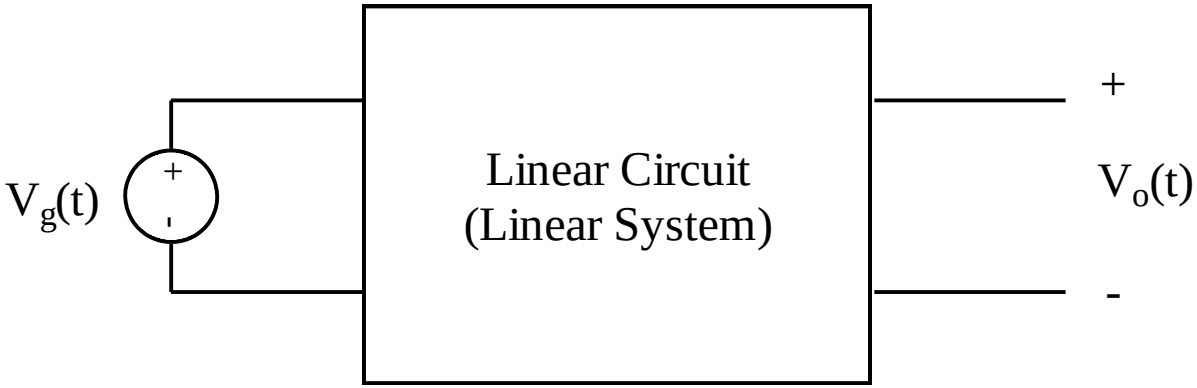
FREQUENCY RESPONSE AND FILTERS I

Fred Terry, Winter 2010

Transfer Function H(ω)



Assume

$$V_g(t) = |V_g| \cos(\omega t)$$

$$V_g = |V_g| e^{j\omega t}$$

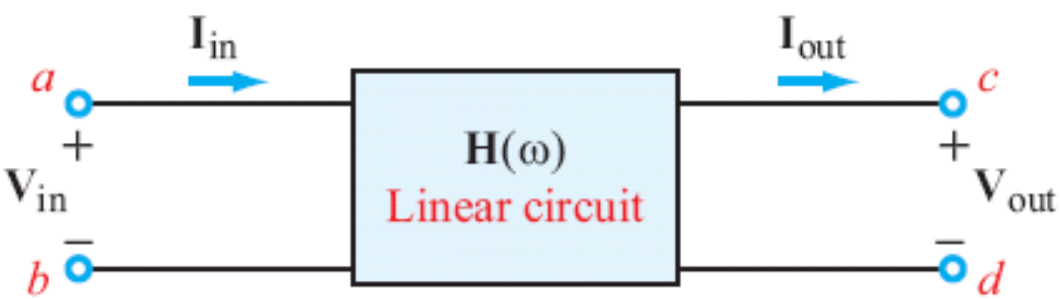
$$V_o = H(\omega) V_g = |V_g| |H(\omega)| e^{j(\omega t + \angle H(\omega))}$$

$$V_o(t) = |V_g| |H(\omega)| \cos(\omega t + \angle H(\omega))$$

Transfer Function

How does circuit respond to changing frequency?

Transfer function: relates output to input



$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

$$M(\omega) = |H(\omega)|$$

$$\phi(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}[H(\omega)]}{\mathrm{Re}[H(\omega)]}\right)$$

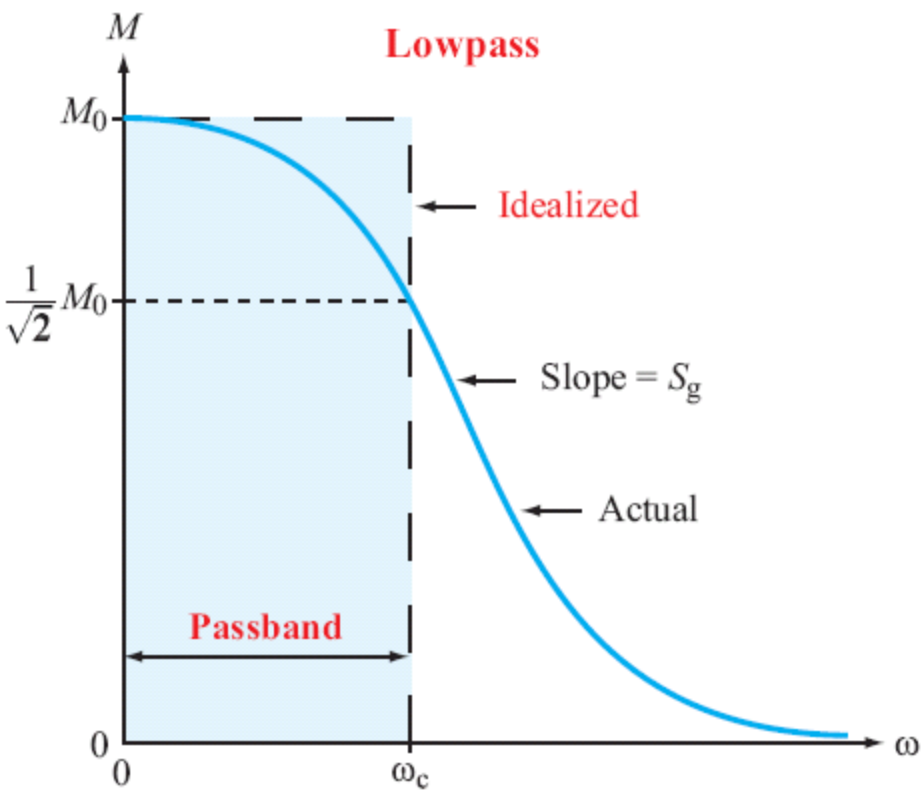
$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \text{Voltage gain}$$

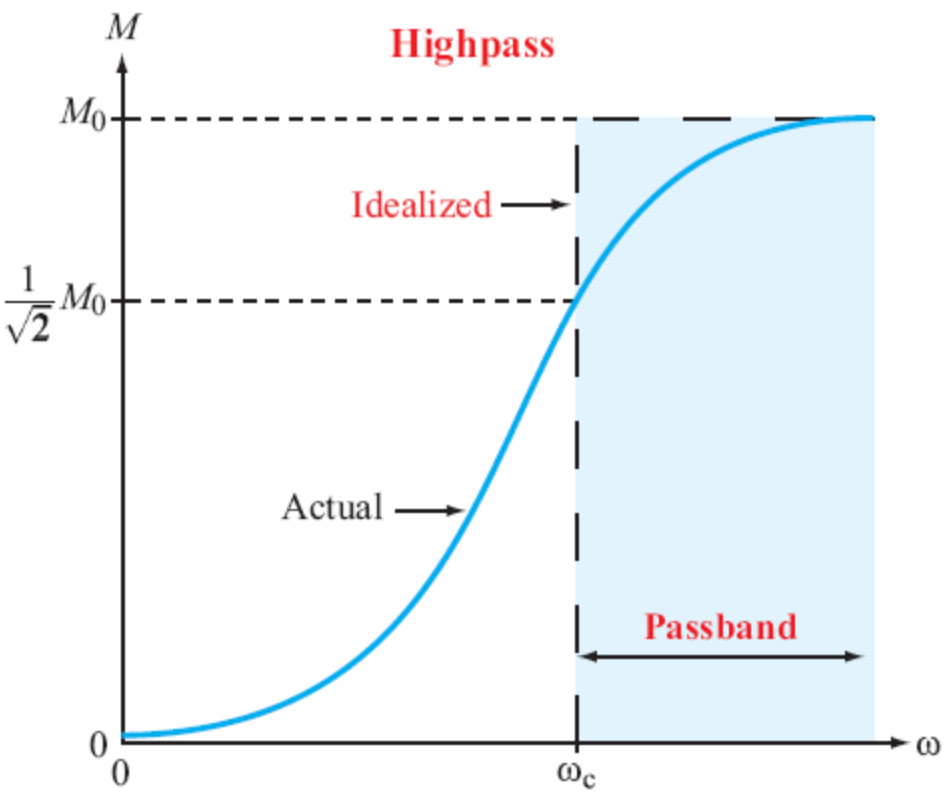
$$H(\omega) = \frac{V_o(\omega)}{I_i(\omega)} = \text{Transfer Impedance}$$

$$H(\omega) = \frac{I_o(\omega)}{I_i(\omega)} = \text{Current gain}$$

$$H(\omega) = \frac{I_o(\omega)}{V_i(\omega)} = \text{Transfer Admittance}$$

Lowpass and Highpass Filters

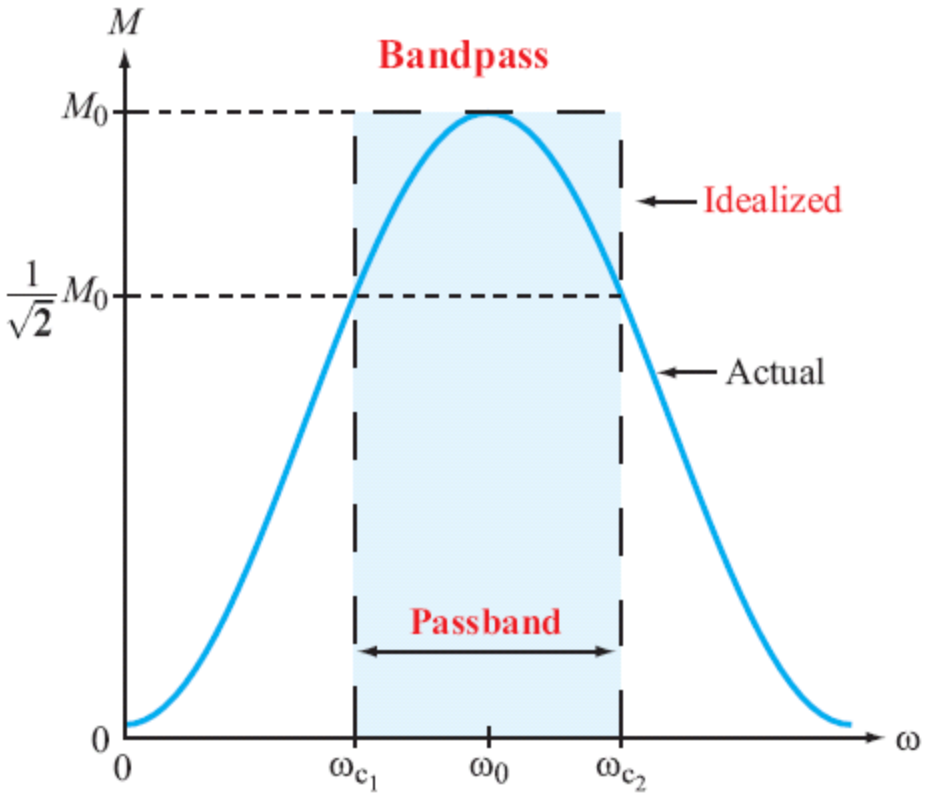


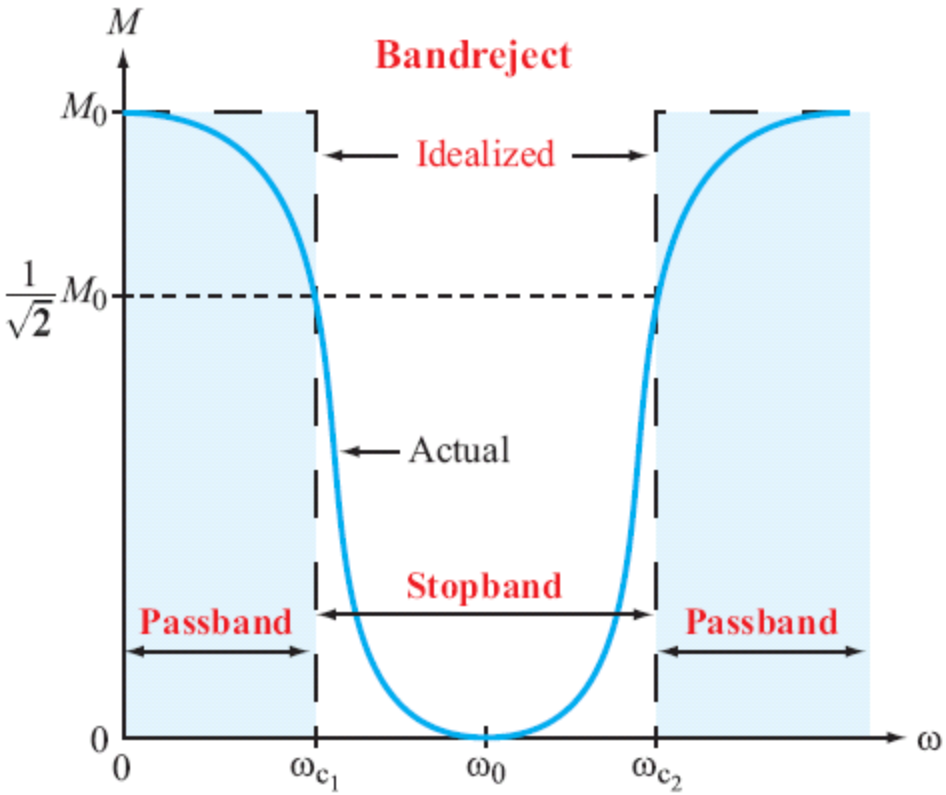


M_0 is the gain factor

ω_c is the corner frequency

Bandpass and Bandreject Filters





Fourier Series & Transfer Functions

Any Periodic Signal f(t) Can Be Represented in A Fourier Series Expansion:

$$If\ \ f(t)\, is\ any\ periodic\ function\, with\ period\, T$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\ \ where\ \omega_0 = \frac{2\pi}{T}$$

$$if$$

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t} dt$$

So by Superposition, if we have a periodic input f(t) (voltage or current) into our linear circuit with transfer function H(ω), we will have an output g(t) given by:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n H\left(n\omega_0\right) e^{jn\omega_0 t}\ \ where\ \omega_0 = \frac{2\pi}{T}$$

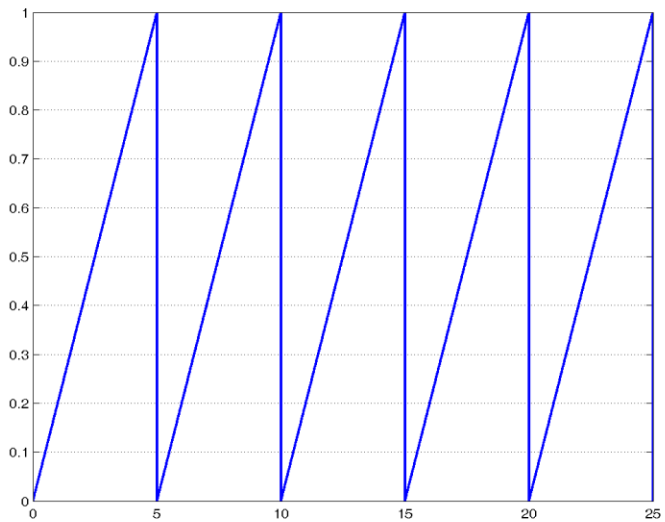
Fourier Series & Transfer Functions

So by Superposition, if we have a periodic input f(t) (voltage or current) into our linear circuit with transfer function H(ω), we will have an output g(t) given by:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n H\left(n\omega_0\right) e^{jn\omega_0 t} \;\; where \; \omega_0 = \frac{2\pi}{T}$$

This is powerful, since it means that by doing AC Steady State Analysis on a Circuit, We Can Easily Calculate the Response to ANY Periodic Signal!!

