ANALYSIS OF SOC AND HIGHER ORDER CIRCUITS USING LAPLACE TRANSFORMS PART 2 : PARTIAL FRACTION EXPANSION, MORE EXAMPLES

Corrected Examples

Credit to F. Ulaby

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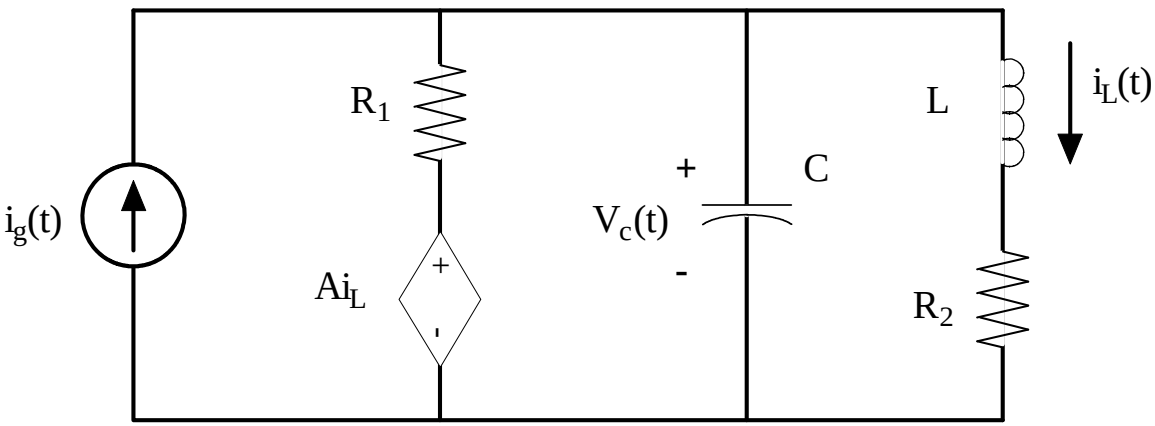
Mapping Back To Differential Equation



Time Domain Result:

$$\frac{d^2 v_c}{dt^2} + \left(\frac{1}{R_1 C_1} + \frac{R_2}{L} \right) \frac{dv_c}{dt} + \left(\frac{R_1 + R_2 - A}{R_1 L C_1} \right) v_c = \frac{R_2}{L C_1} i_g + \frac{1}{C_1} \frac{di_g}{dt}$$

$$R_1 L C_1 \frac{d^2 v_c}{dt^2} + \left(L + R_1 R_2 C_1 \right) \frac{dv_c}{dt} + \left(R_1 + R_2 - A \right) v_c = R_1 R_2 i_g + L R_1 \frac{di_g}{dt}$$

General Solution Using Mathematica

$$V_c(s) = \left\{ \frac{(R_1R_2 + LR_1s)\left[I_G(s)\right] + (A - R_1)LI_{L0} + C_1(R_1R_2 + LR_1s)V_{C0}}{(R_1 + R_2 - A) + (L + C_1R_1R_2)s + C_1LR_1s^2} \right\}$$

$$= \left\{ \begin{aligned} &\left[\frac{R_1R_2 + LR_1s}{(R_1 + R_2 - A) + (L + C_1R_1R_2)s + C_1LR_1s^2} \right]\left[I_G(s)\right] + \\ &\left[\frac{(A - R_1)L}{(R_1 + R_2 - A) + (L + C_1R_1R_2)s + C_1LR_1s^2} \right] i_L(0) + \\ &\left[\frac{C_1(R_1R_2 + LR_1s)}{(R_1 + R_2 - A) + (L + C_1R_1R_2)s + C_1LR_1s^2} \right] v_C(0) \end{aligned} \right\}$$

Mapping Back To Differential Equation

Initial conditions do not effect the differential equation,

only its solution, so ignore the $v_c(0), i_L(0)$ pieces

$$V_c(s) = \left[\frac{R_1R_2 + LR_1s}{(R_1 + R_2 - A) + (L + C_1R_1R_2)s + C_1LR_1s^2}\right]\left[I_G(s)\right]$$

$$\left[(R_1 + R_2 - A) + (L + C_1R_1R_2)s + C_1LR_1s^2\right]V_c(s) = \left[R_1R_2 + LR_1s\right]\left[I_G(s)\right]$$

$$s \leftrightarrow \frac{d}{dt}$$

$$s^2 \leftrightarrow \frac{d^2}{dt^2}$$

$$(R_1 + R_2 - A)v_c + (L + C_1R_1R_2)\frac{dv_c}{dt} + C_1LR_1\frac{d^2v_c}{dt^2} = R_1R_2i_g + LR_1\frac{di_g}{dt}$$

Inverse Transforms: Partial Fraction Expansion

- We will do inverse transforms by the partial fraction expansion method and matching the results to our table of transform pairs
- The Laplace transform of our circuit terms are always of the form N(s)/D(s) where N(s) and D(s) are polynomials in s
- The roots of N(s)=0 are called zeroes
- The roots of D(s)=0 are called poles

$$F(s)=\frac{N(s)}{D(s)}=\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

Partial Fraction Expansion

- We will expand the transform result in terms of the poles $p_1 \ldots p_m$
- There are really 2 cases:
  - Each pole is distinct
  - Repeated poles
- Complex Poles are not really a unique case, but are usually given special attention so we will break up the partial expansion problem into 4 cases
  - Distinct Real Poles
  - Repeated Real Poles
  - Complex Poles
  - Repeated Complex Poles

Distinct Real Poles & Residue Method

$$F(s)=\frac{s^2-4s+3}{s^3+4s^2+3s}=\frac{s^2-4s+3}{s\left(s^2+4s+3\right)}=\frac{s^2-4s+3}{s(s+1)(s+3)}$$

3 real poles $p_1=0,\quad p_2=-1,\quad p_3=-3$

$$=\frac{A_1}{s}+\frac{A_2}{(s+1)}+\frac{A_3}{(s+3)}$$ Partial Fraction Expansion

Residue Method: Multiple by each denominator term and evaluate at pole

$$sF(s)=\frac{s\left(s^2-4s+3\right)}{s(s+1)(s+3)}=\frac{\left(s^2-4s+3\right)}{(s+1)(s+3)}=A_1+\frac{sA_2}{(s+1)}+\frac{sA_3}{(s+3)}$$

$$sF(s)\Big|_{s=0}=\frac{\left(s^2-4s+3\right)}{(s+1)(s+3)}\Bigg|_{s=0}=\frac{3}{(1)(3)}=1=\left[A_1+\frac{sA_2}{(s+1)}+\frac{sA_3}{(s+3)}\right]\Bigg|_{s=0}=A_1$$

$$A_1=1$$

Distinct Real Poles & Residue Method

$$(s+1)F(s)=\frac{(s+1)(s^2-4s+3)}{s(s+1)(s+3)}=\frac{(s^2-4s+3)}{s(s+3)}=\frac{A_1(s+1)}{s}+A_2+\frac{A_3(s+1)}{(s+3)}$$

$$(s+1)sF(s)\Big|_{s=-1}=\frac{(s^2-4s+3)}{s(s+3)}\Bigg|_{s=-1}=\frac{(-1)^2+4+3}{(-1)(-1+3)}=\frac{8}{-2}$$

$$=\left[\frac{A_1(s+1)}{s}+A_2+\frac{A_3(s+1)}{(s+3)}\right]\Bigg|_{s=-1}=A_2$$

$$A_2=-4$$

Distinct Real Poles & Residue Method

$$(s+3)F(s)=\frac{(s+3)(s^2-4s+3)}{s(s+1)(s+3)}=\frac{(s^2-4s+3)}{s(s+1)}=\frac{A_1(s+3)}{s}+\frac{A_2(s+3)}{(s+1)}+A_3$$

$$(s+3)F(s)\Big|_{s=-3}=\frac{(s^2-4s+3)}{s(s+1)}\Bigg|_{s=-3}=\frac{(-3)^2+12+3}{(-3)(-3+1)}=\frac{24}{6}=4$$

$$=\left[\frac{A_1(s+3)}{s}+\frac{A_2(s+3)}{(s+1)}+A_3\right]\Bigg|_{s=-3}=A_3$$

$$A_3=+4$$

Verify

$$F(s)=\frac{s^2-4s+3}{s(s+1)(s+3)}=\frac{1}{s}+\frac{-4}{(s+1)}+\frac{4}{(s+3)}$$

Verify

$$\frac{1}{s}+\frac{-4}{(s+1)}+\frac{4}{(s+3)}=\frac{(s+1)(s+3)-4s(s+3)+4s(s+1)}{s(s+1)(s+3)}$$

$$=\frac{(s^2+4s+3)-(4s^2+12s)+(4s^2+4s)}{s(s+1)(s+3)}$$

$$=\frac{(s^2-4s+3)}{s(s+1)(s+3)}$$

Inverse Transform

$$L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{1}{s} + \frac{-4}{(s+1)} + \frac{4}{(s+3)}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{-4}{(s+1)}\right] + L^{-1}\left[\frac{4}{(s+3)}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] - 4L^{-1}\left[\frac{1}{(s+1)}\right] + 4L^{-1}\left[\frac{1}{(s+3)}\right]$$

$$= \left[1 - 4e^{-t} + 4e^{-3t}\right]u(t)$$

Repeated Real Poles

Example

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s-p_1)(s-p_2)(s-p_3)^m}$$

$$=\frac{A_1}{(s-p_1)}+\frac{A_2}{(s-p_2)}+\frac{B_1}{(s-p_3)}+\frac{B_2}{(s-p_3)^2}+\ldots\frac{B_m}{(s-p_3)^m}$$

