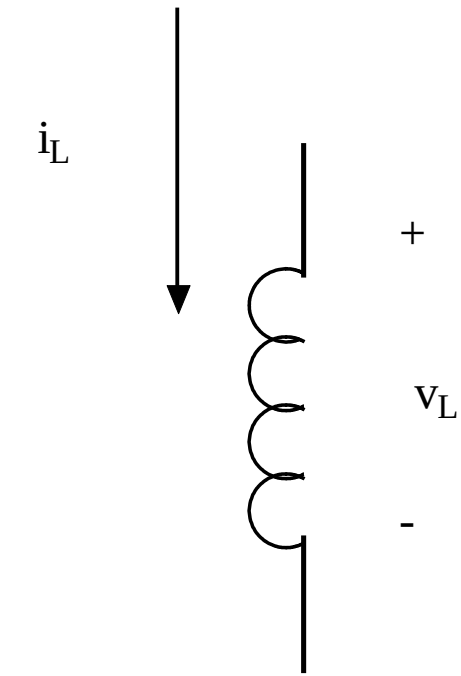
Capacitors/Inductors in jω -Domain



Suppose $V_C(t) = Ae^{j\omega t}$

$$i_C = C\frac{\partial V_C}{\partial t} = j\omega CAe^{j\omega t}$$

$$V_C = Z_C i_C \rightarrow Z_C = \frac{1}{j\omega C}$$

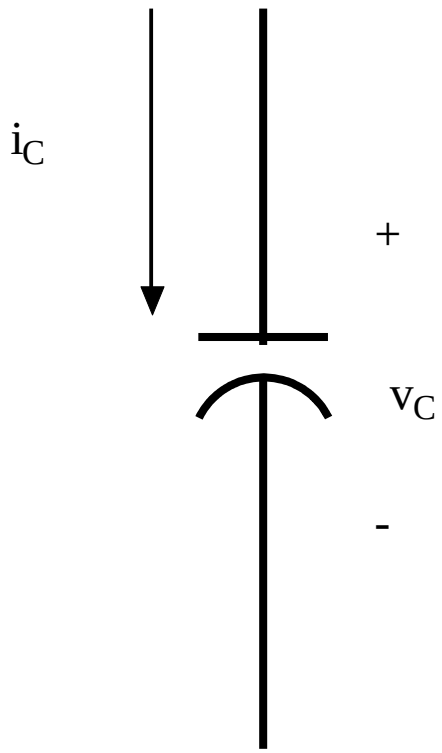


Suppose $i_L(t) = Ae^{j\omega t}$

$$V_L = L\frac{\partial i_L}{\partial t} = j\omega LAe^{j\omega t}$$

$$V_L = Z_L i_L \rightarrow Z_L = j\omega L$$

Capacitors/Inductors in s-Domain



Suppose $V_C(t) = Ae^{st}$

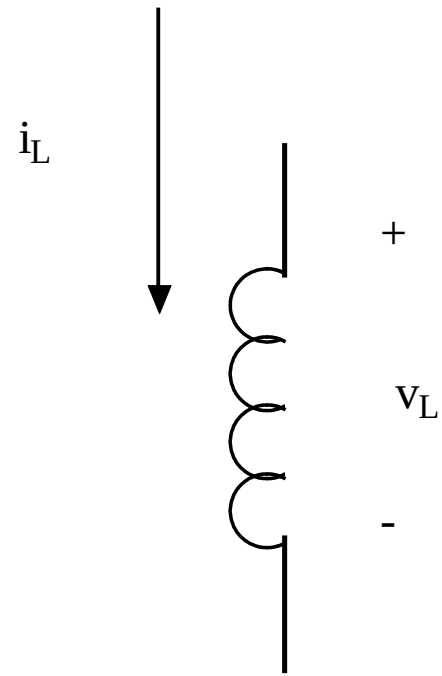
$$i_C = C\frac{\partial V_C}{\partial t} = sCAe^{st}$$