Sawtooth Wave Example



$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}dt$$

$$f(t) = \frac{t}{T} \quad \textbf{for 1st period } t = 0 \textbf{ to } T$$

$$c_n = \frac{1}{T}\int_0^T \left(\frac{t}{T}\right)e^{-jn\omega_0 t}dt = \left(\frac{1}{T^2}\right)\int_0^T te^{-jn\omega_0 t}dt$$

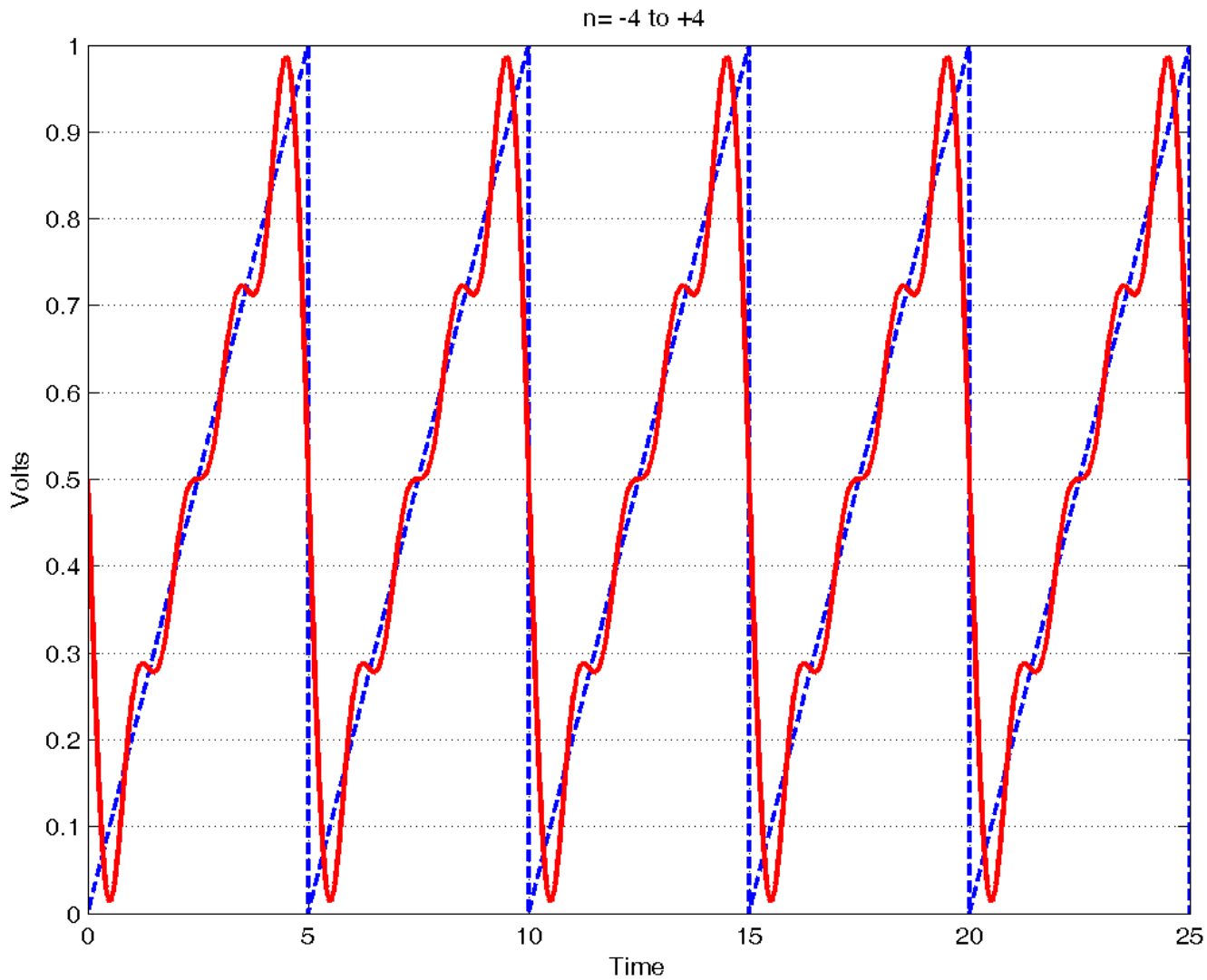
$$\textbf{using } \int xe^{-ax}dx = -\frac{1}{a^2}\left[ax+1\right]e^{-ax} + \textbf{constant} \quad \textbf{if } a \neq 0$$

we get:

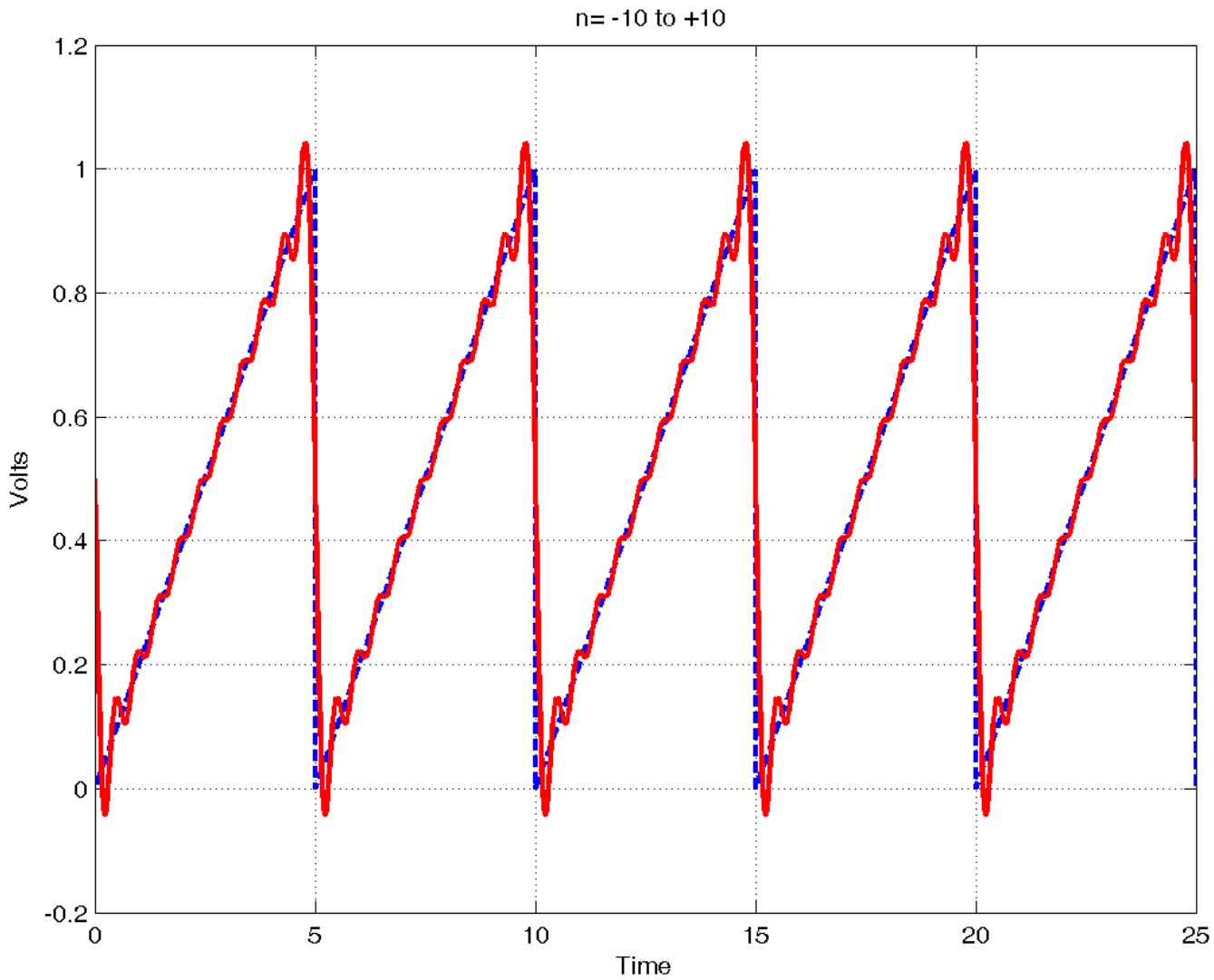
$$c_n = 1/2 \ if \ n = 0$$

$$c_n = \frac{1}{(2n\pi)^2}\left[-1+\left(1+2jn\pi\right)e^{-2jn\pi}\right] \ for \ n \neq 0$$

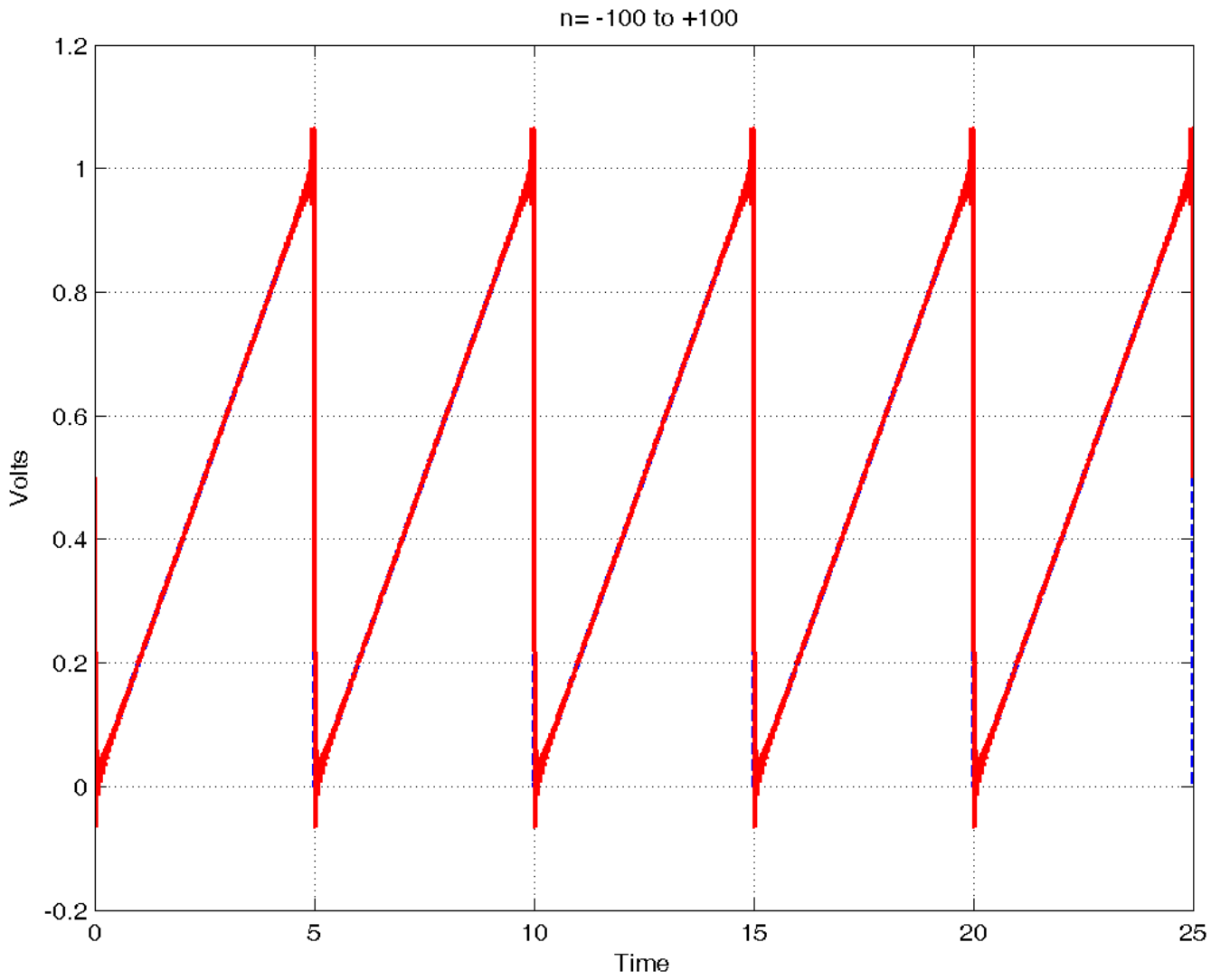
Sawtooth Wave



Sawtooth Wave



Sawtooth Wave



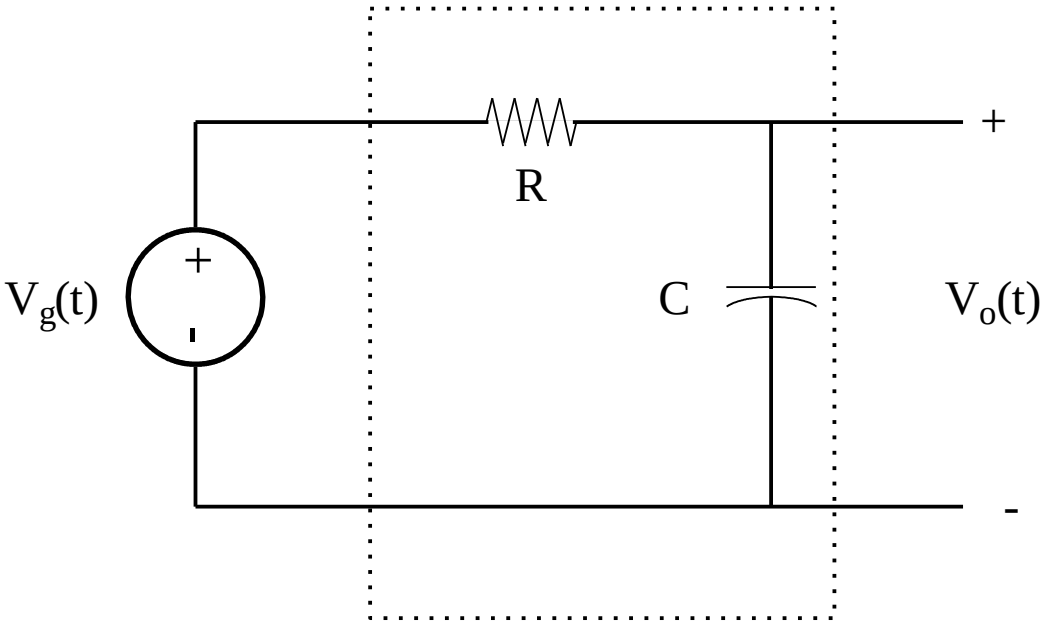
Example: 1st Order Low Pass Filter

$$V_o = \frac{1/j\omega C}{1/j\omega C + R} V_g = \frac{1}{1 + j\omega RC} V_g$$

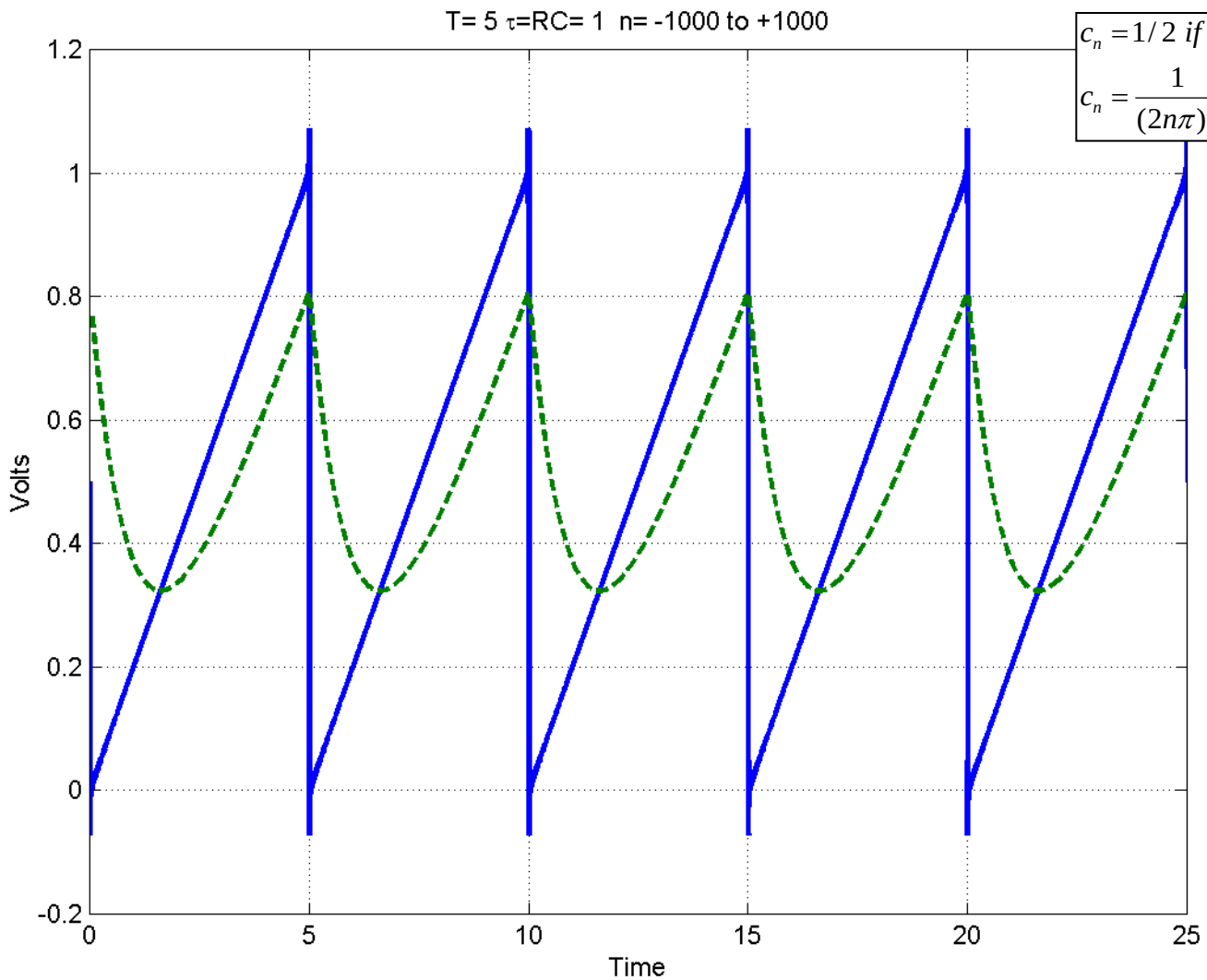
$$H(\omega) = \frac{V_o}{V_g} = \frac{1}{1 + j\omega RC}$$

$$|H(\omega)| = \frac{1}{\left[1 + (\omega RC)^2\right]^{1/2}}$$

$$\angle H(\omega) = -\tan^{-1}(\omega RC)$$



Example: Sawtooth Wave into 1st Order Low Pass Filter



$$c_n = 1/2 \ if \ n = 0$$

$$c_n = \frac{1}{(2n\pi)^2}\left[-1+(1+2jn\pi)e^{-2jn\pi}\right] \ for \ n \neq 0$$