Particular Example:

$$V_c(s)=\frac{5(90000000+22200s+s^2)}{s(3000+s)^2}\ \text{from circuit example}$$

$$=\frac{A}{s}+\frac{B_1}{(3000+s)}+\frac{B_2}{(3000+s)^2}$$

Repeated Real Poles

$$V_c(s) = \frac{5(90000000 + 22200s + s^2)}{s(3000+s)^2} = \frac{A}{s} + \frac{B_1}{(3000+s)} + \frac{B_2}{(3000+s)^2}$$

$$A = sV_c(s)\Big|_{s=0} = \frac{5(90000000 + 22200s + s^2)}{(3000+s)^2}\Bigg|_{s=0} = \frac{5(90000000)}{(3000)^2}$$

$$B_2 = (3000+s)^2 V_c(s)\Big|_{s=-3000}$$

$$= \left[A\frac{(3000+s)^2}{s} + B_1(3000+s) + B_2 \right]\Bigg|_{s=-3000}$$

$$= \frac{5(90000000 + 22200s + s^2)}{s}\Bigg|_{s=-3000}$$

$$= \frac{5(90000000 - 66600000 + 9000000)}{-3000}$$

$$= -54000$$

$$B_1 = ?$$

Repeated Real Poles

$$V_c(s) = \frac{5(90000000 + 22200s + s^2)}{s(3000+s)^2} = \frac{A}{s} + \frac{B_1}{(3000+s)} + \frac{B_2}{(3000+s)^2}$$

$$B_1 = \frac{d}{ds}\left\{(3000+s)^2 V_c(s)\right\}\Big|_{s=-3000}$$

$$= \frac{d}{ds}\left\{A\frac{(3000+s)^2}{s} + B_1(3000+s) + B_2\right\}\Big|_{s=-3000}$$

$$= \frac{d}{ds}\left\{\frac{5(90000000 + 22200s + s^2)}{s}\right\}\Big|_{s=-3000}$$

$$= \left\{\frac{5(22200+2s)}{s} - \frac{5(90000000 + 22200s + s^2)}{s^2}\right\}\Big|_{s=-3000}$$

$$= -45$$

Repeat Real Poles: General

Repeated Real Poles

Expansion coefficients B_1 to B_m are determined through a procedure that involves multiplication by $(\mathbf{s}+p)^m$, differentiation with respect to s, and evaluation at $\mathbf{s}=-p$:

$$B_j = \left\{ \frac{1}{(m-j)!} \frac{d^{m-j}}{d\mathbf{s}^{m-j}} [(\mathbf{s}+p)^m \, \mathbf{F}(\mathbf{s})] \right\} \Bigg|_{\mathbf{s}=-p}, \quad j = 1, 2, \ldots, m. \tag{6.92}$$

For the m, $m-1$, and $m-2$ terms, Eq. (6.92) reduces to:

$$B_m = (\mathbf{s}+p)^m \, \mathbf{F}(\mathbf{s})|_{\mathbf{s}=-p}, \tag{6.93a}$$

$$B_{m-1} = \left\{ \frac{d}{d\mathbf{s}} [(\mathbf{s}+p)^m \, \mathbf{F}(\mathbf{s})] \right\} \Bigg|_{\mathbf{s}=-p}, \tag{6.93b}$$

$$B_{m-2} = \left\{ \frac{1}{2!} \frac{d^2}{d\mathbf{s}^2} [(\mathbf{s}+p)^m \, \mathbf{F}(\mathbf{s})] \right\} \Bigg|_{\mathbf{s}=-p}. \tag{6.93c}$$

Repeated Real Poles

$$V_c(s) = \frac{5(90000000 + 22200s + s^2)}{s(3000+s)^2} = \frac{50}{s} + \frac{-45}{(3000+s)} + \frac{-54000}{(3000+s)^2}$$

$$L^{-1}\left[V_c(s)\right] = \left(50 - 45e^{-3000t} - 54000te^{-3000t}\right)u(t)$$

Two Complex Poles

- Really a special case of 2 distinct poles but were the 2 distinct poles are complex conjugates of each other

$$F(s) = \frac{4s+1}{(s+1)(s^2+4s+13)}$$

$$poles\ \ p_1 = -1,\ p_2 = -2+3j,\ p_3 = -2-3j \quad p_2 = p_3^*$$

$$F(s) = \frac{4s+1}{(s+1)(s-(-2+3j))(s-(-2+3j))}$$

$$= \frac{4s+1}{(s+1)(s+2-3j)(s+2+3j)}$$

$$= \frac{A}{(s+1)} + \frac{B_1}{(s+2-3j)} + \frac{B_2}{(s+2+3j)}$$

Two Complex Poles

$$F(s)=\frac{4s+1}{(s+1)(s+2-3j)(s+2+3j)}=\frac{A}{(s+1)}+\frac{B_1}{(s+2-3j)}+\frac{B_2}{(s+2+3j)}$$

$$A=(s+1)F(s)\Big|_{s=-1}=\frac{-3}{(1-3j)(1+3j)}=\frac{-3}{10}=-0.3$$

$$B_1=(s+2-3j)F(s)\Big|_{s=-2+3j}=\frac{4(-2+3j)+1}{(-2+3j+1)(-2+3j+2+3j)}$$

$$=\frac{-7+12j}{(-1+3j)(6j)}=\frac{-7+12j}{-18-6j}=\left(\frac{3}{20}-\frac{43}{60}j\right)$$

$$B_2=(s+2+3j)F(s)\Big|_{s=-2-3j}=\frac{4(-2-3j)+1}{(-2-3j+1)(-2-3j+2-3j)}$$

$$=\frac{-7-12j}{(-1-3j)(-6j)}=\frac{-7-12j}{-18+6j}=\left(\frac{3}{20}+\frac{43}{60}j\right)$$

Complex Numbers

We will find it is useful to alternatively represent complex numbers in rectangular and polar form

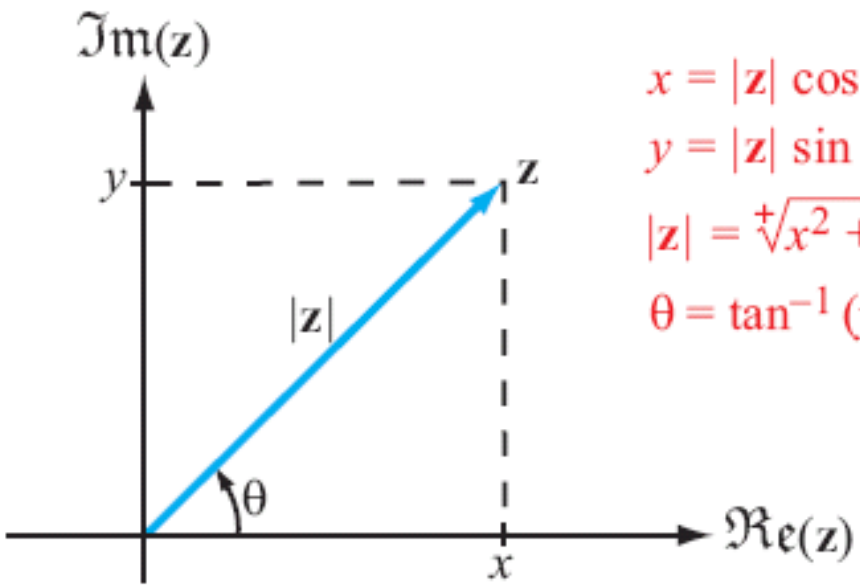
$$j = \sqrt{-1}$$

$z = x + jy$ **Rectangular coordinates**

$z = |z|\angle\theta = |z|e^{j\theta}$ **Polar coordinates**

$\mathrm{Re}(z) = x$

$\mathrm{Im}(z) = y$



$x = |z| \cos\theta$

$y = |z| \sin\theta$

$|z| = \sqrt[+]{x^2 + y^2}$

$\theta = \tan^{-1}(y/x)$

Relations based on Euler Identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Euler's Relation