$$V_C = Z_C i_C \rightarrow Z_C = \frac{1}{sC}$$

Suppose $i_L(t) = Ae^{st}$

$$V_L = L\frac{\partial i_L}{\partial t} = sLAe^{st}$$

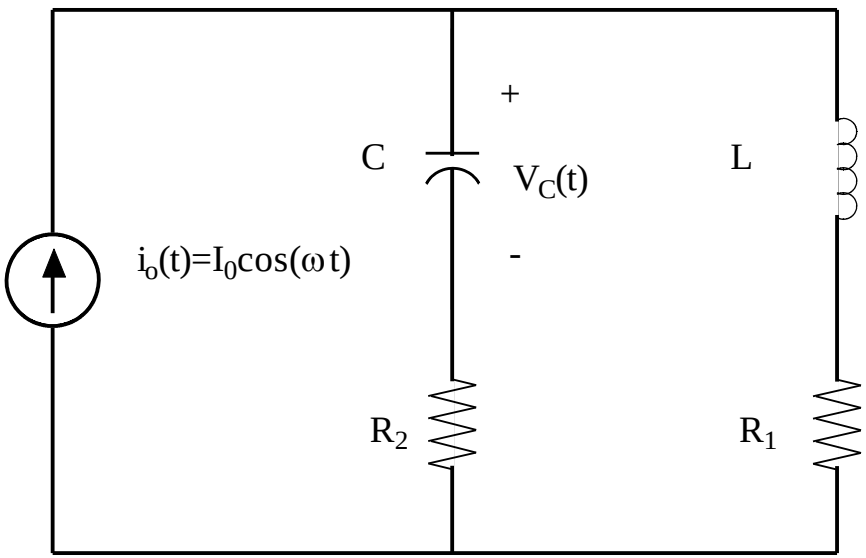
$$V_L = Z_L i_L \rightarrow Z_L = sL$$

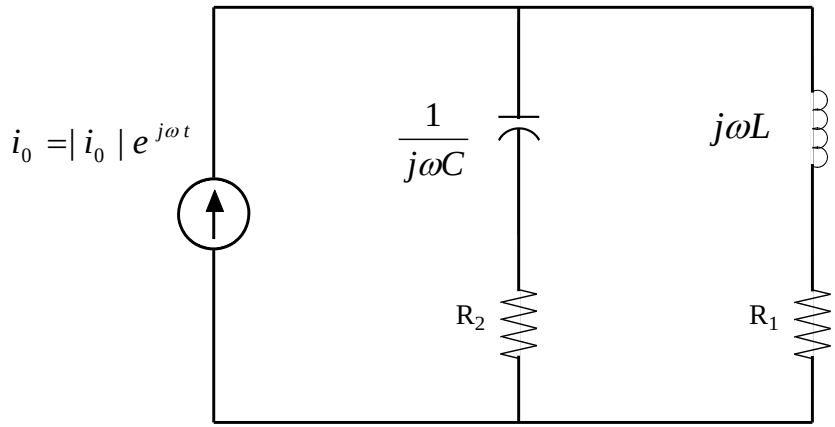


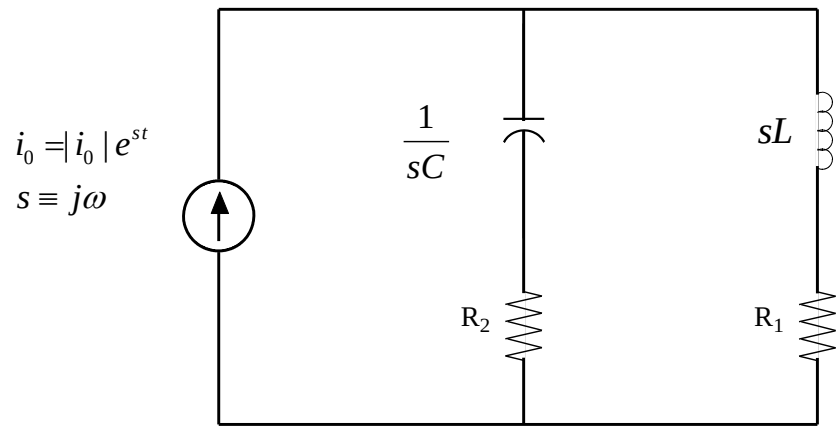
Get all the previous results simply by taking $s = j\omega$