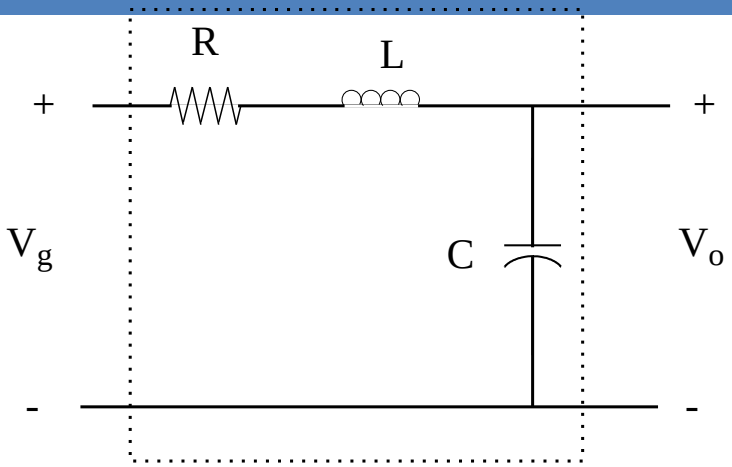
Matlab Code For Example


```
%sawtooth wave into RC filter
clear;
tau=1; % assume 1 WLOG
T=5*tau; % vary this for fun, the period of the sawtooth vs. tau
t=0:1e-3:5*max(T,tau);
N=1000;
v0=0*t;
vg=0*t;
for jj=-N:N,
   if abs(jj)>0,
      cn=-1/(4*(jj*pi)^2)+(1+2*j*jj*pi)*1/(4*(jj*pi)^2)*exp(-2*j*jj*pi);
   else
      cn=0.5;
   end;
   w=2*pi/T*jj;
   H=1/(1+j*tau*w); % transfer function
   v0=v0+cn*H*exp(j*w*t);
   vg=vg+cn*exp(j*w*t);
end;
figure(1);
h=plot(t,vg,t,v0,'--');
grid on;
set(h,'linewidth',2);
title(['T= ' num2str(T) ' \tau=RC= ' num2str(tau) '  n= -' num2str(N) ' to +'
    num2str(N)]);
xlabel('Time');
ylabel('Volts');
% As check, rounding produces imaginary part
figure(2);
plot(t,imag(vg),t,imag(v0));
```

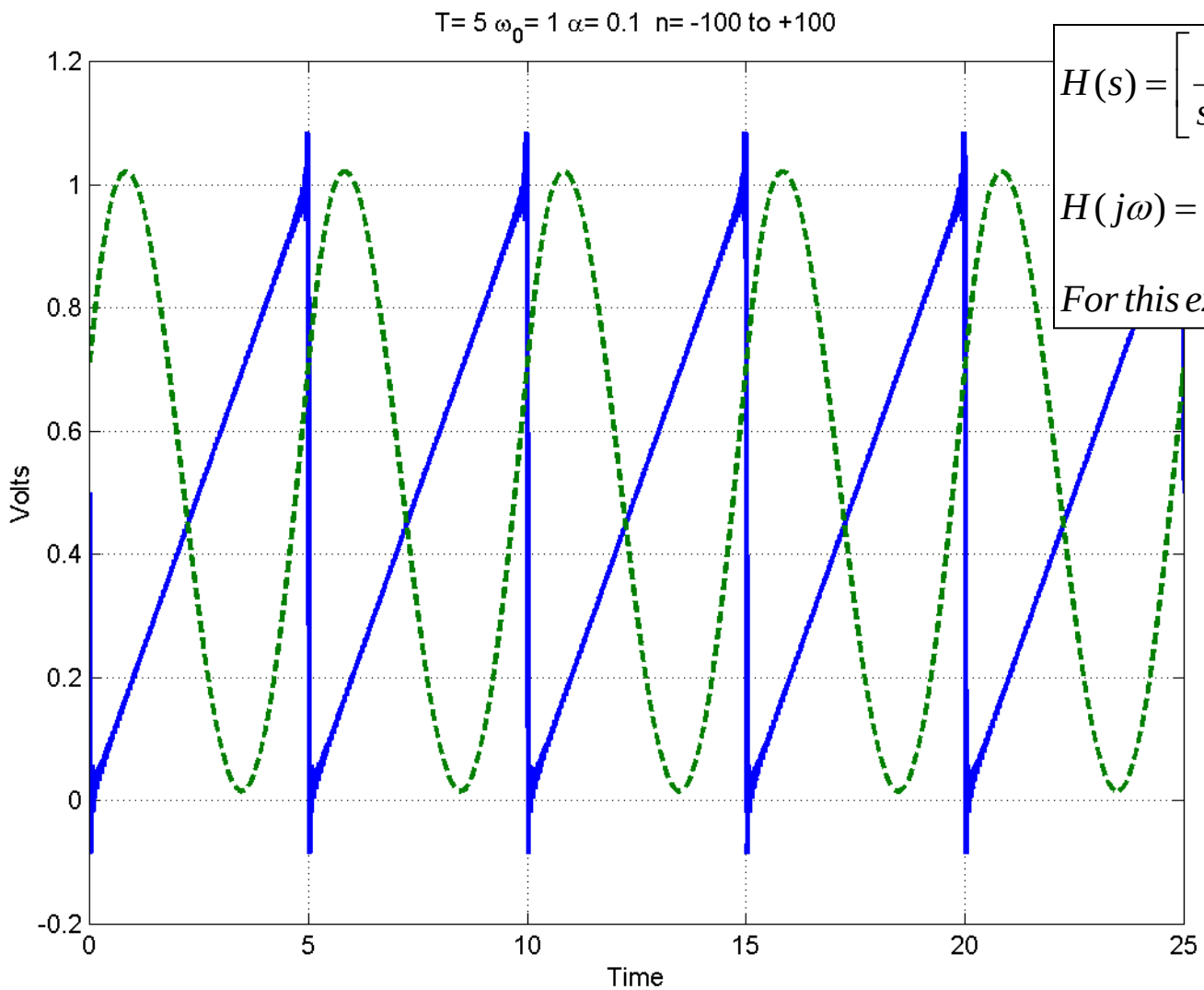
2nd Order Example

Find $H(s)=V_0/V_g$

Get $H(j\omega)=H(\omega)$ by setting $s= j\omega$



Example 2: Second Order System with Saw Tooth Input

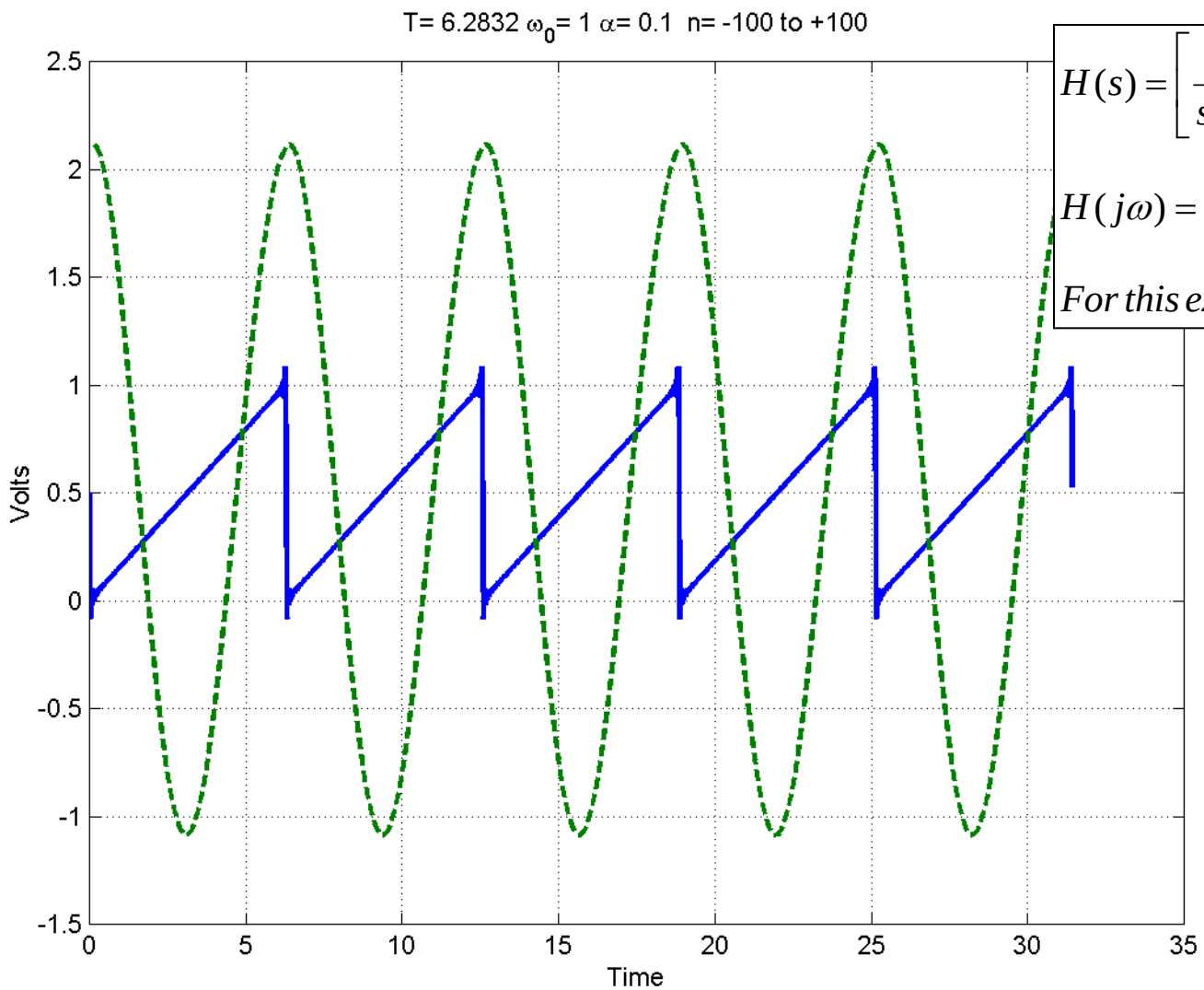


$$H(s) = \left[\frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2} \right]$$

$$H(j\omega) = H(\omega) = \left[\frac{\omega_0^2}{\omega_0^2 - \omega^2 + 2j\alpha\omega} \right]$$

For this example let $\omega_0 = 1 \;\; \alpha = 0.1$

Example 2: Second Order System with Saw Tooth Input



$$H(s) = \left[\frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2} \right]$$

$$H(j\omega) = H(\omega) = \left[\frac{\omega_0^2}{\omega_0^2 - \omega^2 + 2j\alpha\omega} \right]$$

For this example let $\omega_0 = 1 \; \alpha = 0.1$

Systematically Examining Frequency Response: Bode Plots

- Write Transfer Function in form $H(s)=N(s)/D(s)$
- ***Zeros*** are the roots of $N(s)=0$
- ***Poles*** are the roots of $D(s)=0$
- Poles & Zeros Have Direct Significance in Laplace Transform-Based Systems Theory
- Here We Will See that Interesting, Predictable Things Happen to $|H|$ and $\angle H$ at the MAGNITUDE of the Pole and Zero Frequencies ($s=j\omega$)

The Decibel Scale

- To Examine Strongly Varying Functions of Frequency, It Makes Sense to Use a Log Scale
- Bels, Decibels Originally Defined for Power
- Voltage Definitions Assume Resistive Loads

$$G = Number\ of\ bels = \log_{10}\left(P_2/P_1\right)$$

$$G_{dB} = Number\ of\ decibels = 10\log_{10}\left(P_2/P_1\right)$$

$$Special\ Useful\ Case\ P_2 = 0.5P_1$$

$$G_{dB} = 10\log_{10}(0.5) \approx -3.0103C \approx -3dB$$

$$If\ we\ assume\ a\ resistive\ load\ R$$

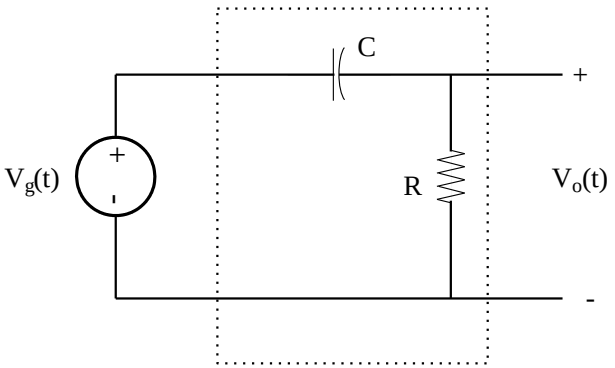
$$G_{dB} = 10\log_{10}\left(P_2/P_1\right) = 10\log_{10}\left(\frac{V_2^2/R}{V_1^2/R}\right) = 20\log_{10}\left(\frac{V_2}{V_1}\right)$$

Bode Plots

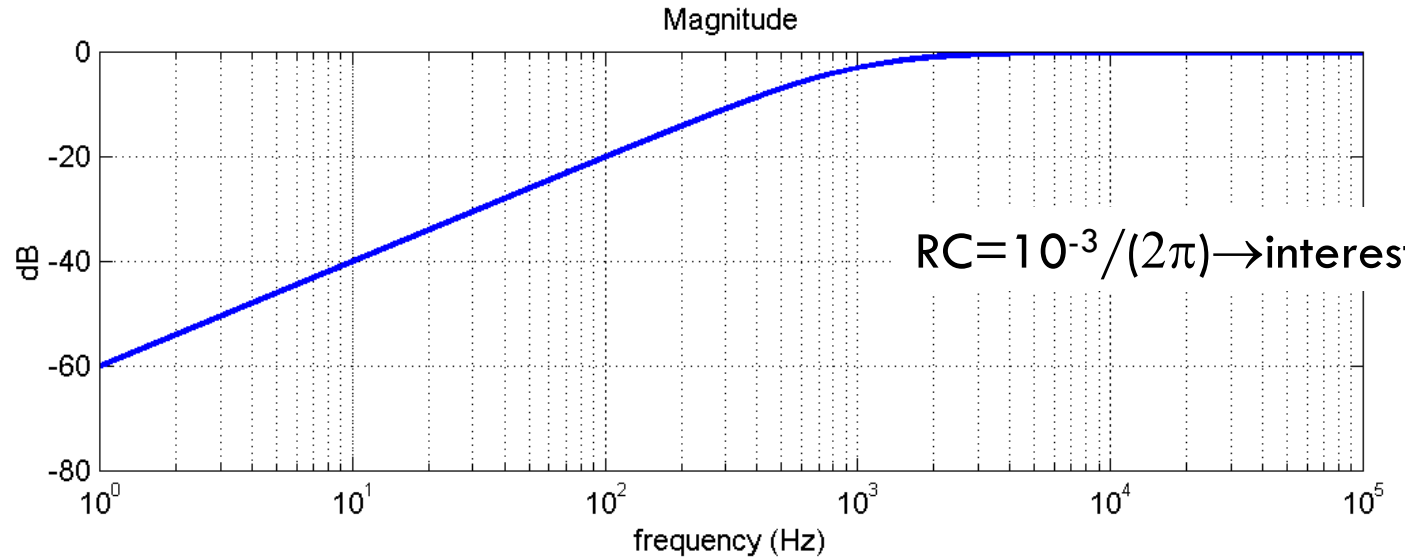
- Bode Plots are Plots (semilog plots) of
  - $|H(\omega)|$ in dB vs. $\log_{10}(\omega)$
  - $\angle H(\omega)$ vs. $\log_{10}(\omega)$
- We Can Do the Plots Very Accurately and Easily on Computers (Matlab) but There Are Also Simple Approximation Techniques Working from Poles and Zeros
- Examining the Approximate Construction Methods
  - Lets Us Check the Computer Results
  - Aids in Systems Intuition Development

Example: 1st Order High Pass Filter

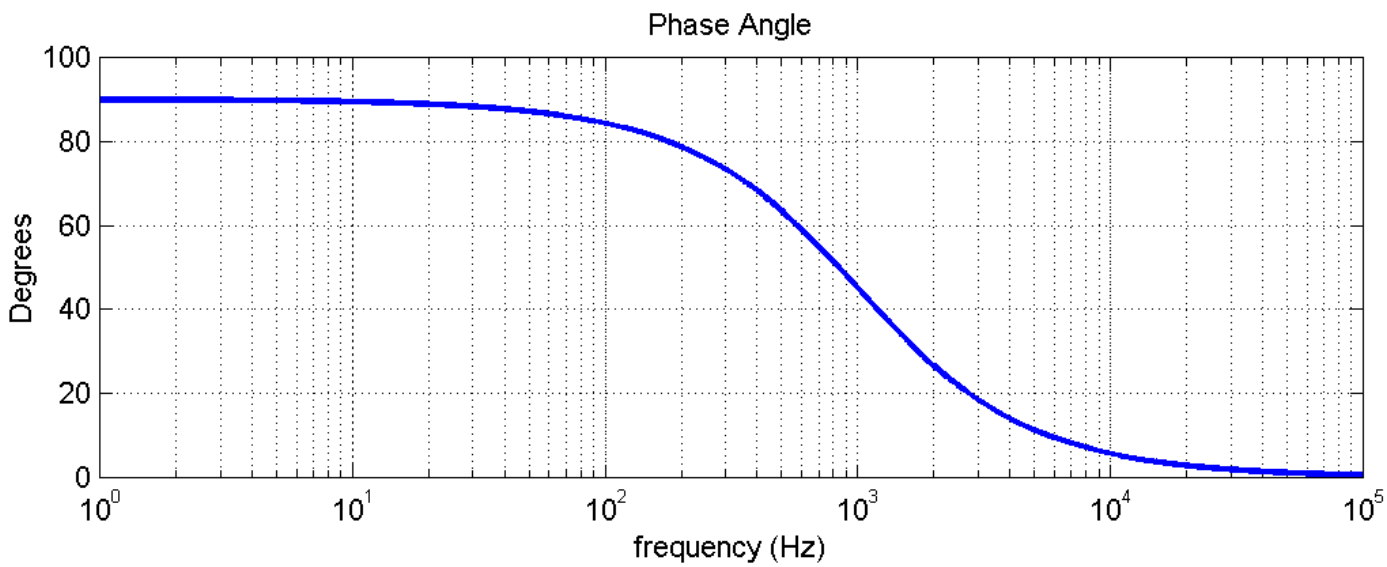
Find H(ω), Poles, Zeros



Example: 1st Order High Pass Filter



$RC=10^{-3}/(2\pi)\rightarrow$interesting $f\sim10^3$ Hz



Bode Plot Estimation Rules

- Look at Behavior of $H(\omega)$ around Poles and Zeros
- Relatively Easy to Derive Rule Set Results
- 1st Look at a Poles, Zeroes
- Construct Simple Rules Set
- Good Tool for What is *Possible* to Build!