We will derive Euler's Relation Later Using Taylor's Theorem

$$e^{j\theta} = \left[\cos(\theta) + j\sin(\theta)\right]$$

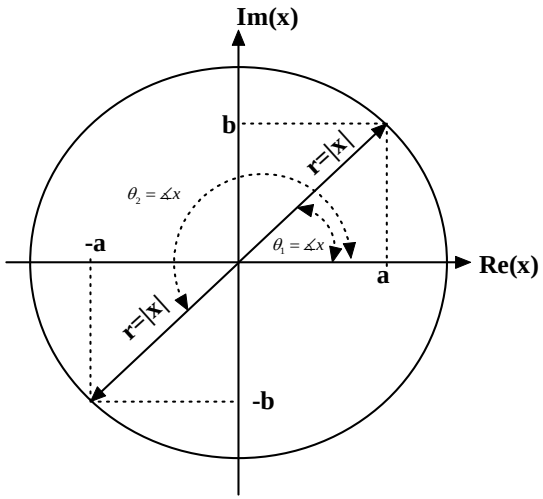
$$e^{j\theta} + e^{-j\theta} = \left[\cos(\theta) + j\sin(\theta)\right] + \left[\cos(\theta) - j\sin(\theta)\right] = 2\cos(\theta)$$

$$e^{j\theta} - e^{-j\theta} = \left[\cos(\theta) + j\sin(\theta)\right] - \left[\cos(\theta) - j\sin(\theta)\right] = 2j\cos(\theta)$$

$$Ae^{j\theta} + A^{*}e^{-j\theta} = (a + jb)e^{j\theta} + (a - jb)e^{j\theta}$$

$$= 2a\cos(\theta) - 2b\sin(\theta)$$

$$= 2|A|\cos(\theta + \angle A)$$



$$|A| = \left(a^2 + b^2\right)^{1/2}$$

$$\angle A = \tan^{-1}\left(\frac{b}{a}\right)$$ with attention to correct quadrant

Two Complex Poles

$$B_1 = \left(\frac{3}{20} - \frac{43}{60}j\right) = \frac{1}{6}\sqrt{\frac{193}{10}}e^{jArcTan[-\frac{43*20}{60*3}]} = \frac{1}{6}\sqrt{\frac{193}{10}}e^{-jArcTan[\frac{43}{9}]} = \frac{1}{6}\sqrt{\frac{193}{10}}e^{-j\phi}$$

$$B_2 = B_1^* = \left(\frac{3}{20} + \frac{43}{60}j\right) = \frac{1}{6}\sqrt{\frac{193}{10}}e^{+jArcTan[\frac{43}{9}]} = \frac{1}{6}\sqrt{\frac{193}{10}}e^{j\phi}$$

$$\phi = \tan^{-1}\left(\frac{43}{9}\right) \approx 1.364472432765179 2\ radians$$

$$\approx 78.17851165939275°$$

Two Complex Poles

$$F(s)=\frac{4s+1}{(s+1)(s+2-3j)(s+2+3j)}=\left\{\frac{-0.3}{(s+1)}+\frac{\frac{1}{6}\sqrt{\frac{193}{10}}e^{-j\phi}}{(s+2-3j)}+\frac{\frac{1}{6}\sqrt{\frac{193}{10}}e^{+j\phi}}{(s+2+3j)}\right\}$$

$$L^{-1}\left[F(s)\right]=\left[-0.3e^{-t}+\left(\frac{1}{6}\sqrt{\frac{193}{10}}e^{-j\phi}\right)e^{-(2-3j)t}+\left(\frac{1}{6}\sqrt{\frac{193}{10}}e^{+j\phi}\right)e^{-(2+3j)t}\right]u(t)$$

$$=\left[-0.3e^{-t}+\left(\frac{1}{6}\sqrt{\frac{193}{10}}\right)\left(e^{j(3t-\phi)}+e^{-j(3t-\phi)}\right)e^{-2t}\right]u(t)$$

Two Complex Poles

$$e^{j\theta}+e^{-j\theta}=\left[\cos(\theta)+j\sin(\theta)\right]+\left[\cos(\theta)-j\sin(\theta)\right]=2\cos(\theta)$$

$$\left(e^{j(3t-\phi)}+e^{-j(3t-\phi)}\right)=2\cos(3t-\phi)$$

$$L^{-1}\left[F(s)\right]=\left[-0.3e^{-t}+\left(\frac{1}{6}\sqrt{\frac{193}{10}}\right)\left(e^{j(3t-\phi)}+e^{-j(3t-\phi)}\right)e^{-2t}\right]u(t)$$

$$=\left[-0.3e^{-t}+2\left(\frac{1}{6}\sqrt{\frac{193}{10}}\right)\cos(3t-\phi)e^{-2t}\right]u(t)$$

$$=\left[-0.3e^{-t}+\left(\frac{1}{3}\sqrt{\frac{193}{10}}\right)\cos(3t-\phi)e^{-2t}\right]u(t)$$

$$\phi=\tan^{-1}\left(\frac{43}{9}\right)\approx 1.364472432765179 2\ radians$$

$$\approx 78.17851165939275^{\circ}$$

Two Complex Poles: General

$$F(s) = \frac{Ae^{j\theta}}{s+(a+jb)} + \frac{Ae^{-j\theta}}{s+(a-jb)}$$

$$L^{-1}\left[F(s)\right] = \left[Ae^{j\theta}e^{-(a+jb)t} + Ae^{-j\theta}e^{-(a-jb)t}\right]u(t)$$

$$= \left[e^{j\theta}e^{-jbt} + e^{-j\theta}e^{+jbt}\right]Ae^{-at}u(t)$$

$$= \left[e^{-j(bt-\theta)} + e^{+j(bt-\theta)}\right]Ae^{-at}u(t)$$

$$= 2\cos(bt-\theta)Ae^{-at}u(t)$$

$$= 2Ae^{-at}\cos(bt-\theta)u(t)$$

Repeated Complex Poles

- Direct Extension of Methods of repeated real poles and distinct complex pole pairs
- See Chapter 6, section 6-9.4 for examples

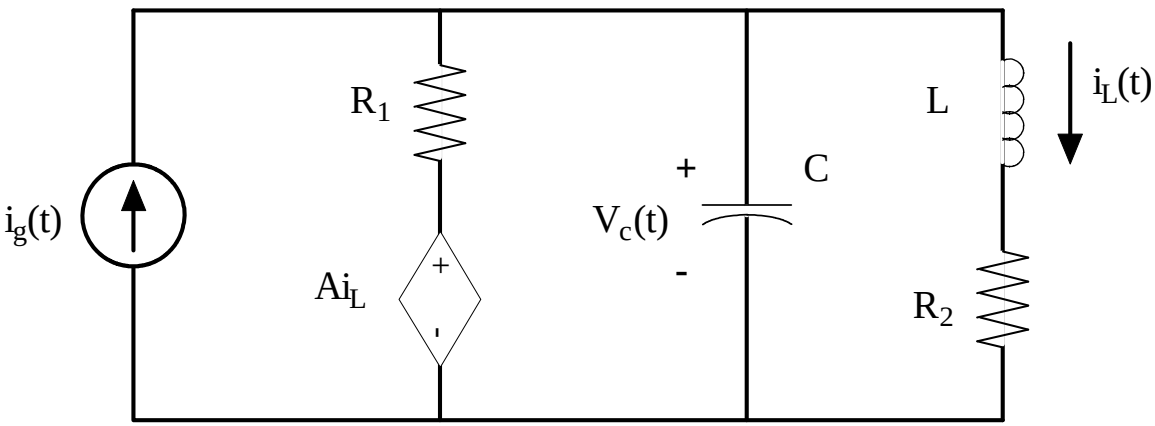
Table 10-3: Transform pairs for four types of poles.

Pole	$\mathbf{F}(\mathbf{s})$	$f(t)$
1. Distinct real	$\dfrac{A}{\mathbf{s}+a}$	$Ae^{-at}\,u(t)$
2. Repeated real	$\dfrac{A}{(\mathbf{s}+a)^n}$	$A\,\dfrac{t^{n-1}}{(n-1)!}\,e^{-at}\,u(t)$
3. Distinct complex	$\left[\dfrac{Ae^{j\theta}}{\mathbf{s}+a+jb}+\dfrac{Ae^{-j\theta}}{\mathbf{s}+a-jb}\right]$	$2Ae^{-at}\cos(bt-\theta)\,u(t)$
4. Repeated complex	$\left[\dfrac{Ae^{j\theta}}{(\mathbf{s}+a+jb)^n}+\dfrac{Ae^{-j\theta}}{(\mathbf{s}+a-jb)^n}\right]$	$\dfrac{2At^{n-1}}{(n-1)!}\,e^{-at}\cos(bt-\theta)\,u(t)$

Laplace Transform Circuit Analysis Examples

- Transform circuit to s-domain (including initial conditions
- Use Nodal/Mesh to find transform of desired variables
  - This automatically interconnects the initial conditions
- Table-look-up inverse transform to get results
- Examples heavily use Mathematica to save tedium
- Mathematica does not always fully simplify complex results, some work by hand needed to achieve elegance
- In some cases, slides show line-by-line detail

Example Numerical Solution



$$i_g = 90\cos(\omega t)u(t)\ mA \quad R_1 = 1K\Omega \quad R_2 = 5\Omega$$

$$\omega = 3000s^{-1} \quad C = 1\mu F \quad L = 1mH \quad A = 800\Omega$$

$$i_L(0) = i_{L0} = 0 \quad v_C(0) = v_{C0} = 0$$

General Solution Using Mathematica

$$V_c(s)=\left\{\frac{(R_1R_2+LR_1s)\left[I_G(s)\right]+(A-R_1)Li_L(0)+C_1(R_1R_2+LR_1s)v_C(0)}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right\}$$

$$=\left\{\begin{array}{l}\left[\dfrac{R_1R_2+LR_1s}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]\left[I_G(s)\right]+\\ \left[\dfrac{(A-R_1)L}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]i_L(0)+\\ \left[\dfrac{C_1(R_1R_2+LR_1s)}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]v_C(0)\end{array}\right\}$$