General RLC Example 1: Current Divider in $j\omega$- Domain

Find $V_c(t)$




$i_0 = | i_0 | e^{j\omega t}$
$\frac{1}{j\omega C}$
$j\omega L$
R_2
R_1


$i_0 = | i_0 | e^{st}$
$s \equiv j\omega$
$\frac{1}{sC}$
sL
R_2
R_1

$$Z_1 = Ls + R_1$$

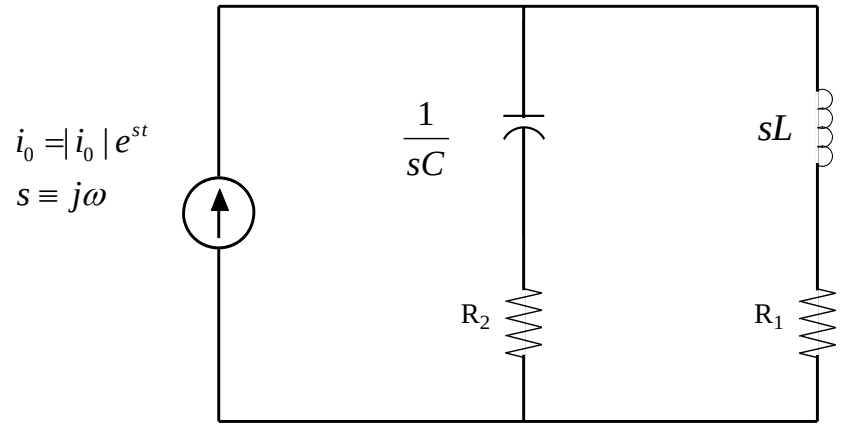
$$Z_2 = \frac{1}{Cs} + R_2$$

$$i_c = \left[\frac{1/Z_2}{1/Z_2 + 1/Z_1}\right] i_0 = \left[\frac{Z_1}{Z_1 + Z_2}\right] i_0$$

$$V_c = i_c / Cs = \left[\frac{Z_1}{Z_1 + Z_2}\right] \frac{1}{Cs} i_0$$

$$= \frac{Ls + R_1}{Ls + R_1 + \frac{1}{Cs} + R_2} \left(\frac{1}{Cs}\right) i_0$$

$$= \frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1} i_0$$



$$V_c = \frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1} i_0$$

$$\left[LCs^2 + (R_1 + R_2)Cs + 1\right] V_c = \left[Ls + R_1\right] i_0$$

$$\left[s^2 + \frac{(R_1 + R_2)}{L} s + \frac{1}{LC}\right] V_c = \left[\frac{1}{C} s + \frac{R_1}{L}\right] i_0$$

which maps back to:

$$\frac{d^2 V_c}{dt^2} + \frac{(R_1 + R_2)}{L} \frac{dV_c}{dt} + \frac{1}{LC} V_c = \frac{1}{C} \frac{di_0}{dt} + \frac{R_1}{L} i_0$$

If we want, we can get the governing differential equation from this s-domain expression (see above box) but our main point here will be to find the AC steady state response