Magnitude Response Around Pole

$$let\ H(\omega) = \frac{1}{1 + j\omega/\omega_p}$$

$$|H| = \left[\frac{1}{1+(\omega/\omega_p)^2}\right]^{1/2} = \left[1+(\omega/\omega_p)^2\right]^{-1/2}$$

$$20\log_{10}(|H|) = 20(-1/2)\log_{10}\left(1+(\omega/\omega_p)^2\right)$$

$$|H|_{dB} = -10\log_{10}\left(1+(\omega/\omega_p)^2\right)$$

$$\omega \ll \omega_p \rightarrow |H|_{dB} \approx 0\,dB$$

$$\omega = \omega_p \rightarrow |H|_{dB} \approx -3\,dB$$

$$\omega \gg \omega_p \rightarrow |H|_{dB} \approx -10\log_{10}\left((\omega/\omega_p)^2\right) = -20\log_{10}(\omega/\omega_p) = -20\log_{10}(\omega) + 20\log_{10}(\omega_p)$$

$$so\ for\ \omega \gg \omega_p$$

$$\frac{d|H|_{dB}}{d\log_{10}(\omega)} \approx -20\,dB/decade$$

Phase Response Around Pole

$$let\ H(\omega) = \frac{1}{1 + j\omega / \omega_p}$$

$$\angle H = -\tan^{-1}\left(\omega / \omega_p\right)$$

Magnitude Response Around Zero

$$let\ H(\omega) = 1 + j\omega/\omega_z$$

$$|H| = \left[1 + (\omega/\omega_z)^2\right]^{1/2}$$

$$20\log_{10}(|H|) = 20(+1/2)\log_{10}\left(1 + (\omega/\omega_z)^2\right)$$

$$|H|_{dB} = +10\log_{10}\left(1 + (\omega/\omega_z)^2\right)$$

$$\omega \ll \omega_z \rightarrow |H|_{dB} \approx 0\,dB$$

$$\omega = \omega_z \rightarrow |H|_{dB} \approx +3\,dB$$

$$\omega \gg \omega_z \rightarrow |H|_{dB} \approx +10\log_{10}\left((\omega/\omega_z)^2\right) = +20\log_{10}(\omega/\omega_z) = +20\log_{10}(\omega) - 20\log_{10}(\omega_z)$$

$$so\ for\ \omega \gg \omega_z$$

$$\frac{d|H|_{dB}}{d\log_{10}(\omega)} \approx +20\,dB/decade$$

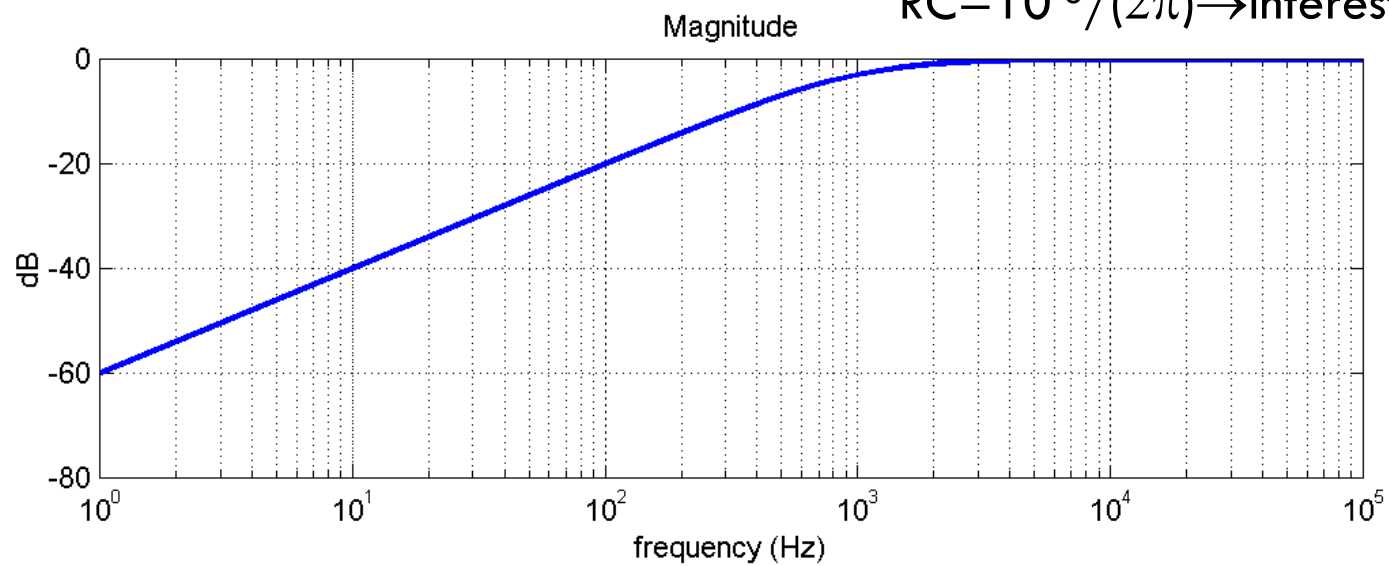
Phase Response Around Zero

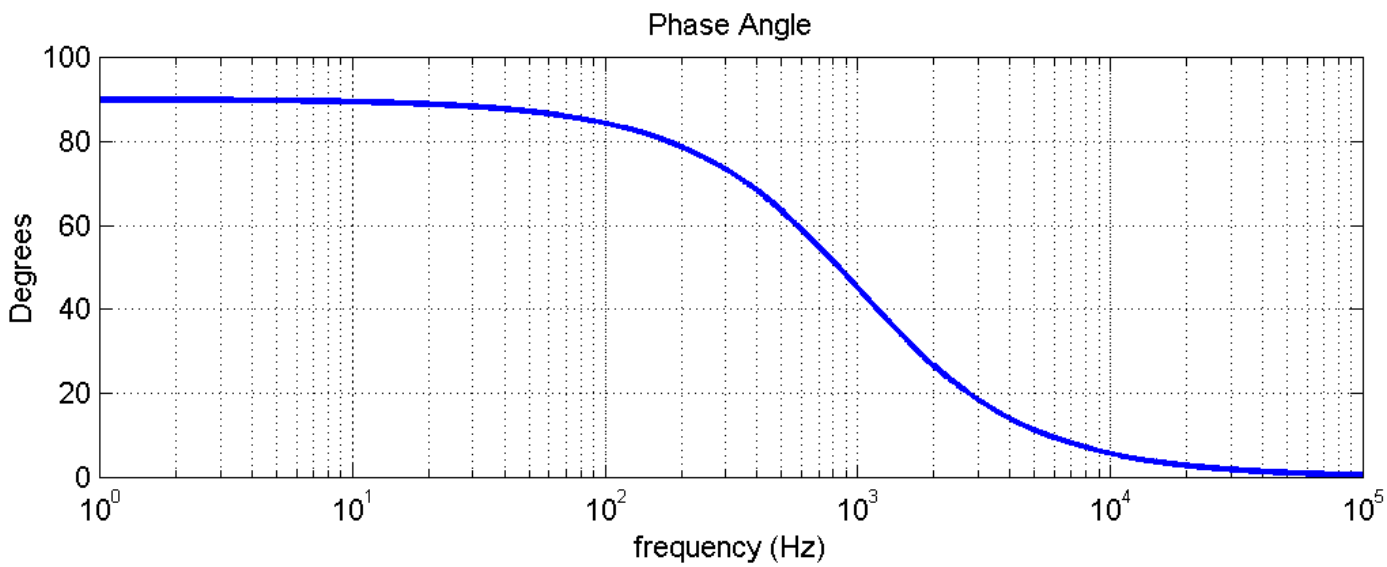
$$let\ H(\omega) = H(\omega) = 1 + j\omega / \omega_z$$

$$\angle H = +\tan^{-1}\left(\omega / \omega_p\right)$$

Example: 1st Order High Pass Filter

$RC=10^{-3}/(2\pi) \rightarrow$ interesting $f \sim 10^3$ Hz



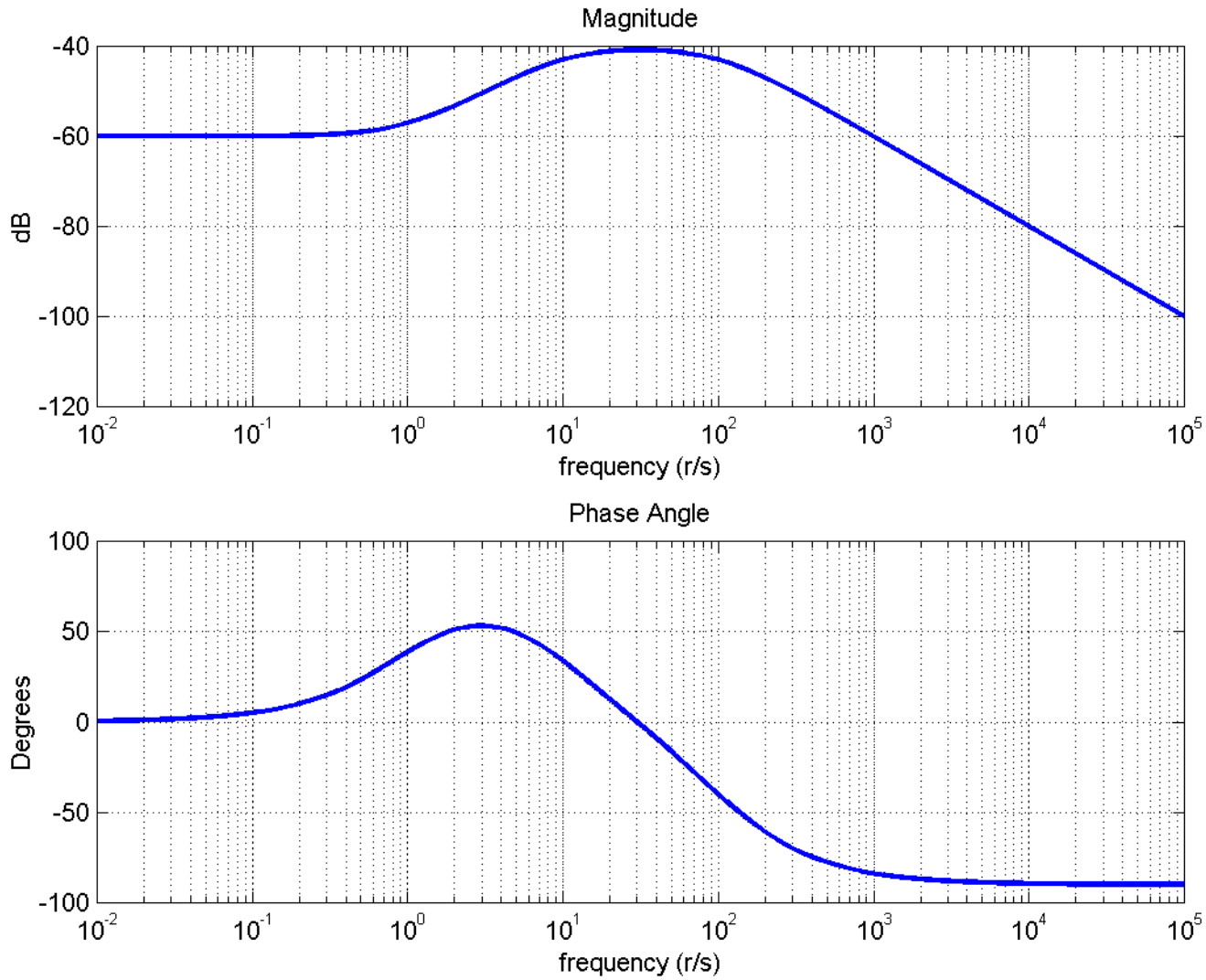


Simple Poles & Zeros

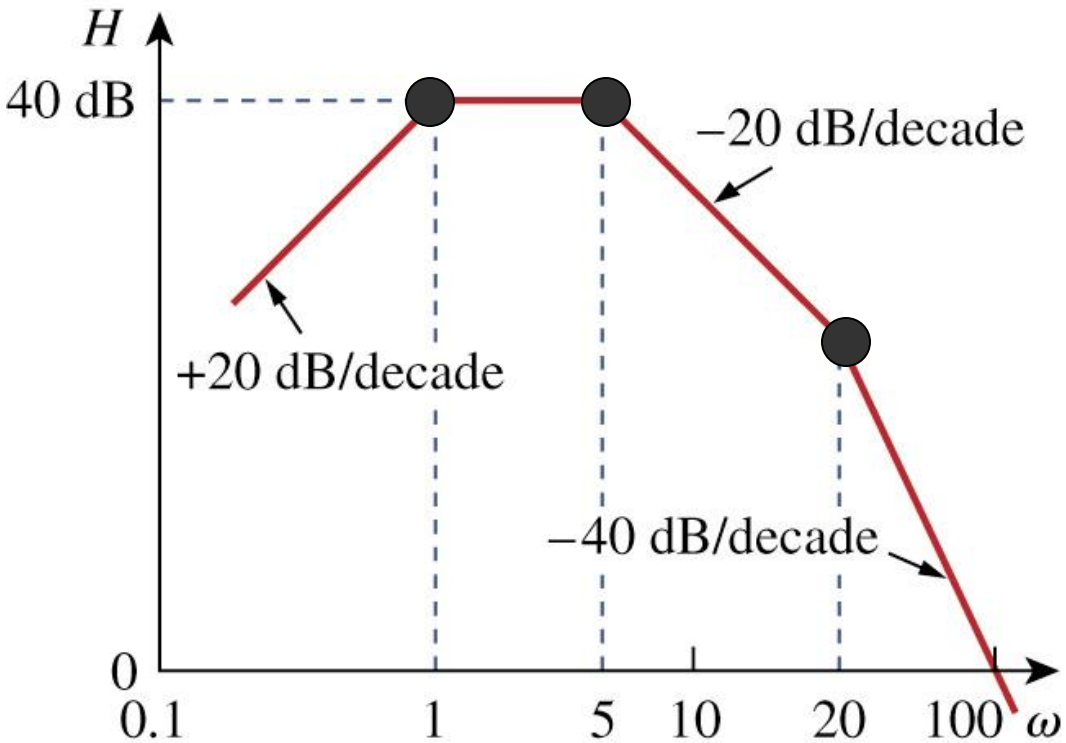
- Every Pole
  - Changes the Slope of H_{dB} by –20 dB/dec Starting at the Magnitude of the Pole Frequency ω_p
  - Total Phase Shift of –90°
  - Changes the Slope of $\angle H$ by -45°/dec ~ between $0.1\omega_p$ and $10\omega_p$
- Every Zero
  - Changes the Slope of H_{dB} by +20 dB/dec Starting at the Magnitude of the Zero Frequency ω_Z
  - Total Phase Shift of +90°
  - Changes the Slope of $\angle H$ by +45°/dec ~ between $0.1\ \omega_x$ and $10\ \omega_x$

Example

$$H(s) = \frac{(s+1)}{(s+10)(s+100)}$$



Interpreting Bode Plots



Zeros?

s=0

Poles?

s=1

s=5

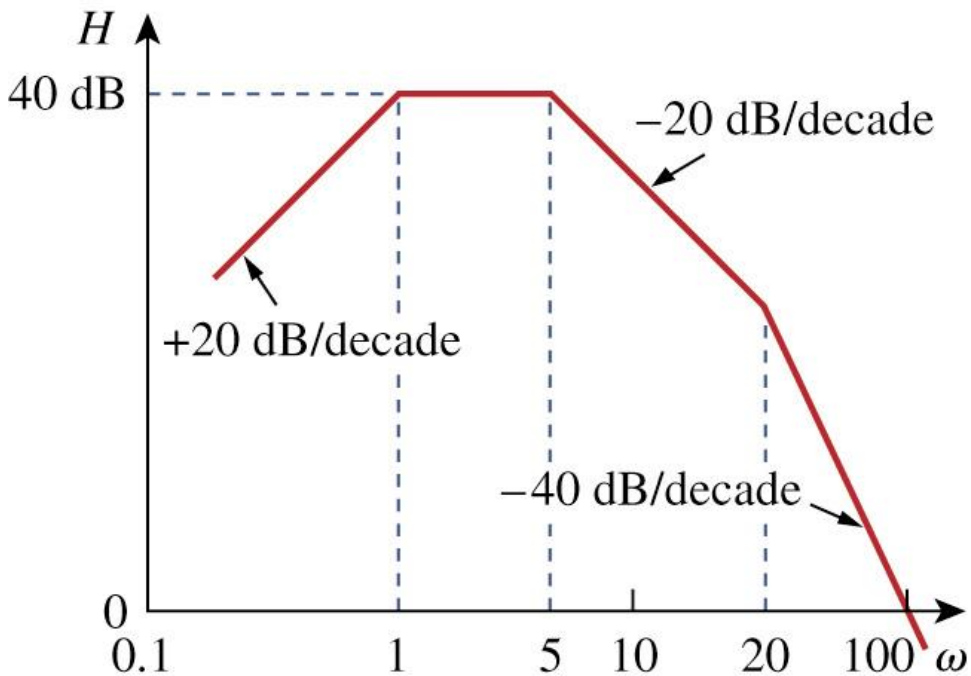
s=20

Gain?