General Solution Using Mathematica

$$V_c(s)=\left\{\frac{(R_1R_2+LR_1s)\left[I_G(s)\right]+(A-R_1)Li_{L0}+C_1(R_1R_2+LR_1s)v_{C0}}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right\}$$

$$=\left[\frac{R_1R_2+LR_1s}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]\left[I_G(s)\right]$$

$$=\left[\frac{R_1R_2+LR_1s}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]\left[\left(\frac{s}{s^2+\omega^2}\right)90mA\right]$$

$$=\left[\frac{(5000+s)}{205+6x10^{-3}s+10^{-6}s^2}\right]\left[\frac{9x10^{-2}s}{s^2+9000000}\right]$$

$$=\frac{90000s(5000+s)}{(9000000+s^2)(205000000+6000s+s^2)}$$

$$=\frac{90000s(5000+s)}{(s+j3000)(s-j3000)(s+(3000+j14000))(s+(3000-j14000))}$$

Inverse Transform: Raw Mathematica+Hand Analysis

$$v_c(t)=\left\{\left(-\frac{4653}{3874}+\frac{83547i}{27118}\right)e^{(-3000-14000i)t}-\left(\frac{4653}{3874}+\frac{83547i}{27118}\right)e^{(-3000+14000i)t}\right\}u(t)$$

$$+\left\{\left(\frac{4653}{1937}\right)\mathrm{Cos}[3000t]-\left(\frac{2241}{1937}\right)\mathrm{Sin}[3000t]\right\}u(t)$$

$$=\left\{\left(-\frac{4653}{3874}+\frac{83547i}{27118}\right)e^{(-14000i)t}-\left(\frac{4653}{3874}+\frac{83547i}{27118}\right)e^{(14000i)t}\right\}e^{-3000t}u(t)$$

$$+\left\{\left(\frac{4653}{1937}\right)\mathrm{Cos}[3000t]-\left(\frac{2241}{1937}\right)\mathrm{Sin}[3000t]\right\}u(t)$$

$$=\left\{-2\left(\frac{4653}{3874}\right)Cos\left[14000t\right]+2\left(\frac{83547}{27118}\right)Sin\left[14000t\right]\right\}e^{-3000t}u(t)$$

$$+\left\{\left(\frac{4653}{1937}\right)\mathrm{Cos}[3000t]-\left(\frac{2241}{1937}\right)\mathrm{Sin}[3000t]\right\}u(t)$$

$$=\left\{-\left(\frac{4653}{1937}\right)Cos\left[14000t\right]+\left(\frac{83547}{27118}\right)Sin\left[14000t\right]\right\}e^{-3000t}u(t)$$

$$+\left\{\left(\frac{4653}{1937}\right)\mathrm{Cos}[3000t]-\left(\frac{2241}{1937}\right)\mathrm{Sin}[3000t]\right\}u(t)$$

Inverse Transform: Elegant

$$V_C(s)=\frac{90000s(5000+s)}{(s+j3000)(s-j3000)(s+(3000+j14000))(s+(3000-j14000))}$$

$$=\frac{\frac{4653}{3874}+\frac{2241}{3874}j}{-3000j+s}+\frac{\frac{4653}{3874}-\frac{2241}{3874}j}{3000j+s}-\frac{\frac{4653}{3874}+\frac{83547}{27118}j}{(3000-14000j)+s}-\frac{\frac{4653}{3874}-\frac{83547}{27118}j}{(3000+14000j)+s}$$

$$=\left(\frac{A_1e^{j\theta_1}}{s+j3000}+\frac{A_1e^{-j\theta_1}}{s-j3000}\right)-\left(\frac{A_2e^{j\theta_2}}{s+3000+j14000}+\frac{A_2e^{-j\theta_2}}{s+3000+j14000}\right)$$

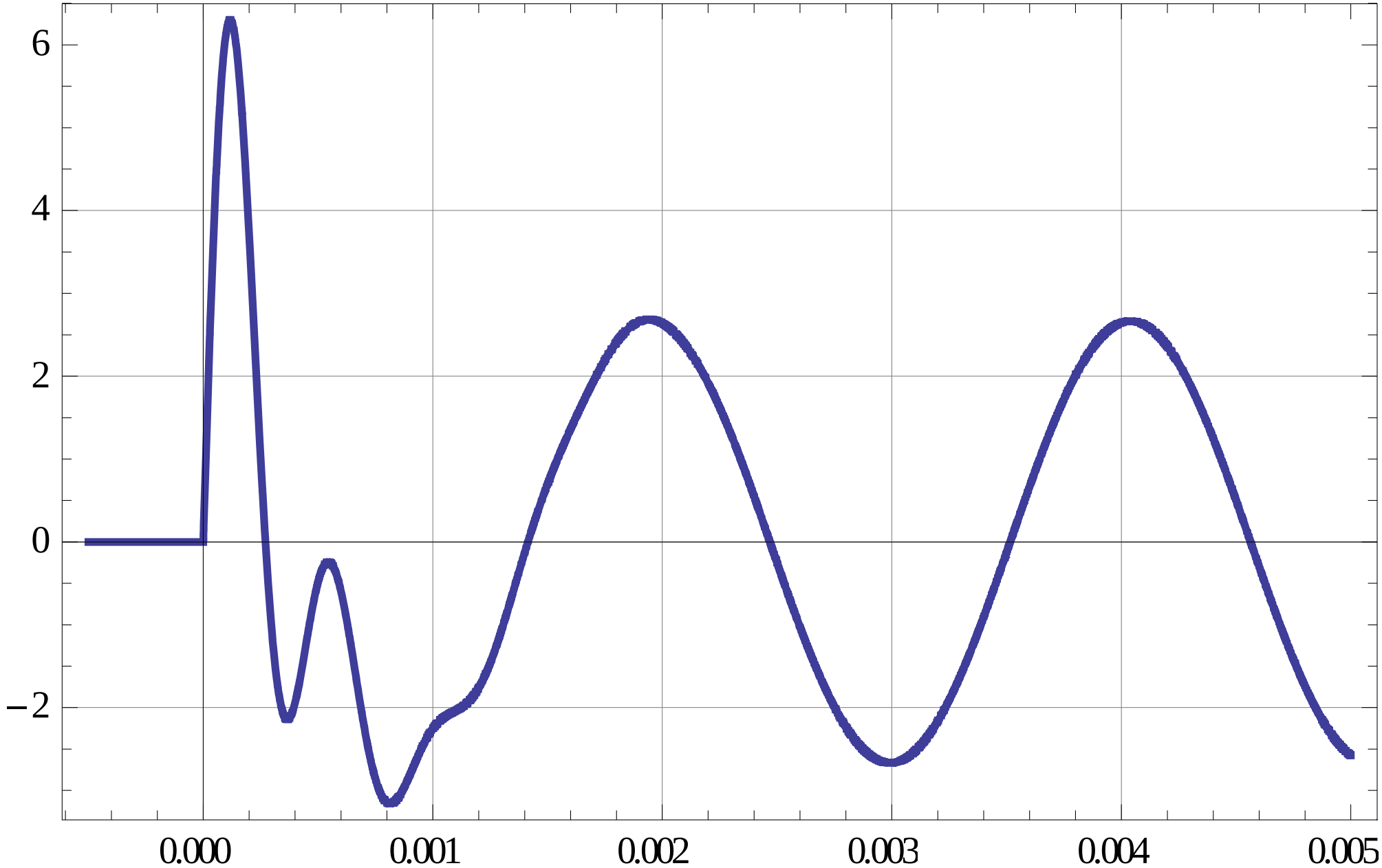
$$A_1=\left[\left(\frac{4653}{3874}\right)^2+\left(\frac{2241}{3874}\right)^2\right]^{1/2}=9\sqrt{\frac{85}{3874}}$$

$$\theta_1=\tan^{-1}\left(-\frac{2241}{3874}\Big/\frac{4653}{3874}\right)\approx-25.7166°$$

$$A_2=\left[\left(\frac{4653}{3874}\right)^2+\left(\frac{83547}{27118}\right)^2\right]^{1/2}=\frac{225}{7}\sqrt{\frac{41}{3874}}$$

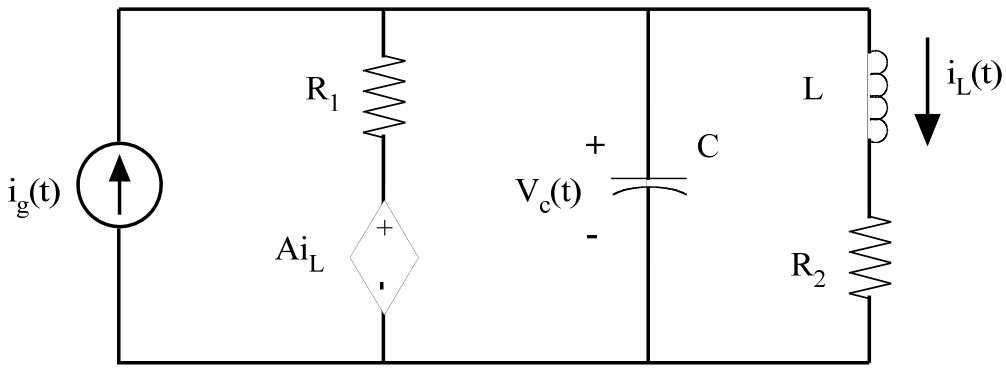
$$\theta_2=\tan^{-1}\left(-\frac{83547}{27118}\Big/\frac{4653}{3874}\right)\approx-68.701556°$$