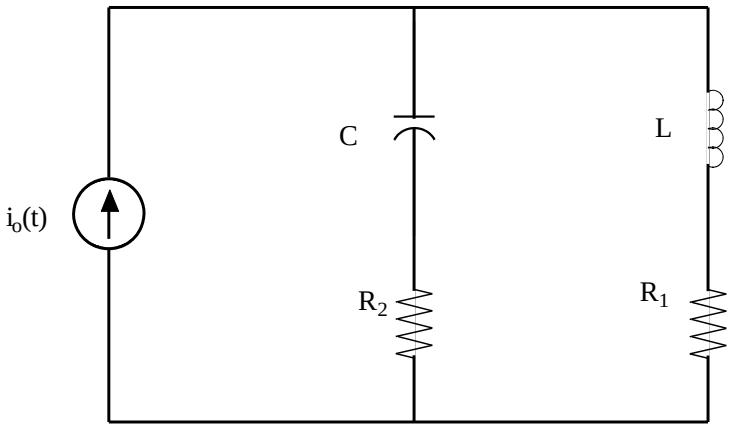
Final Result

$$V_C = H(j\omega) i_0(t)$$

$$H(s) = \frac{V_C(t)}{i_0(t)} = \left[\frac{sL + R_1}{\left(1 + s^2 LC\right) + s\left(R_1 + R_2\right)C} \right]$$

To get $H(j\omega)$ just take $s = j\omega$

$$H(j\omega) = H(\omega) = \left[\frac{j\omega L + R_1}{\left(1 - \omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C} \right]$$



H(**s**) is called a ***transfer function***

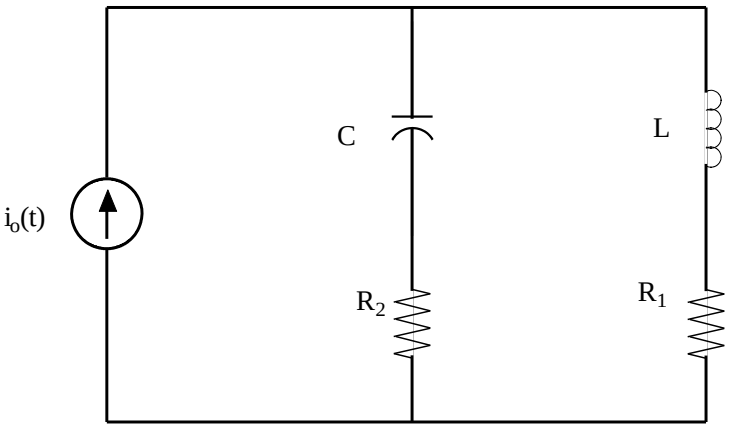
Final Result

$$V_C(t) = \left[\frac{j\omega L + R_1}{\left(1-\omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C} \right] i_0(t)$$

$$= H(\omega) i_0(t)$$

$$H(\omega) = \frac{V_C(t)}{i_0(t)}$$

$$= \left[\frac{j\omega L + R_1}{\left(1-\omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C} \right]$$



H(ω) is called a ***transfer function***

Numerical Example

For the example circuit, suppose:

$$L = 100nH \quad C = 0.01nF = 10\,pF \quad R_1 = 10\Omega \quad R_2 = 10\Omega \quad I_0 = 1A \quad \omega = 10^9\ s^{-1}$$

Find $V_c(t)$

$$H(\omega) = \left[\frac{j\omega L + R_1}{\left(1-\omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C}\right] = \left[\frac{j100\Omega + 10\Omega}{(1-1) + j0.2}\right] = \left(500 - 50j\right)\Omega$$

$$V_c(t) = \mathrm{Re}\left[\left(500-50j\right)\Omega \cdot 1e^{j10^9 t} A\right] = \mathrm{Re}\left[\left(500-50j\right)\left(\cos(10^9 t) + j\sin(10^9 t)\right)V\right]$$