40 dB = 100

$$H(s) = \frac{100s}{\left(1+\frac{s}{1}\right)\left(1+\frac{s}{5}\right)\left(1+\frac{s}{20}\right)} = 100\frac{(1)(5)(20)s}{(s+1)(s+5)(s+20)} = \frac{10^4 s}{(s+1)(s+5)(s+20)}$$

How Did I Determine Gain=100 (40dB)?



Evaluate the function!

$$H(s) = \frac{(Gain)(s)}{\left(1+\frac{s}{1}\right)\left(1+\frac{s}{5}\right)\left(1+\frac{s}{20}\right)}$$

Choose a frequency, s=1

$$H(1) = 40dB = \frac{(Gain)(1)}{\left(1+\frac{1}{1}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{20}\right)}$$

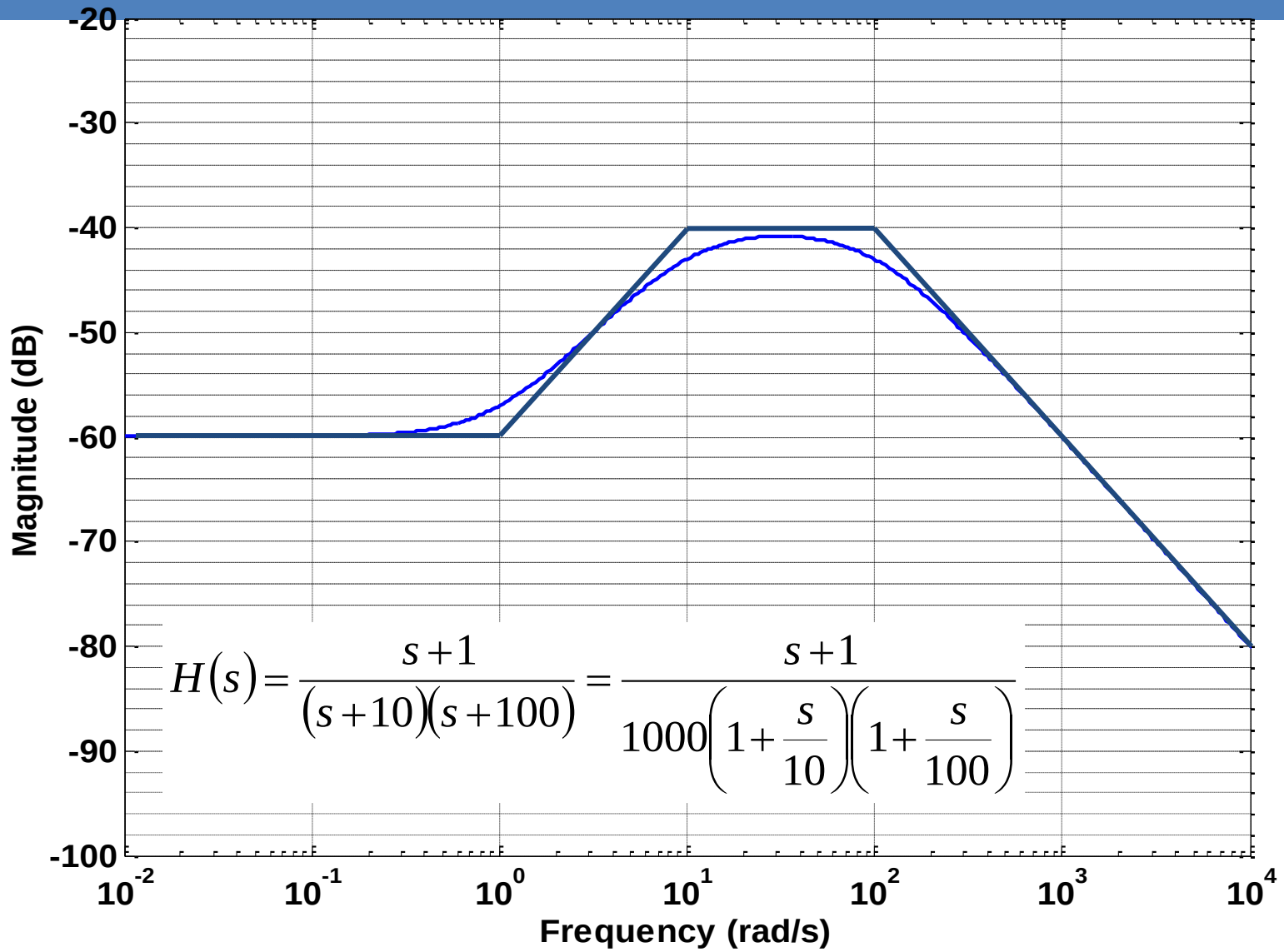
This plot used linear approximation: magnitude =1 until ω>z,p

$$H(1) = 40dB = 100 = \frac{(Gain)(1)}{(1)(1)(1)}$$

$$Gain = 100 = 40dB$$

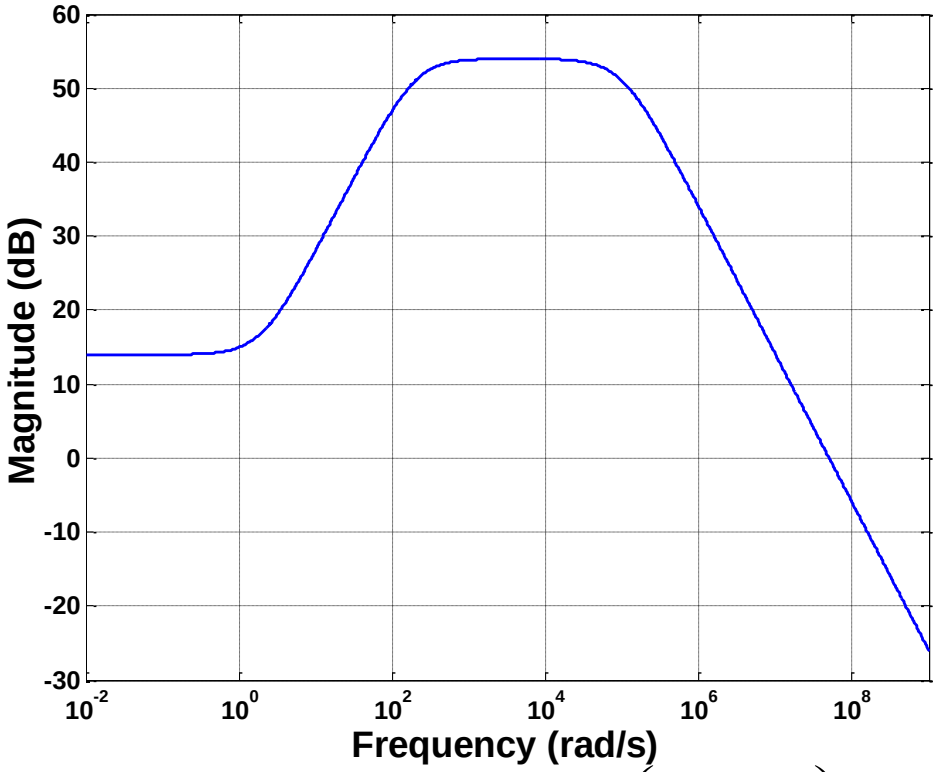
Use same approach to interpret phase!

Bode Plot using linear approximation



$$H(s) = \frac{s+1}{(s+10)(s+100)} = \frac{s+1}{1000\left(1+\frac{s}{10}\right)\left(1+\frac{s}{100}\right)}$$

Try again for "real" Bode plot



Given

Zeros	Poles
s=2	s=200
	s=10^5

(you could not necessarily read these off of plot)

$$H(s) = \frac{(Gain)\left(1+\frac{s}{2}\right)}{\left(1+\frac{s}{200}\right)\left(1+\frac{s}{10^5}\right)}$$

$$H(10^4) = 497 = \frac{(Gain)\left(1+\frac{10^4}{2}\right)}{\left(1+\frac{10^4}{200}\right)\left(1+\frac{10^4}{10^5}\right)}$$

$$Gain = 557.5 = 14.9dB$$

Second Order Functions: Quadratic Poles & Zeros

Terms in H(s) of the form: $$s^2 + 2\alpha s + \omega_0^2$$

3 Cases of Interest:

1. "Overdamped" Roots s_1,s_2 negative Reals
2. "Critically Damped" Roots s_1=s_2 negative Reals
3. "Underdamped" Roots s_1,s_2 Complex

Quad Pole/Zero: Real Roots

- Factor into Product of 1st Order Terms & Treat Like 2 Separate Poles/Zeros
- Example:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\alpha s + \omega_n^2}$$

$$s_1 = p_1 = -\alpha + \left[\alpha^2 - \omega_n^2\right]^{1/2} \qquad s_2 = p_2 = -\alpha - \left[\alpha^2 - \omega_n^2\right]^{1/2}$$

$$H(s) = \frac{\omega_n^2}{(s - p_1)(s - p_2)} \rightarrow H(\omega) = \frac{\omega_n^2}{(j\omega - p_1)(j\omega - p_2)}$$

$$if\ \ p_1 = p_2\ \ this\ \ simply\, becomes:$$

$$H(s) = \frac{\omega_n^2}{(s - p_1)(s - p_1)} = \frac{\omega_n^2}{(s - p_1)^2} \rightarrow H(\omega) = \frac{\omega_n^2}{(j\omega - p_1)^2}$$

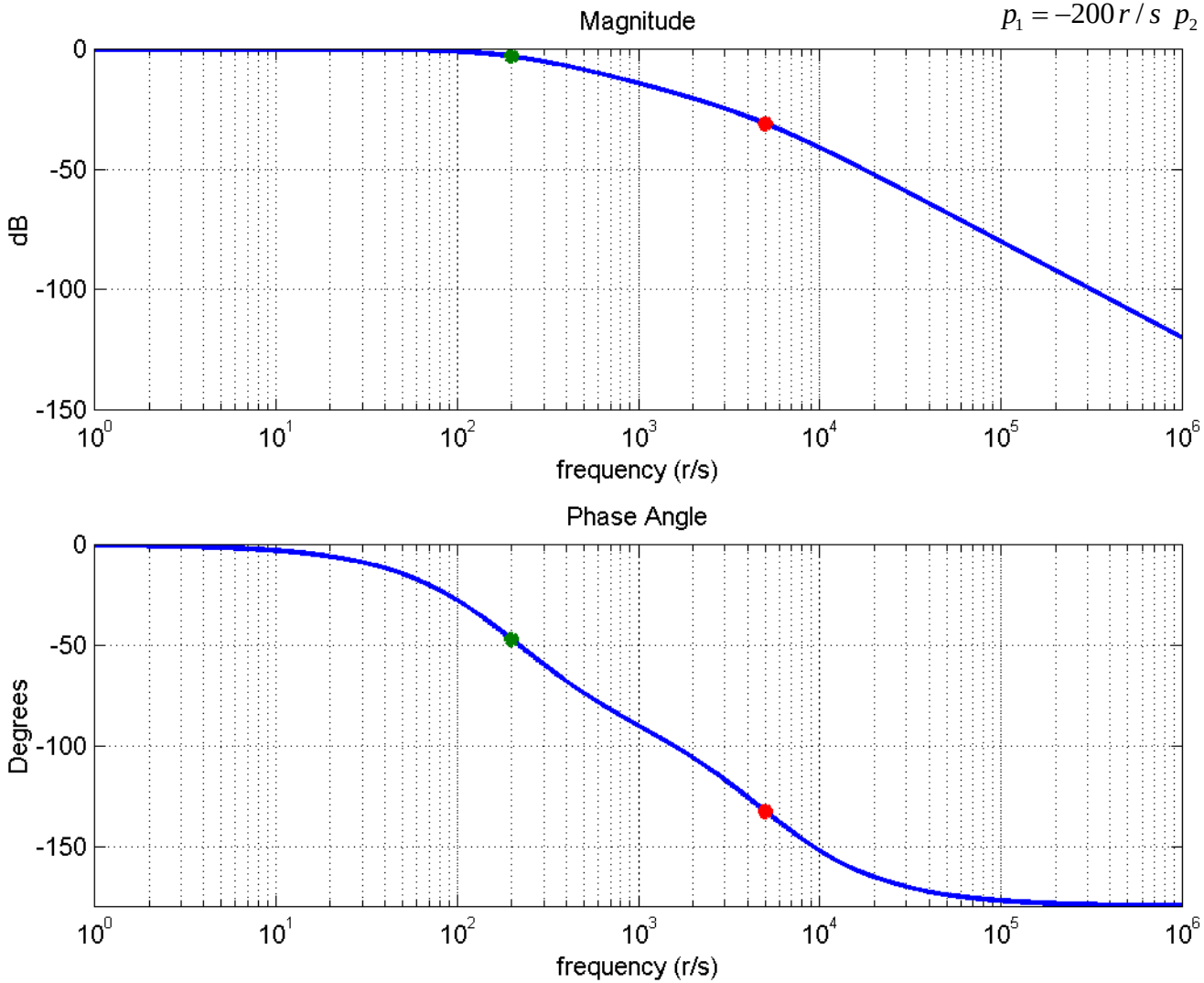
Example

$$H(s) = \frac{\omega_n^2}{s^2 + 2\alpha s + \omega_n^2} = \frac{1}{(s/\omega_n)^2 + (2\alpha/\omega_n)s + 1}$$