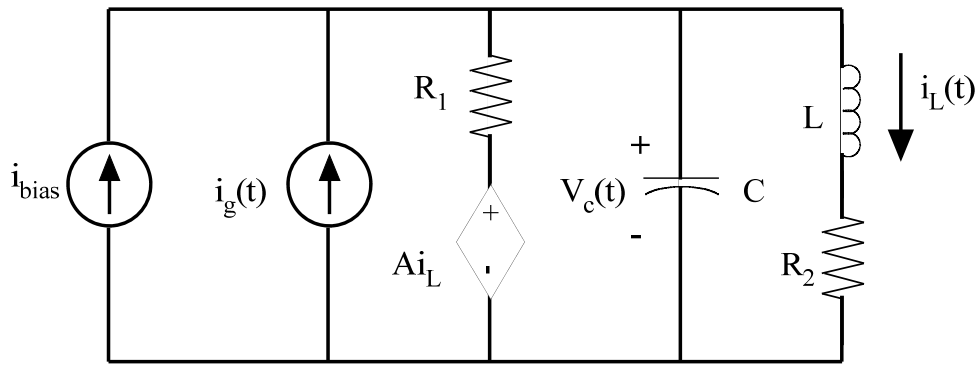
$$v_c(t)=\left\{18\sqrt{\frac{85}{3874}}Cos[3000t-\theta_1]-\frac{450}{7}\sqrt{\frac{41}{3874}}Cos[14000t-\theta_2]e^{-3000t}\right\}u(t)$$



$$\alpha = 3000 s^{-1} \rightarrow \frac{1}{\alpha} \approx 3.333 x 10^{-4} s \rightarrow \frac{6}{\alpha} = 2ms$$

Example Numerical Solution: Non-zero initial conditions





$$i_g = \left(90\cos\left(\omega t\right)u(t) - 410\right)\ mA \quad R_1 = 1K\Omega \quad R_2 = 5\Omega$$

$$\omega = 3000s^{-1} \quad C = 1\mu F \quad L = 1mH \quad A = 800\Omega$$

$$I_{L0} = 0 \quad V_{C0} = 0$$

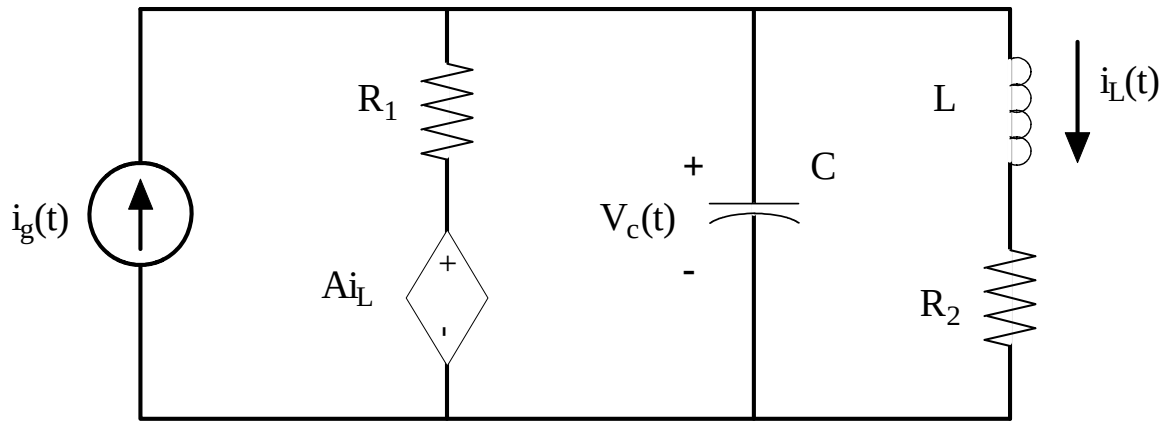
Laplace Transform Circuit Analysis Example: Initial Conditions

Laplace Transform of u(t) and 1 are the same

The world starts at t=0 in Laplace

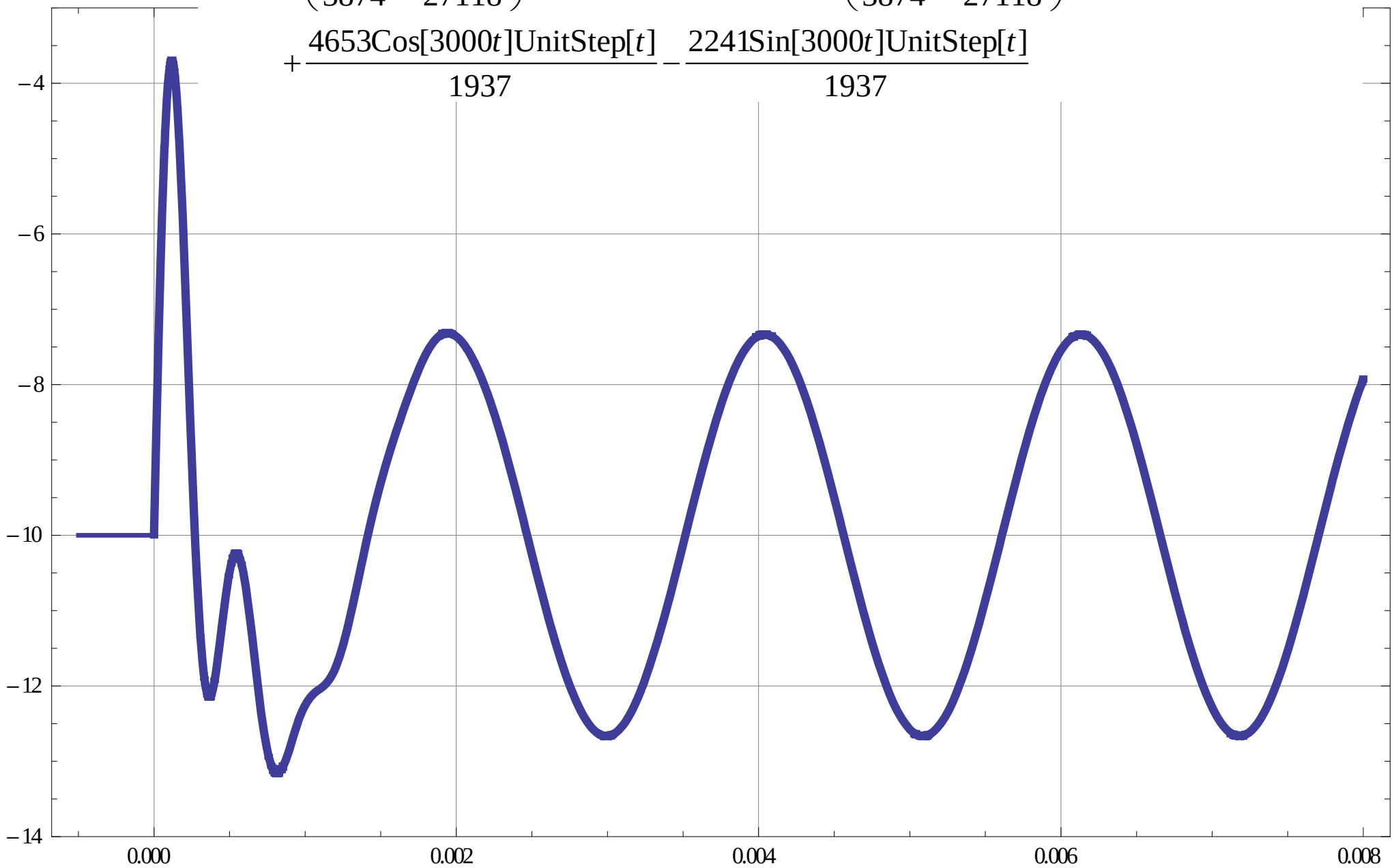
$$I_g(s) = L\left[\left(90\cos(\omega t)u(t) - 410\right)\ mA\right]$$

$$I_g(s) = \left[\frac{90s}{s^2+\omega^2} - \frac{410}{s}\right] mA$$



$$I_g(s) = L\left[i_g(t)\right]$$

$$vc(t) = -10\,\mathrm{UnitStep}[-t] - 10\,\mathrm{UnitStep}[t] - \left(\frac{4653}{3874} - \frac{83547i}{27118}\right) e^{(-3000-14000i)t}\,\mathrm{UnitStep}[t] - \left(\frac{4653}{3874} + \frac{83547i}{27118}\right) e^{(-3000+14000i)t}\,\mathrm{UnitStep}[t] + \frac{4653\,\mathrm{Cos}[3000t]\,\mathrm{UnitStep}[t]}{1937} - \frac{2241\,\mathrm{Sin}[3000t]\,\mathrm{UnitStep}[t]}{1937}$$