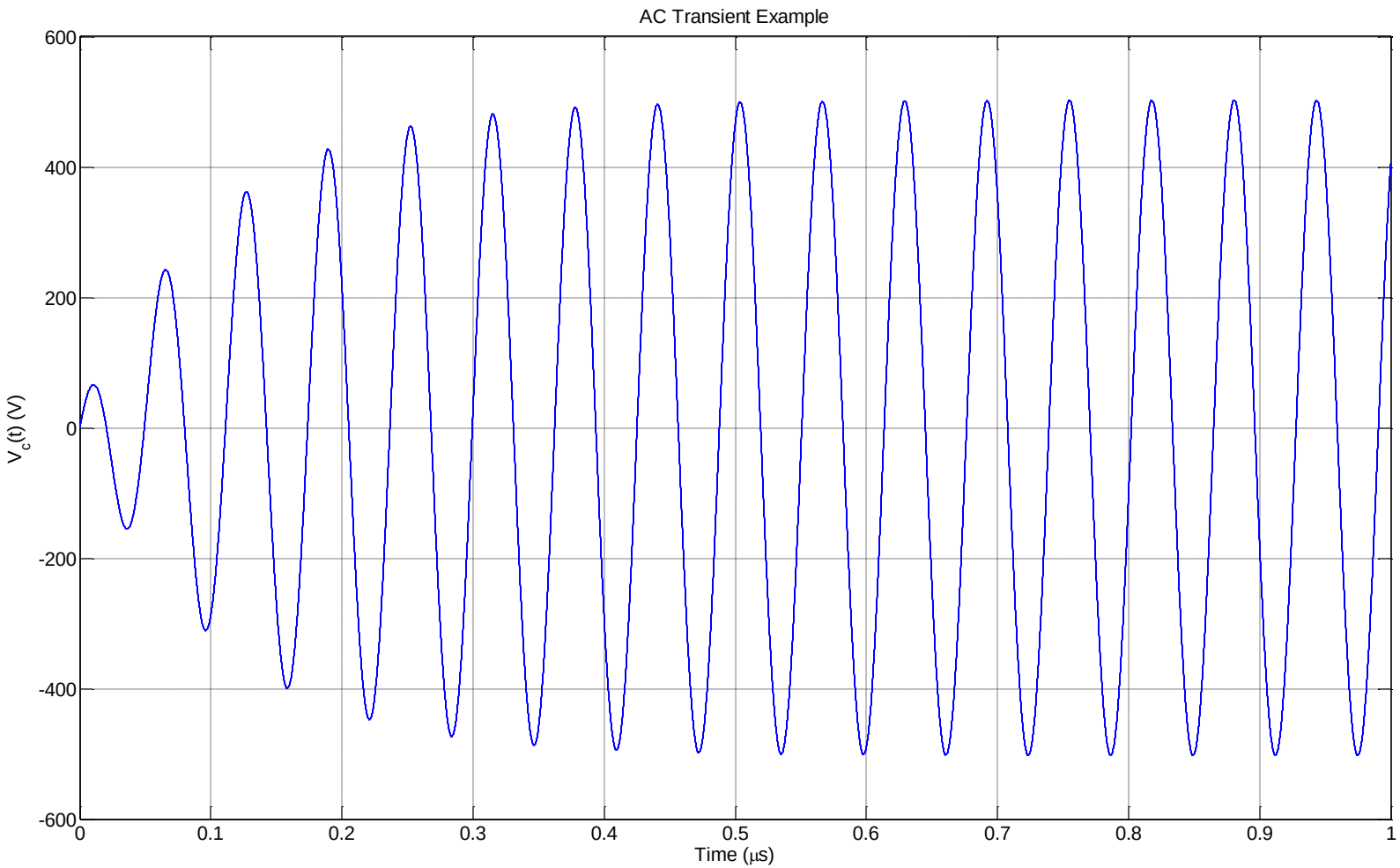
$$V_c(t) = \left[500\cos(10^9 t) + 50\sin(10^9 t)\right]V$$

$$V_c(t) = |H(\omega)| \cdot |i_0| \cos\left(10^9 t + \measuredangle H(\omega)\right)$$

$$|H(\omega)| = \left(500^2 + 50^2\right)^{1/2} \approx 502.494\Omega$$

$$\measuredangle H(\omega) = \tan^{-1}\left(\frac{-50j}{500}\right) \approx -5.7106°$$

Full Transient Solution

Full Transient Solution

Full Transient Solution