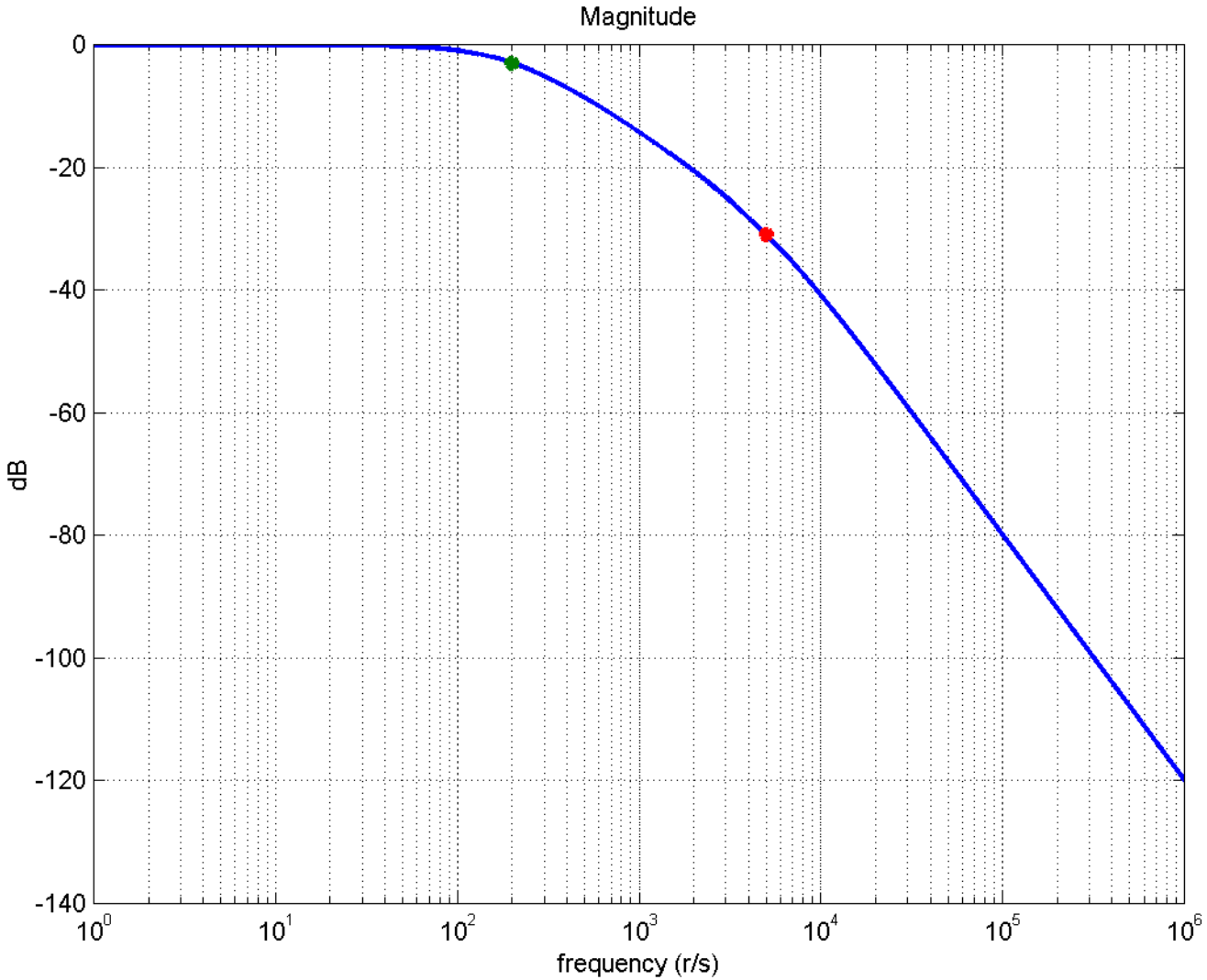
$$\omega_n = 1000\, r/s$$

$$\alpha = 2600\, r/s$$

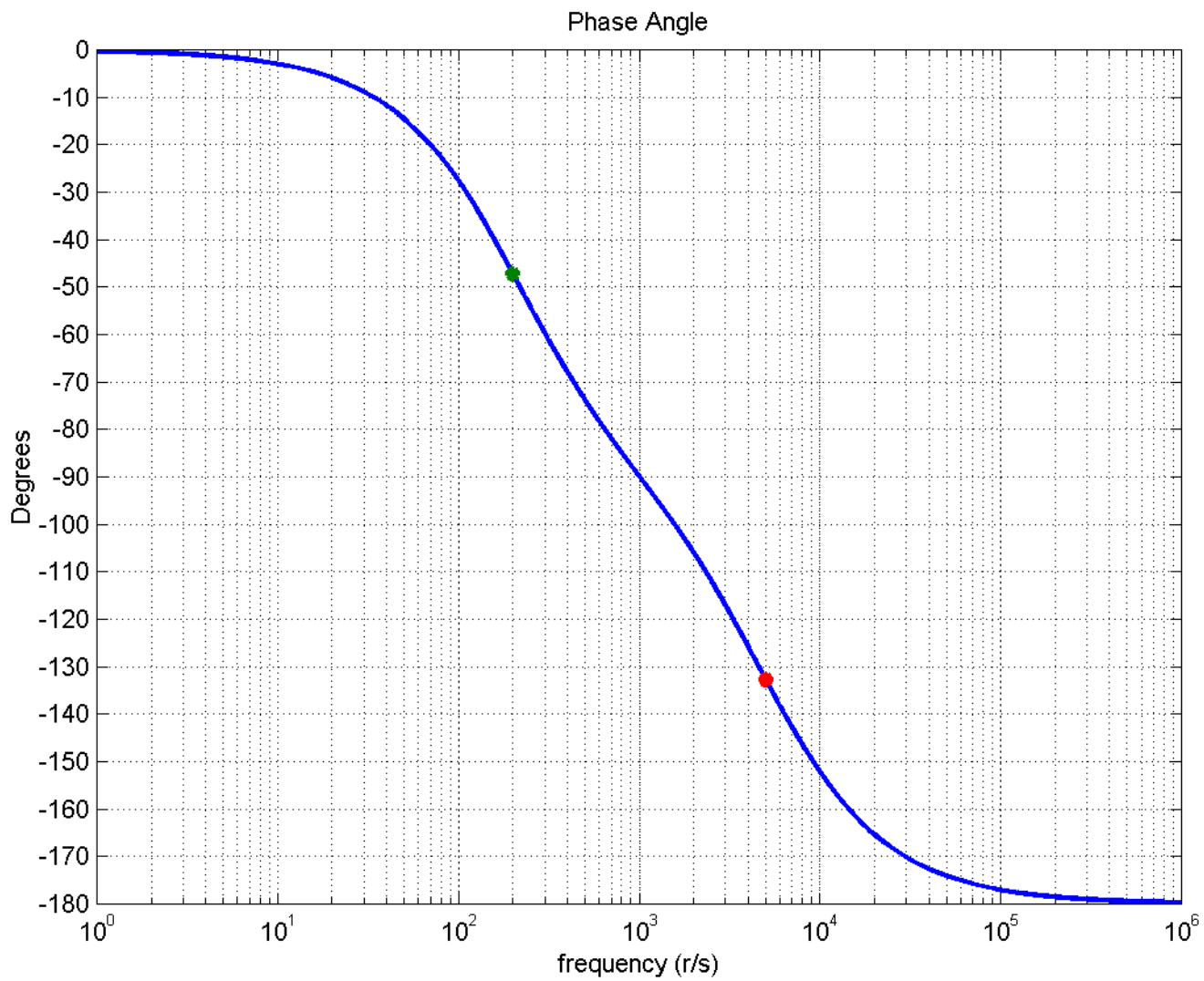
$$p_1 = -200\, r/s \quad p_2 = -5000\, r/s$$



Magnitude Construction



Phase Construction



Quad Pole/Zero: Complex Roots

- All the "Action" is Around ω_n
- Poles: -40 dB/dec, -180° Phase Shift
- Zeros: +40 dB/dec, +180° Phase Shift
- Sharpness of Phase Change is Very Sensitive to $\zeta_2=\alpha/\omega_n$
- Example:

$$H(s)=\frac{\omega_n^2}{s^2+2\alpha s+\omega_n^2}=\frac{1}{\left(s/\omega_n\right)^2+\left(2\zeta_2/\omega_n\right)s+1}$$

$$s_1=p_1=-\alpha+j\omega_d \quad s_2=p_2=-\alpha-j\omega_d$$

$$\omega_d=\left[\omega_n^2-\alpha^2\right]^{1/2}$$

$$\zeta_2\equiv\alpha/\omega_n$$

$$H(\omega)=\frac{\omega_n^2}{\left(\omega_n^2-\omega^2\right)+2j\omega\alpha}=\frac{1}{\left(1-\left(\omega/\omega_n\right)^2\right)+2j\zeta_2\left(\omega/\omega_n\right)}$$

Complex Quadratic Pole

$$H(\omega)=\frac{\omega_n^2}{\left(\omega_n^2-\omega^2\right)+2j\omega\alpha}=\frac{1}{\left(1-\left(\omega/\omega_n\right)^2\right)+2j\zeta_2\left(\omega/\omega_n\right)}$$

$$\log_{10}\left(\left|H\right|\right)=\log_{10}\left\{\left[\left(1-\left(\omega/\omega_n\right)^2\right)^2+4\zeta_2{}^2\left(\omega/\omega_n\right)^2\right]^{-1/2}\right\}$$

$$20\log_{10}\left(\left|H\right|\right)=-10\log_{10}\left[\left(1-\left(\omega/\omega_n\right)^2\right)^2+4\zeta_2{}^2\left(\omega/\omega_n\right)^2\right]$$

For $\omega\gg\omega_n\ \&\ \zeta_2<1$:

$$20\log_{10}\left(\left|H\right|\right)\approx-10\log_{10}\left[\left(\omega/\omega_n\right)^4\right]=-40\log_{10}\left[\left(\omega/\omega_n\right)\right]$$ **40 dB/dec.**

$$\measuredangle H=-\tan^{-1}\left[\frac{2\omega\alpha}{\omega_n^2-\omega^2}\right]=-\tan^{-1}\left[\frac{2\zeta_2\left(\omega/\omega_n\right)}{1-\left(\omega/\omega_n\right)^2}\right]$$

$$\measuredangle H=0\ if\ \omega=0$$

$$\measuredangle H=-90^\circ\ if\ \omega=\omega_n$$

$$\measuredangle H=-180^\circ\ if\ \omega\to\infty$$

Complex Quadratic Pole: Angle

$$\measuredangle H = -\tan^{-1}\left[\frac{2\omega\alpha}{\omega_n^2-\omega^2}\right] = \tan^{-1}\left[-\frac{2\omega\alpha}{\omega_n^2-\omega^2}\right]$$

$$\measuredangle H = 0 \; if \; \omega = 0 \quad \measuredangle H = -90° \; if \; \omega = \omega_n \quad \measuredangle H = -180° \; if \; \omega \to \infty$$

Look at approximate endpoints:

$$\measuredangle H = -45°, \quad \omega < \omega_n \qquad \tan\left(\measuredangle H\right) = -1 = \frac{-2\omega\alpha}{\omega_n^2-\omega^2}$$

$$\omega_n^2-\omega^2 = +2\omega\alpha \to \left[\omega^2+2\alpha\omega-\omega_n^2=0\right] \to \omega_{lower} = -\alpha \pm \left[\alpha^2+\omega_n^2\right]^{1/2}$$

$\omega_{lower} = -\alpha + \left[\alpha^2+\omega_n^2\right]^{1/2}$ **only positive root considered**

$$\measuredangle H \to -135°, \quad \omega > \omega_n \qquad \tan\left(\measuredangle H\right) = +1 = \frac{-2\omega\alpha}{\omega_n^2-\omega^2}$$

$$\omega_n^2-\omega^2 = -\alpha \to \left[\omega^2-2\alpha\omega-\omega_n^2=0\right] \to \omega_{upper} = \alpha \pm \left[\alpha^2+\omega_n^2\right]^{1/2}$$

$\omega_{upper} = \alpha + \left[\alpha^2+\omega_n^2\right]^{1/2}$ **only positive root considered**

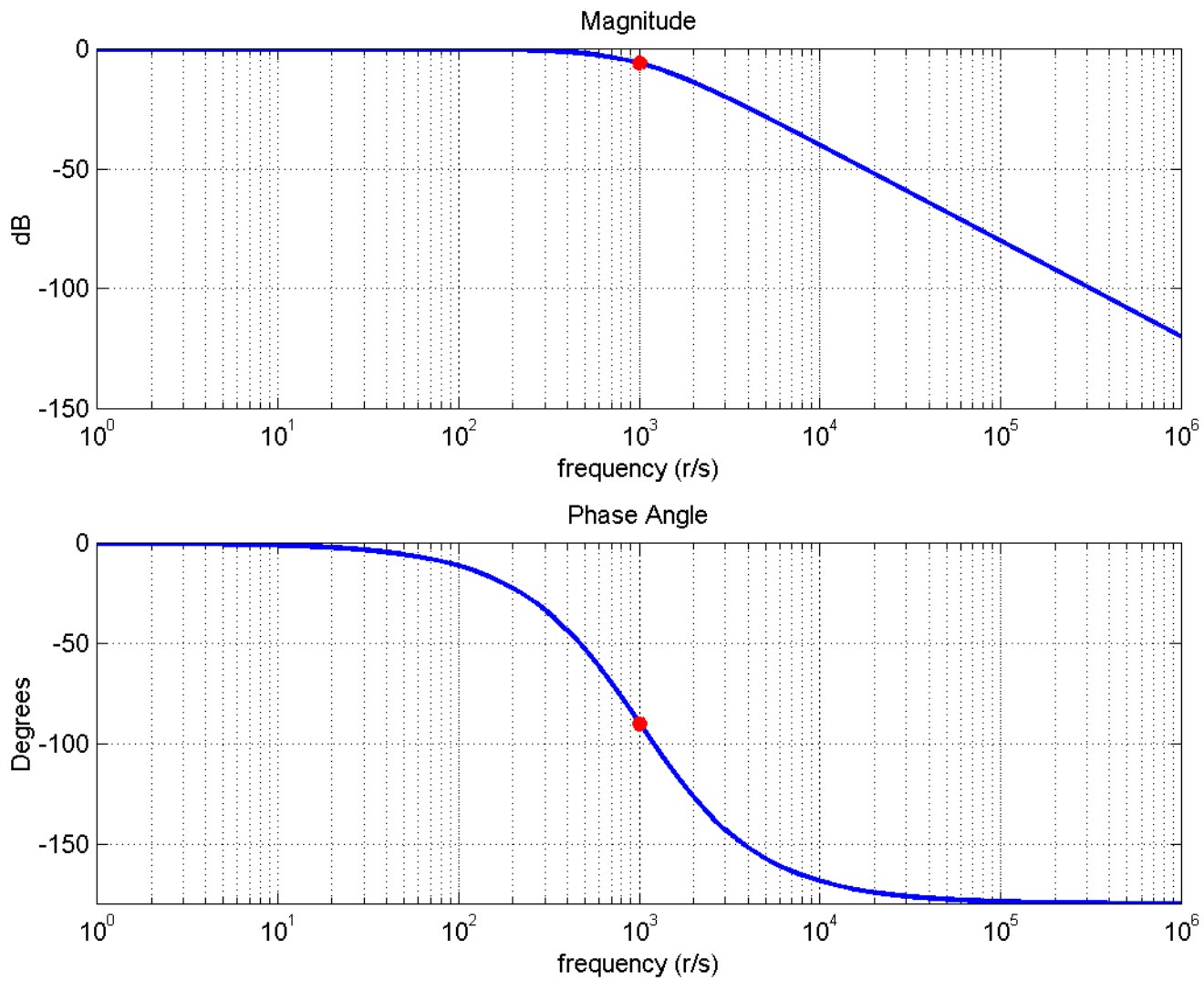
So the -90° transition occurs over $\Delta\omega = \omega_{upper} - \omega_{lower} = 2\alpha$

Slope in degrees/decade $\approx$

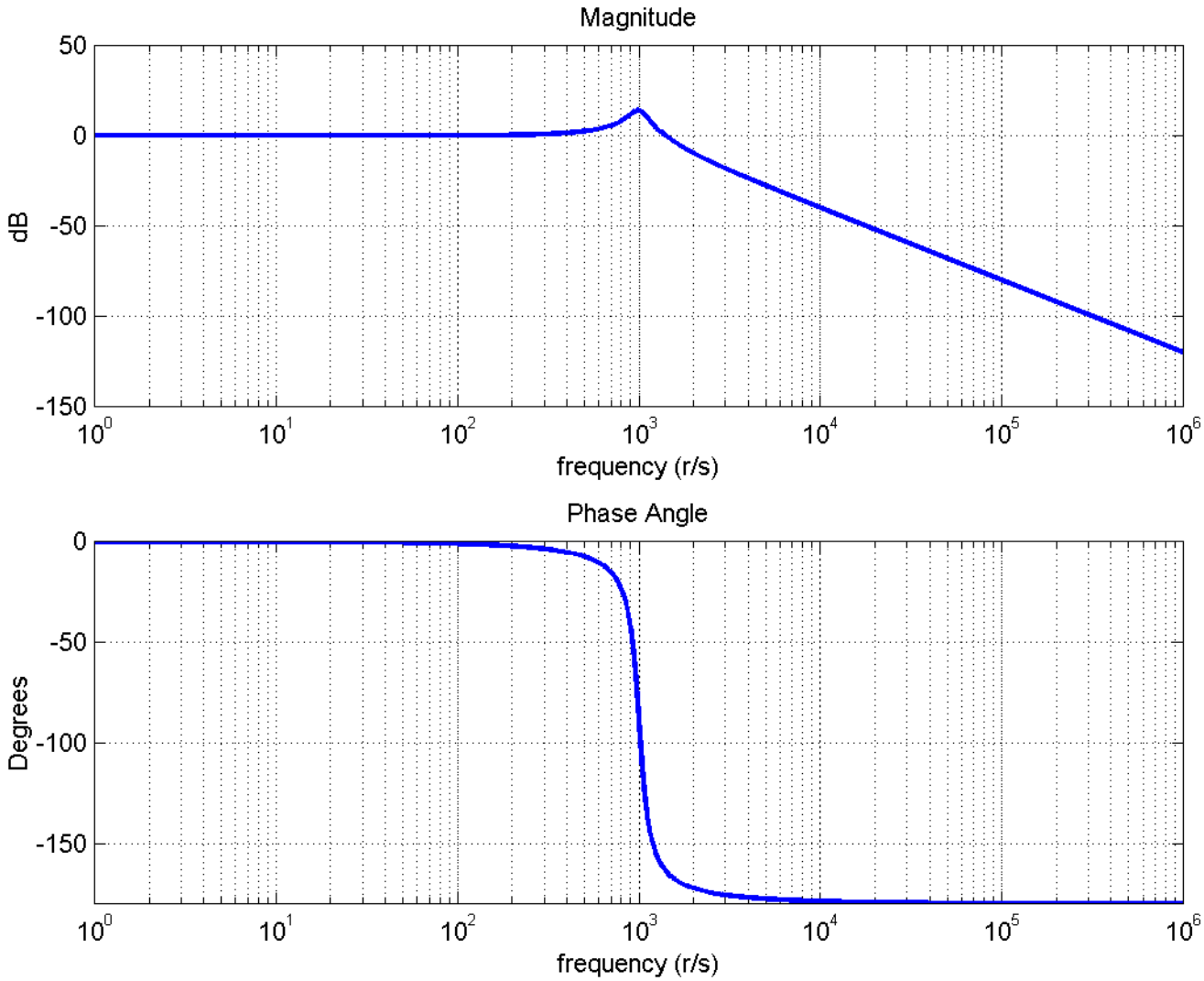
$$\left[\frac{-90°}{\log_{10}\left(\omega_{upper}\right)-\log_{10}\left(\omega_{lower}\right)}\right] = \left[\frac{-90°}{\log_{10}\left(\dfrac{\omega_{upper}}{\omega_{lower}}\right)}\right] = \left[\frac{-90°}{\log_{10}\left(\dfrac{\alpha+\left[\alpha^2+\omega_n^2\right]^{1/2}}{-\alpha+\left[\alpha^2+\omega_n^2\right]^{1/2}}\right)}\right] = \left[\frac{-90°}{\log_{10}\left(\dfrac{\zeta_2+\left[\zeta_2^{\,2}+1\right]^{1/2}}{-\zeta_2+\left[\zeta_2^{\,2}+1\right]^{1/2}}\right)}\right]$$