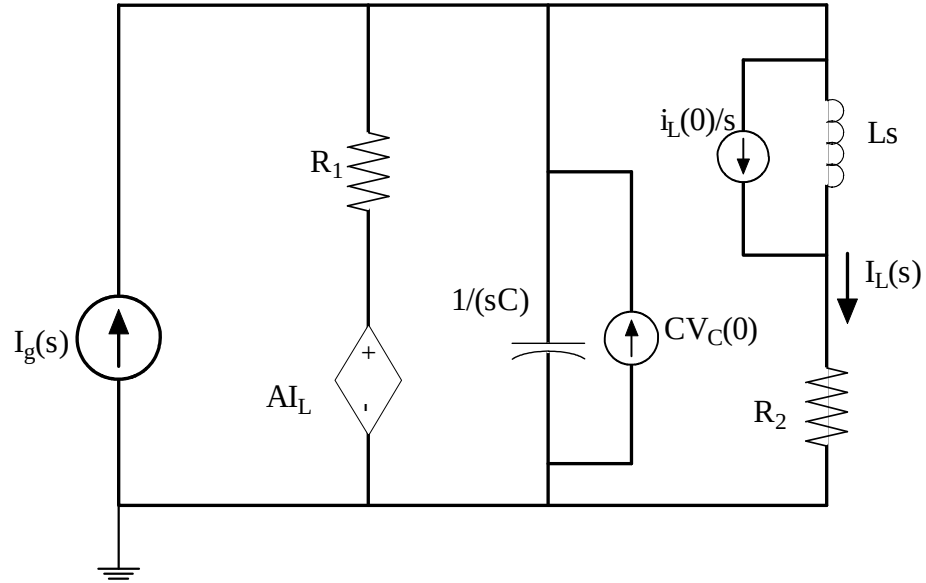


Further Reduction

$$v_c(t) = -10\text{UnitStep}[-t] - 10\text{UnitStep}[t]$$
$$-\left(\frac{4653}{3874} - \frac{83547i}{27118}\right)e^{(-3000-14000i)t}\text{UnitStep}[t] - \left(\frac{4653}{3874} + \frac{83547i}{27118}\right)e^{(-3000+14000i)t}\text{UnitStep}[t]$$
$$+\frac{4653\text{Cos}[3000t]\text{UnitStep}[t]}{1937} - \frac{2241\text{Sin}[3000t]\text{UnitStep}[t]}{1937}$$
$$= -10 + \left[\left(-\frac{9306}{3874}\right)\text{Cos}[14000t] + \left(\frac{167094}{27118}\right)\text{Sin}[14000t]\right]e^{(-3000)t}u(t)$$
$$+\left[\left(\frac{4653}{1937}\right)\text{Cos}[3000t] - \left(\frac{2241}{1937}\right)\text{Sin}[3000t]\right]u(t)$$

Easy way to do this was superposition!

Recap: Last Time Nodal Analysis



In[1]:= $\mathbf{IL} = \dfrac{\mathbf{R2}\,(\mathbf{IL0\,L} + \mathbf{Vc})}{\mathbf{R2} + \mathbf{L\,s}} * \mathbf{1} / \mathbf{R2}$

Out[1]= $\dfrac{\mathrm{IL0\,L} + \mathrm{Vc}}{\mathrm{R2} + \mathrm{L\,s}}$

In[2]:= $\mathbf{k1} = -\mathbf{IGS} + (\mathbf{Vc} - \mathbf{A} * \mathbf{IL}) / \mathbf{R1} + \mathbf{s} * \mathbf{C1} * \mathbf{Vc} - \mathbf{C1} * \mathbf{VC0} + \mathbf{IL}$

Out[2]= $-\mathrm{IGS} + \mathrm{C1\,s\,Vc} + \dfrac{\mathrm{IL0\,L} + \mathrm{Vc}}{\mathrm{R2} + \mathrm{L\,s}} + \dfrac{\mathrm{Vc} - \frac{\mathrm{A}\,(\mathrm{IL0\,L} + \mathrm{Vc})}{\mathrm{R2} + \mathrm{L\,s}}}{\mathrm{R1}} - \mathrm{C1\,VC0}$

In[3]:= `Solve[k1 == 0, Vc]`

Out[3]= $\{\{\mathrm{Vc} \to (\mathrm{A\,IL0\,L} - \mathrm{IL0\,L\,R1} + \mathrm{IGS\,R1\,R2} + \mathrm{IGS\,L\,R1\,s} + \mathrm{C1\,R1\,R2\,VC0} + \mathrm{C1\,L\,R1\,s\,VC0}) / (-\mathrm{A} + \mathrm{R1} + \mathrm{R2} + \mathrm{L\,s} + \mathrm{C1\,R1\,R2\,s} + \mathrm{C1\,L\,R1\,s^2})\}\}$

In[4]:= $\mathbf{VCS} = (\mathbf{A\,IL0\,L} - \mathbf{IL0\,L\,R1} + \mathbf{IGS\,R1\,R2} + \mathbf{IGS\,L\,R1\,s} + \mathbf{C1\,R1\,R2\,VC0} + \mathbf{C1\,L\,R1\,s\,VC0}) / (-\mathbf{A} + \mathbf{R1} + \mathbf{R2} + \mathbf{L\,s} + \mathbf{C1\,R1\,R2\,s} + \mathbf{C1\,L\,R1\,s^2})$

Out[4]= $\dfrac{\mathrm{A\,IL0\,L} - \mathrm{IL0\,L\,R1} + \mathrm{IGS\,R1\,R2} + \mathrm{IGS\,L\,R1\,s} + \mathrm{C1\,R1\,R2\,VC0} + \mathrm{C1\,L\,R1\,s\,VC0}}{-\mathrm{A} + \mathrm{R1} + \mathrm{R2} + \mathrm{L\,s} + \mathrm{C1\,R1\,R2\,s} + \mathrm{C1\,L\,R1\,s^2}}$

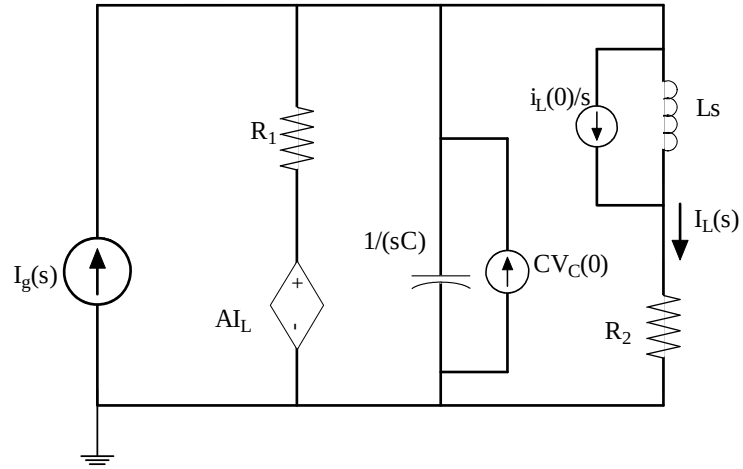
Ramped Signal

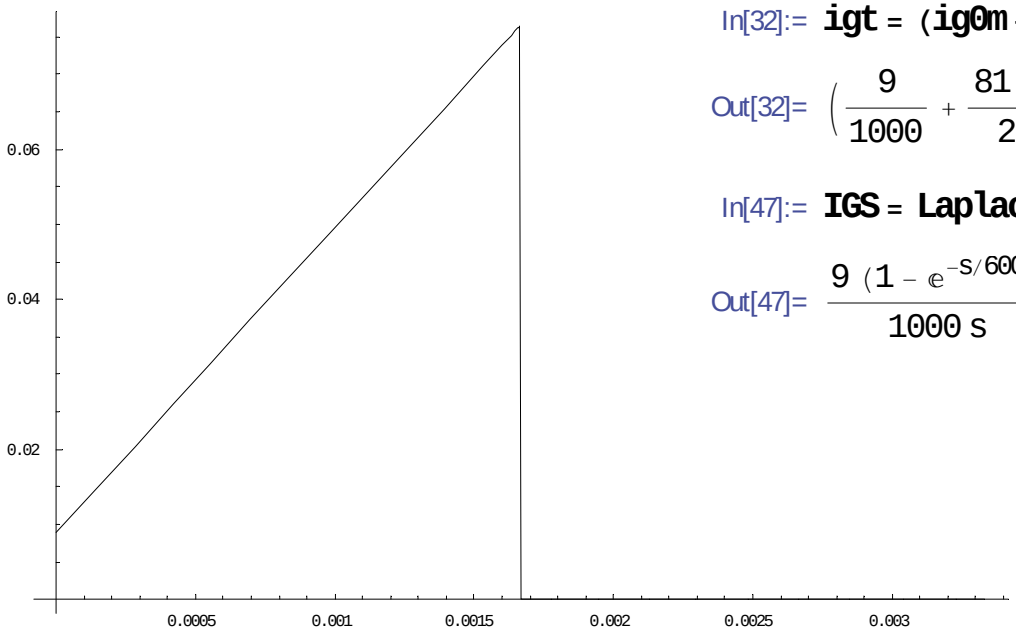
$$R_1 = 1000\Omega \quad R_2 = 5\Omega \quad A = 900\Omega$$

$$i_g(t) = \left(9 + 500 \bullet 81t\right)u(t)u(-t + 5/3000)\, mA$$

$$i_g(t) = \left(9 + 500 \bullet 81t\right)u(t)u(-t + 5/3000)\, mA$$

$$i_g(t) = \left(9 + 40{,}500\, t\right)u(t)u(-t + 5/3000)\, mA$$





In[32]:= `igt = (ig0m + t * 500 * 81 * 10^-3) * UnitStep[t] * UnitStep[-t + 5 / 3000]`