Remember the full transition is -180°

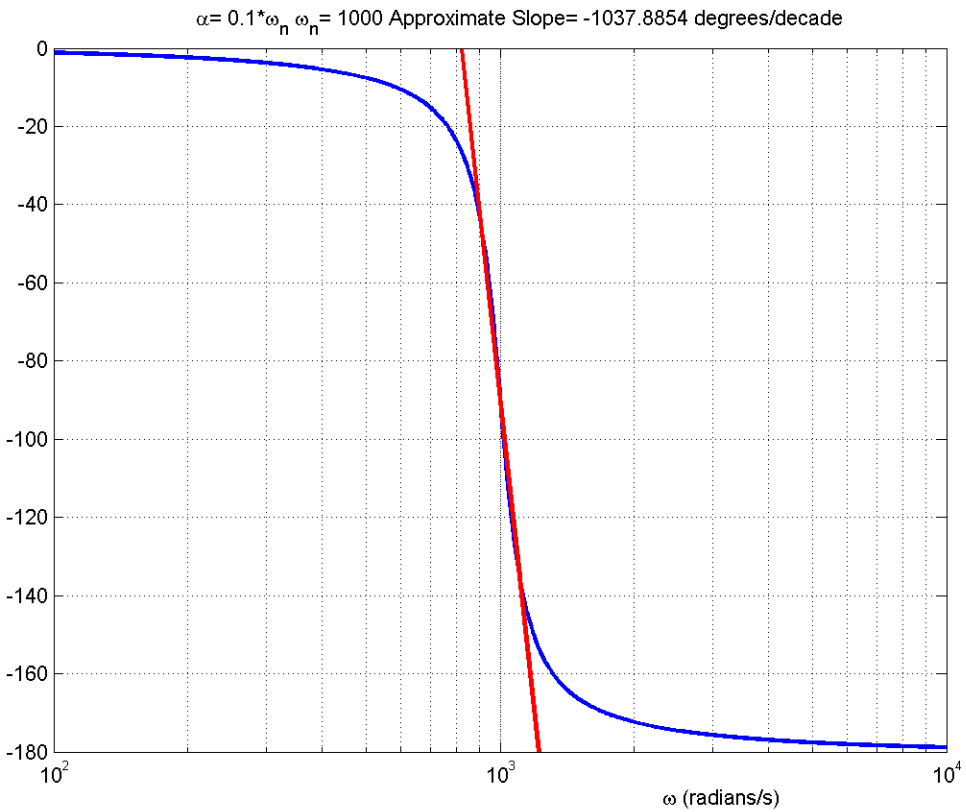
Pole $\alpha=\omega_n$ & $\omega_n=1000$ r/s



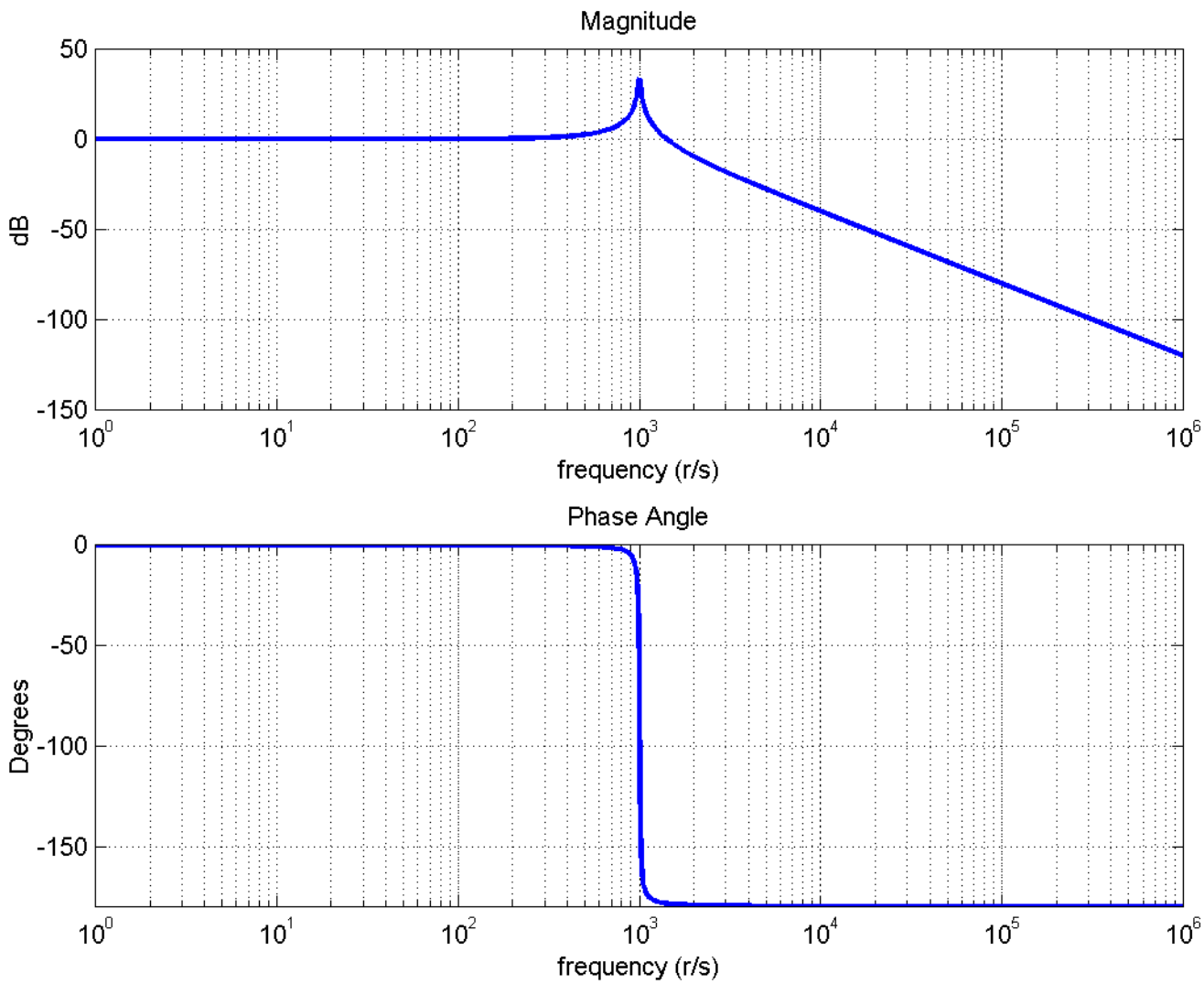
Pole $\alpha=0.1\omega_n$ & $\omega_n=1000$ r/s



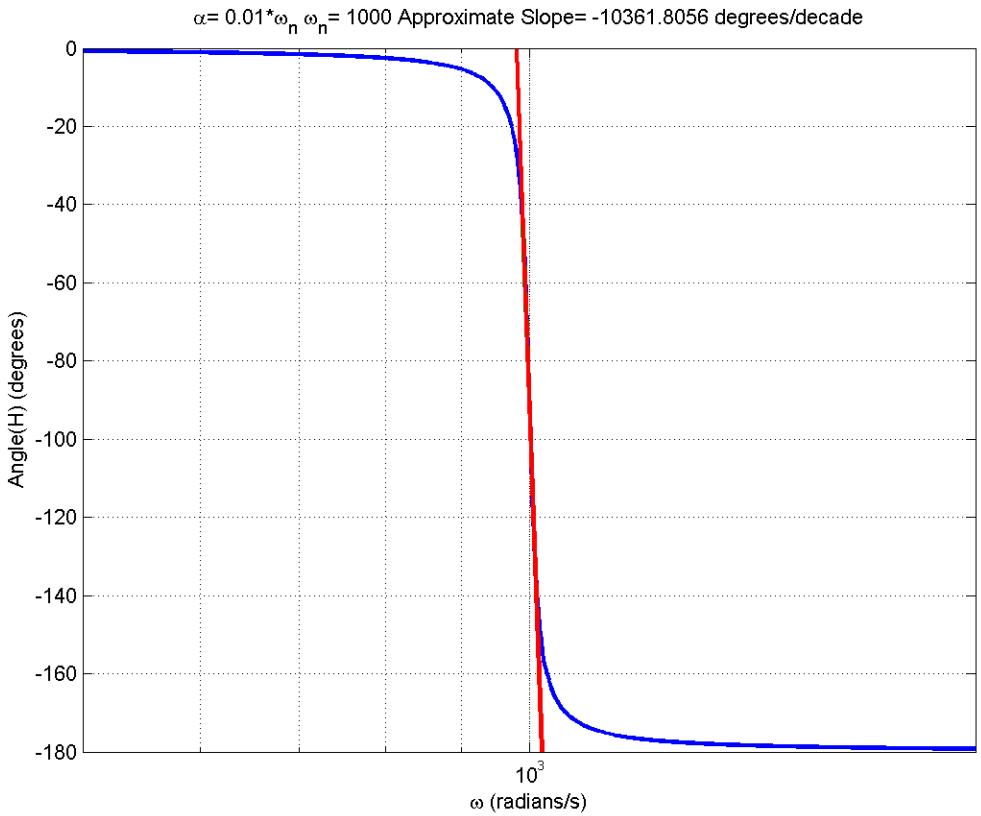
Pole $\alpha=0.1\omega_n$ & $\omega_n=1000$ r/s



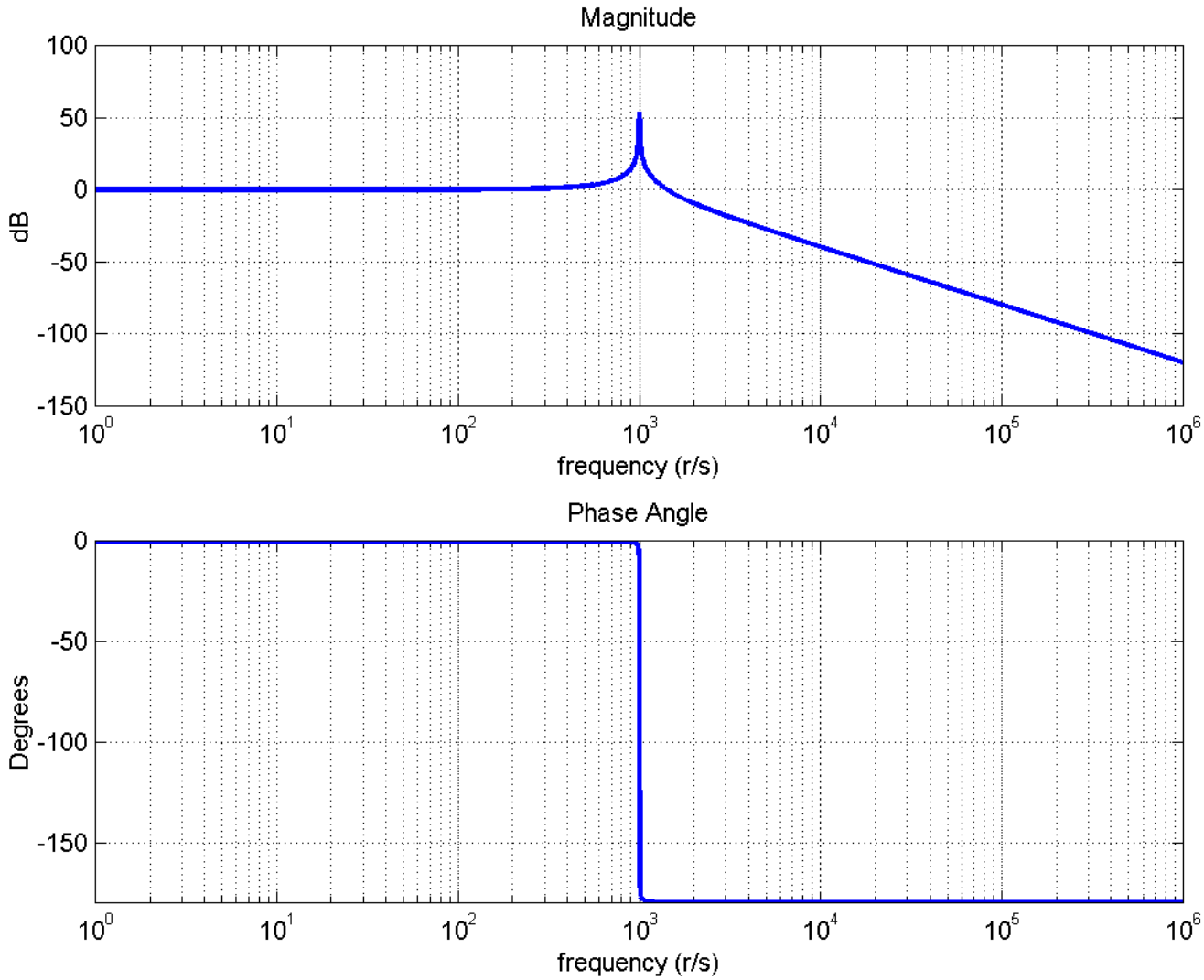
Pole $\alpha=0.01\omega_n$ & $\omega_n=1000$ r/s



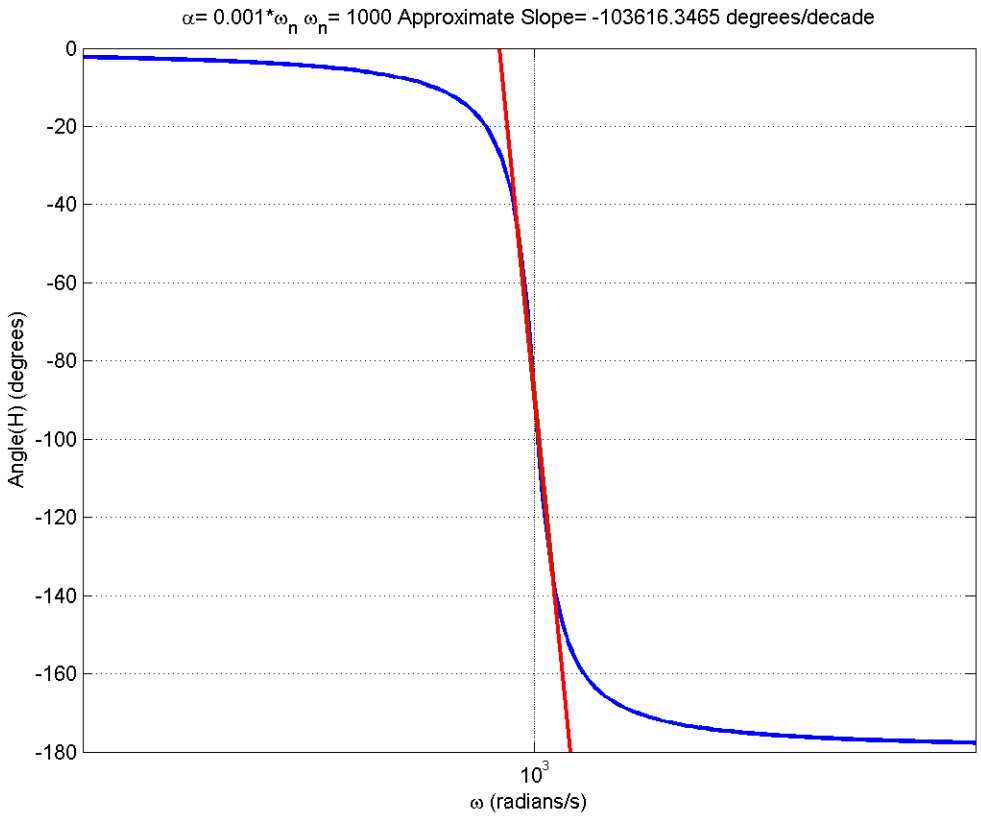
Pole $\alpha=0.01\omega_n$ & $\omega_n=1000$ r/s



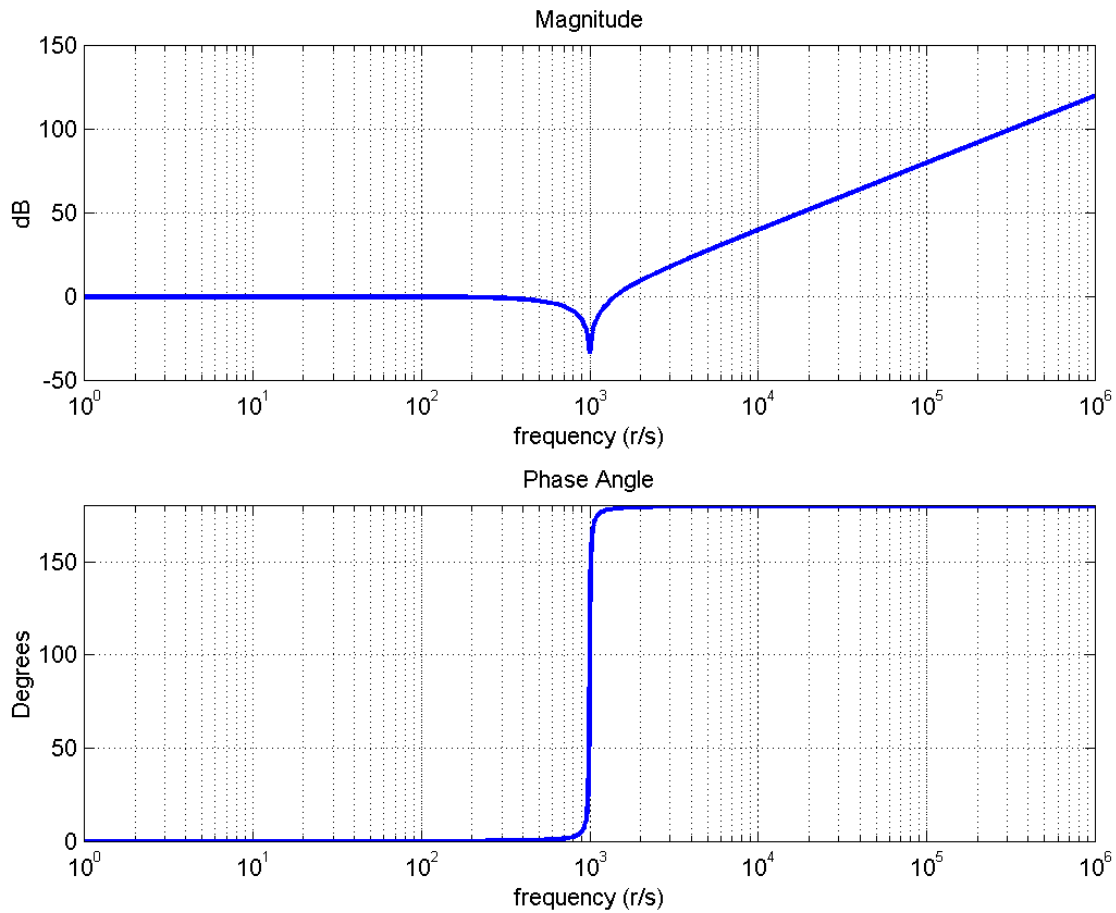
Pole α=0.001ω_n & ω_n=1000 r/s



Pole α=0.001ω_n & ω_n=1000 r/s



Zero α=0.01ω_n & ω_n=1000 r/s



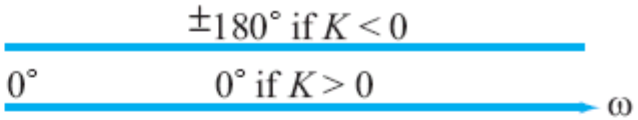
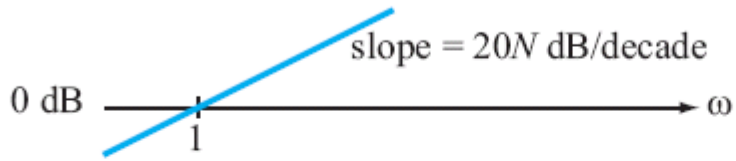
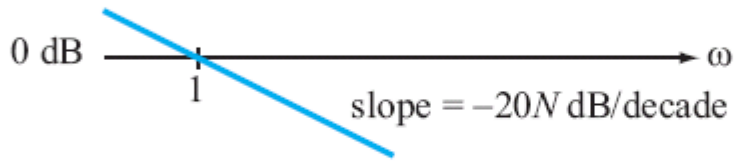
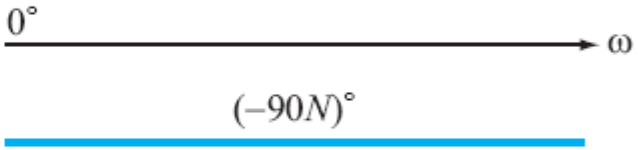
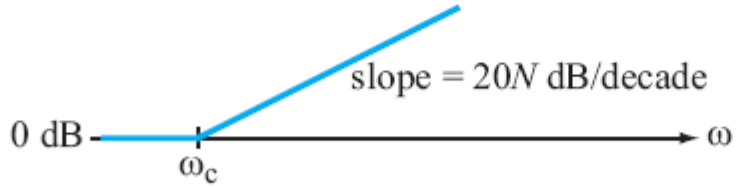
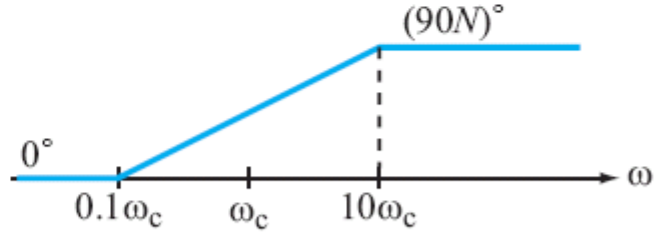
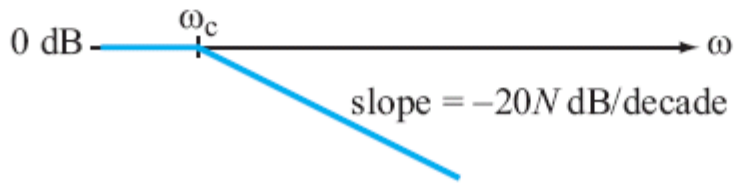
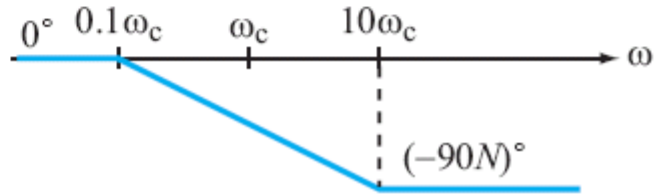
$$H(s) = \frac{s^2 + 2\alpha s + \omega_n^2}{\omega_n^2}$$

$$= \left(s/\omega_n\right)^2 + \left(2\zeta_2/\omega_n\right)s + 1$$

$$H(\omega) = \frac{\left(\omega_n^2 - \omega^2\right) + 2j\omega\alpha}{\omega_n^2}$$

$$= \left(1 - \left(\omega/\omega_n\right)^2\right) + 2j\zeta_2\left(\omega/\omega_n\right)$$

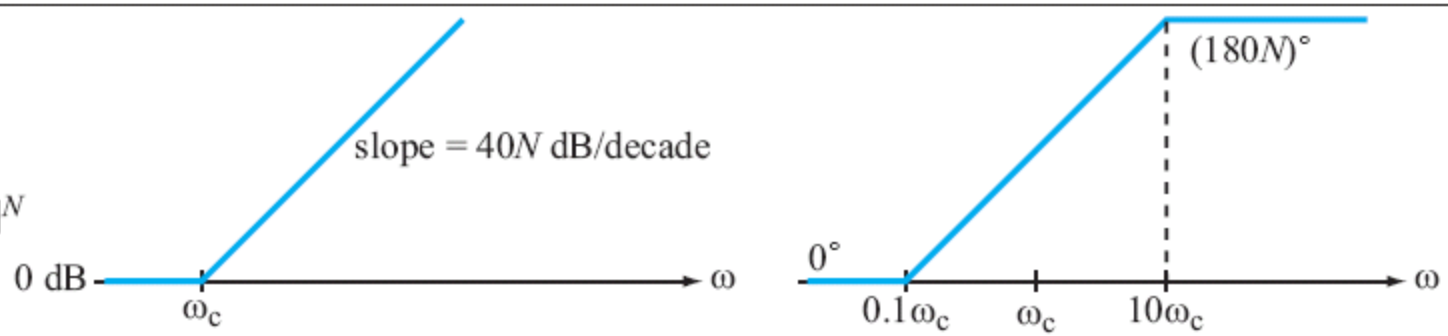
Bode Plot Linear Approximation

Factor	Bode Magnitude	Bode Phase
Constant K		
Zero @ Origin $(j\omega)^N$		
Pole @ Origin $(j\omega)^{-N}$		
Simple Zero $(1 + j\omega/\omega_c)^N$		
Simple Pole $\left(\frac{1}{1 + j\omega/\omega_c}\right)^N$		

Bode Plot Linear Approximation

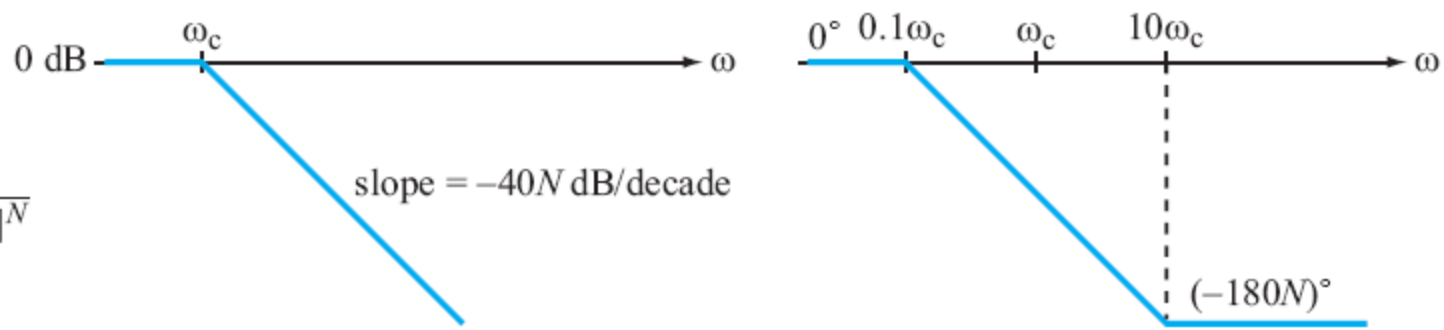
Quadratic Zero

$[1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2]^N$



Quadratic Pole

$$\frac{1}{[1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2]^N}$$



Quadratic Poles

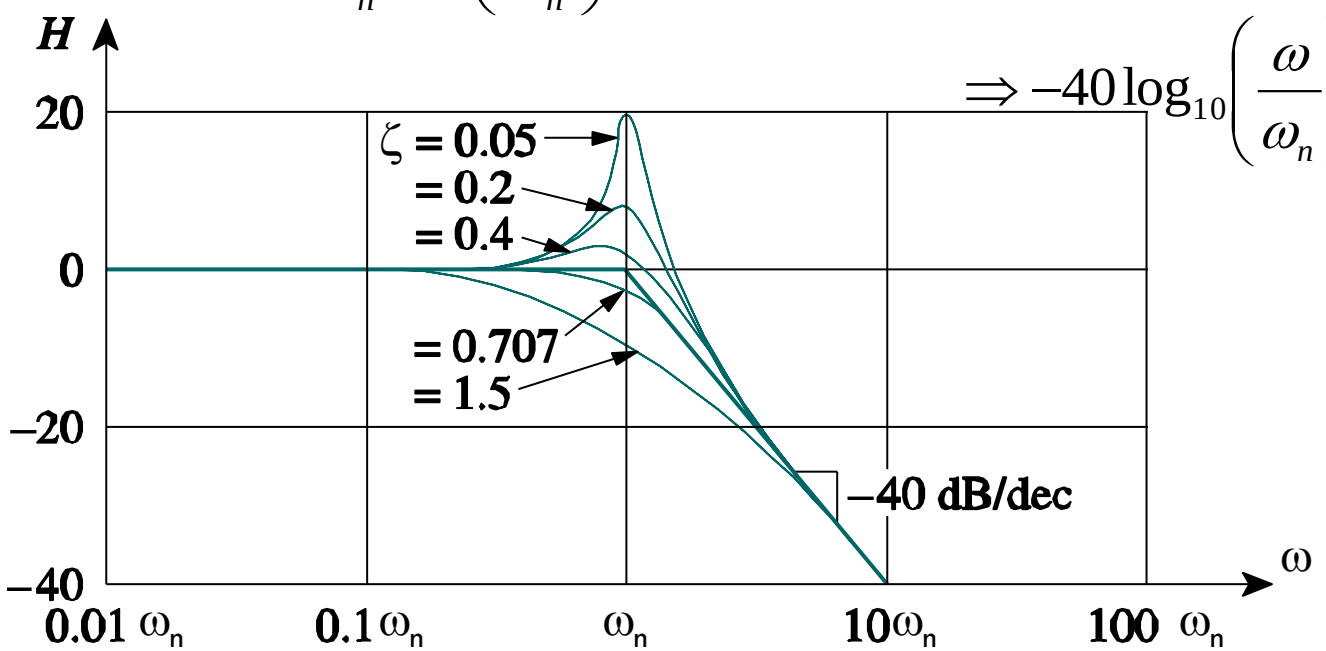
H(s) term in the form of $s^2 + 2\alpha s + \omega_o^2$

$$H(\omega) = \frac{1}{1 + \frac{j2\xi\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2}$$

$$H_{dB} = -20\log_{10}\left|1 + \frac{j2\xi\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right|$$

$$\Rightarrow 0 \text{ as } \omega \to 0$$

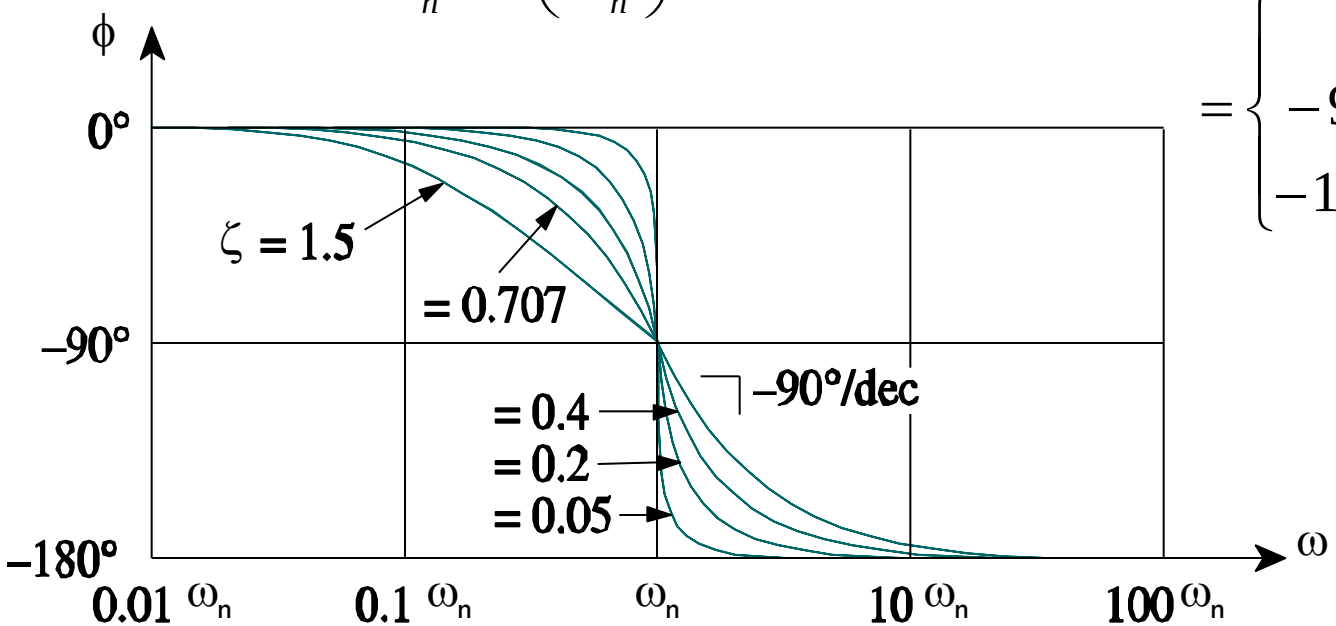
$$\Rightarrow -40\log_{10}\left(\frac{\omega}{\omega_n}\right) 0 \text{ as } \omega \to \infty$$



Quadratic Poles

H(s) term in the form of $s^2 + 2\alpha s + \omega_o^2$

$$H(\omega) = \frac{1}{1 + \frac{j2\xi\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2}$$

$$\phi = -\tan^{-1}\left(\frac{2\xi\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$$

$$= \begin{cases} 0, & \omega = 0 \\ -90^0, & \omega = \omega_n \\ -180^0, & \omega \to \infty \end{cases}$$