Out[32]= $\left(\frac{9}{1000} + \frac{81\,t}{2}\right) \text{UnitStep}\left[\frac{1}{600} - t\right] \text{UnitStep}[t]$

In[47]:= `IGS = LaplaceTransform[igt, t, s]`

Out[47]= $\frac{9\left(1 - e^{-s/600}\right)}{1000\, s} + \frac{81}{2}\left(\frac{1}{s^2} - \frac{e^{-s/600}\,(600 + s)}{600\, s^2}\right)$

Plug in, Get Laplace Transform, Take Inverse Transform

In[42]:= **vc1s =**

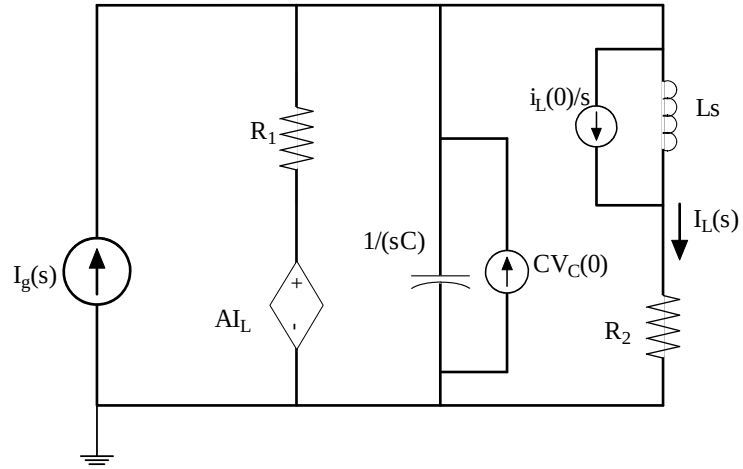
$$\left(3\,e^{-s/600}\left(-472500000000+472500000000\,e^{s/600}-987000000\,s+199500000\,e^{s/600}\,s-178500\,s^2+6000\,e^{s/600}\,s^2+e^{s/600}\,s^3\right)\right)\Big/\left(7\,s^2\left(105000000+6000\,s+s^2\right)\right)$$

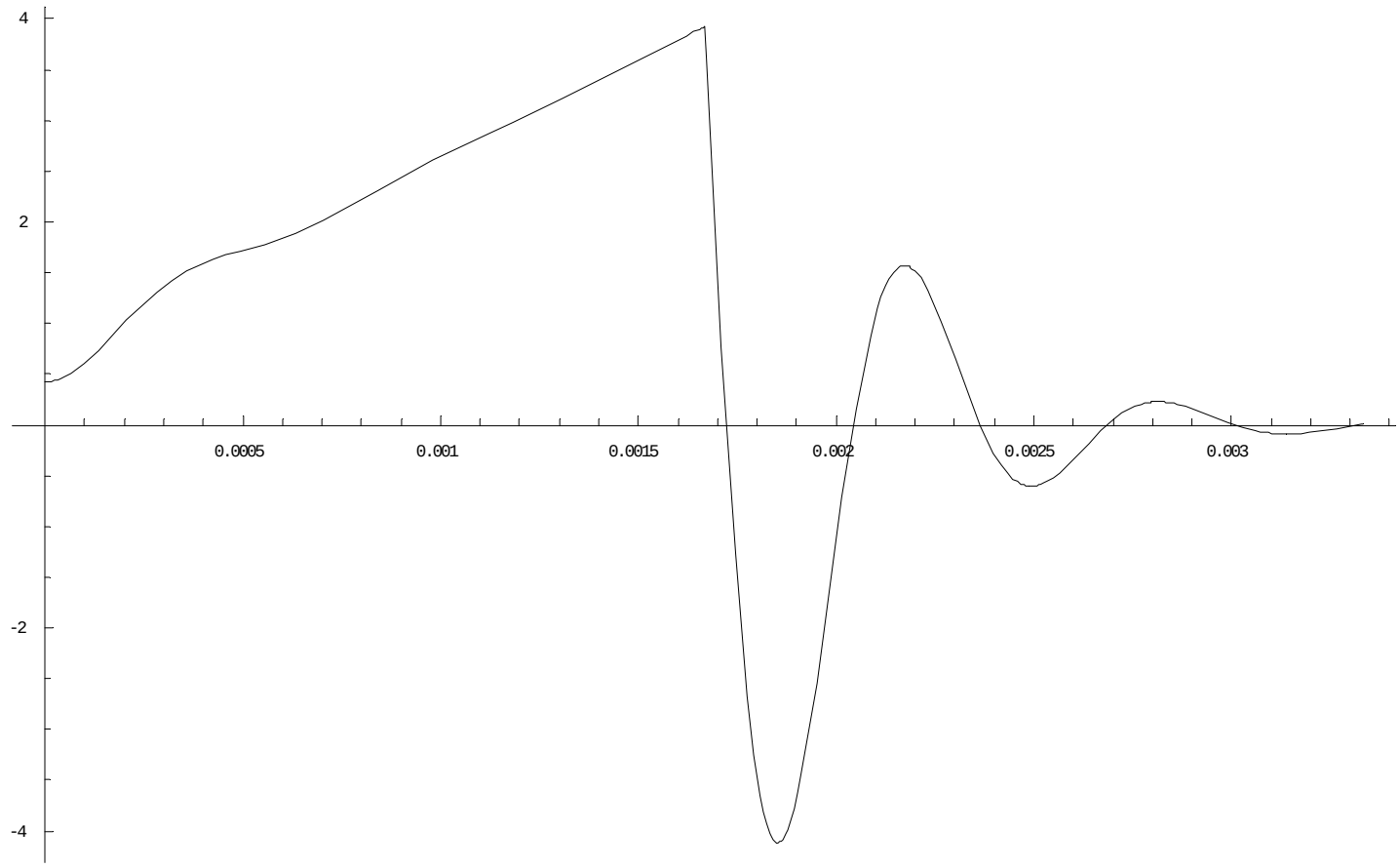
Out[42]= $$\left(3\,e^{-s/600}\left(-472500000000+472500000000\,e^{s/600}-987000000\,s+199500000\,e^{s/600}\,s-178500\,s^2+6000\,e^{s/600}\,s^2+e^{s/600}\,s^3\right)\right)\Big/\left(7\,s^2\left(105000000+6000\,s+s^2\right)\right)$$

In[43]:= **vct = InverseLaplaceTransform[vc1s, s, t]**

Out[43]= $$\frac{3}{7}\Bigg(\frac{1}{2}\sqrt{\frac{3}{2}}\,e^{-3000t}\,\mathrm{Sin}\left[4000\sqrt{6}\,t\right]+\frac{3}{560}\,e^{-3000t}\left(48\,e^{3000t}\left(-1+17500\,t\right)+48\,\mathrm{Cos}\left[4000\sqrt{6}\,t\right]-29\sqrt{6}\,\mathrm{Sin}\left[4000\sqrt{6}\,t\right]\right)+$$
$$\frac{1}{8}\,e^{-3000t}\left(8\,\mathrm{Cos}\left[4000\sqrt{6}\,t\right]-\sqrt{6}\,\mathrm{Sin}\left[4000\sqrt{6}\,t\right]\right)+\frac{19}{80}\left(8-e^{-3000t}\left(8\,\mathrm{Cos}\left[4000\sqrt{6}\,t\right]+\sqrt{6}\,\mathrm{Sin}\left[4000\sqrt{6}\,t\right]\right)\right)-$$
$$\frac{119}{8}\sqrt{\frac{3}{2}}\,e^{-3000\left(-\frac{1}{600}+t\right)}\,\mathrm{Sin}\left[4000\sqrt{6}\left(-\frac{1}{600}+t\right)\right]\mathrm{UnitStep}\left[-\frac{1}{600}+t\right]-\frac{3}{560}\,e^{-3000\left(-\frac{1}{600}+t\right)}$$
$$\left(48\,e^{3000\left(-\frac{1}{600}+t\right)}\left(-1+17500\left(-\frac{1}{600}+t\right)\right)+48\,\mathrm{Cos}\left[4000\sqrt{6}\left(-\frac{1}{600}+t\right)\right]-29\sqrt{6}\,\mathrm{Sin}\left[4000\sqrt{6}\left(-\frac{1}{600}+t\right)\right]\right)\mathrm{UnitStep}\Big[$$
$$-\frac{1}{600}+t\Big]-\frac{47}{40}\left(8-e^{-3000\left(-\frac{1}{600}+t\right)}\left(8\,\mathrm{Cos}\left[4000\sqrt{6}\left(-\frac{1}{600}+t\right)\right]+\sqrt{6}\,\mathrm{Sin}\left[4000\sqrt{6}\left(-\frac{1}{600}+t\right)\right]\right)\right)\mathrm{UnitStep}\left[-\frac{1}{600}+t\right]\Bigg)$$

In[45]:= **Plot[vct, {t, 0, 10/3000}]**



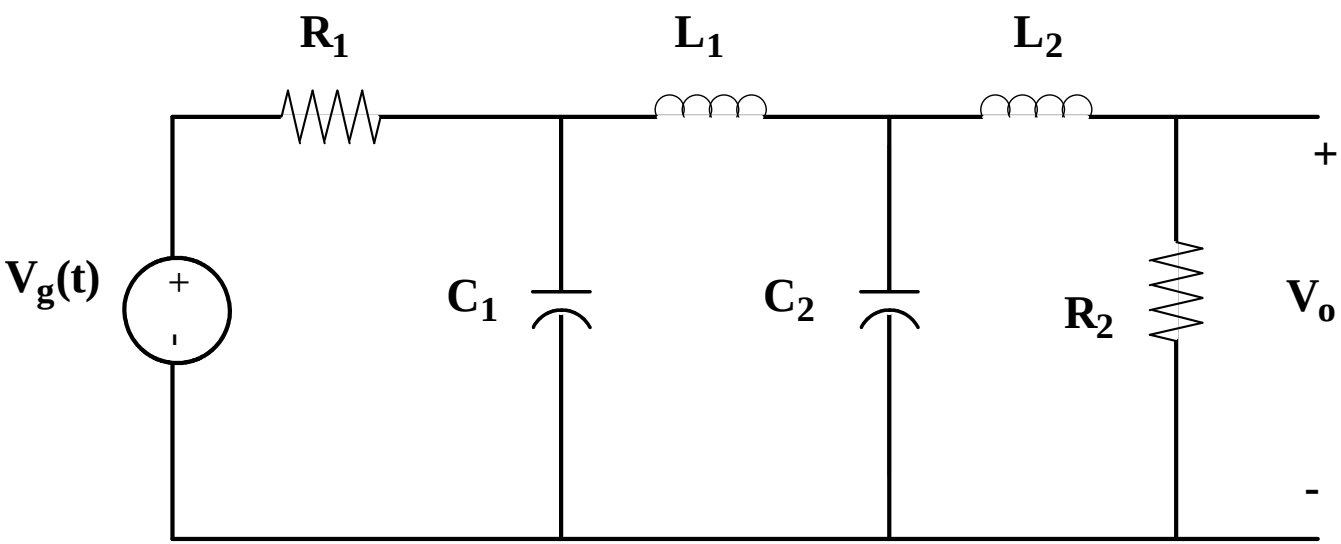


4 Order Butterworth Low Pass Filter

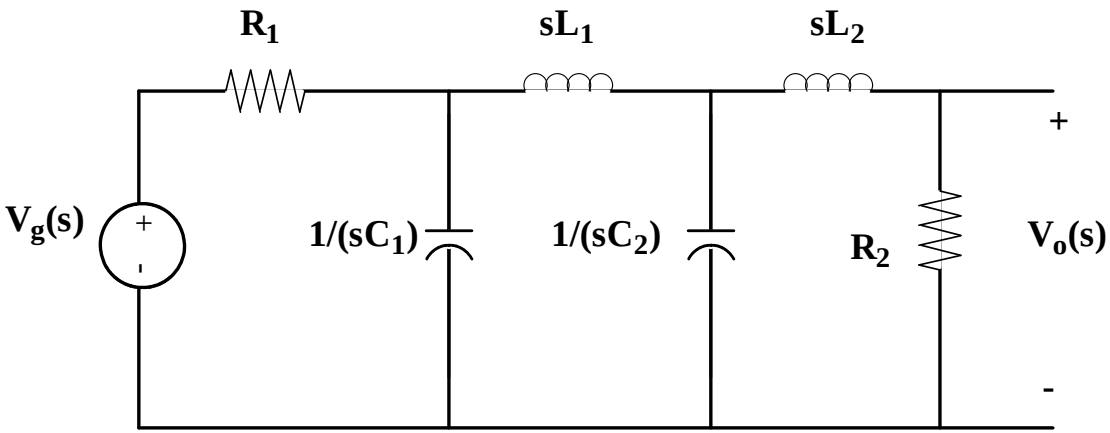
For this circuit

1) Let's find $H(s)=V_o(s)/V_g(s)$ for the general case

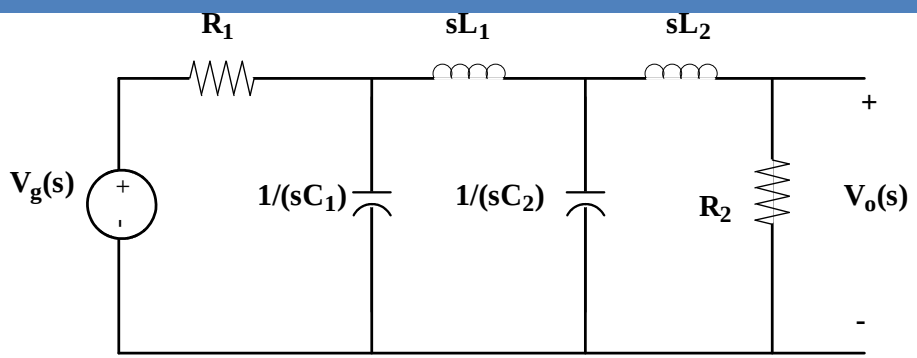
2) Transient Response for a specific case



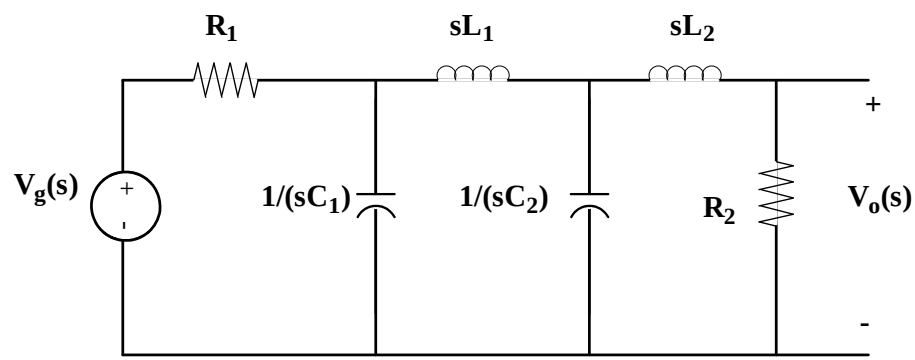
4 Order Butterworth Low Pass Filter: Nodal Analysis



4 Order Butterworth Low Pass Filter: Nodal Analysis



4 Order Butterworth Low Pass Filter: Nodal Analysis



In[51]:= `kcl1 = v1 * (1 / R1 + s * C1 + 1 / (s * L1)) - vg / R1 - v2 / (s * L1)`

Out[51]= $\left(\frac{1}{R1} + \frac{1}{L1\,s} + C1\,s\right) v1 - \frac{v2}{L1\,s} - \frac{vg}{R1}$

In[52]:= `kcl2 = -v1 / (s * L1) + v2 * (1 / (s * L1) + s * C2 + 1 / (s * L2)) - v0 / (s * L2)`

Out[52]= $-\frac{v0}{L2\,s} - \frac{v1}{L1\,s} + \left(\frac{1}{L1\,s} + \frac{1}{L2\,s} + C2\,s\right) v2$

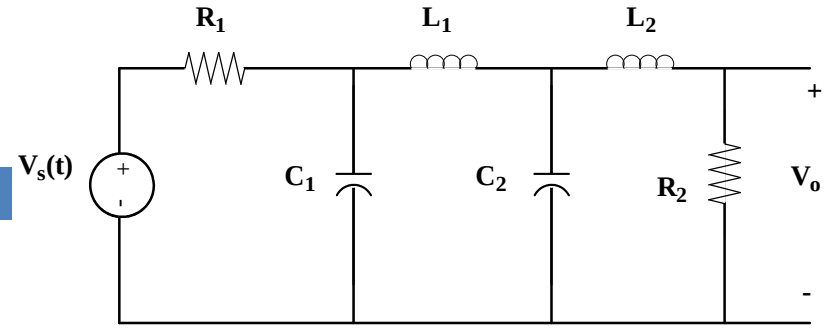
In[53]:= `kcl3 = -v2 / (s * L2) + v0 * (1 / (s * L2) + 1 / R2)`

Out[53]= $\left(\frac{1}{R2} + \frac{1}{L2\,s}\right) v0 - \frac{v2}{L2\,s}$

In[54]:= `Solve[{kcl1 == 0, kcl2 == 0, kcl3 == 0}, {v1, v2, v0}]`

Out[54]= $\{\{v0 \to (R2\,vg) / (R1 + R2 + L1\,s + L2\,s + C1\,R1\,R2\,s + C2\,R1\,R2\,s + C1\,L1\,R1\,s^2 + C1\,L2\,R1\,s^2 + C2\,L2\,R1\,s^2 + C2\,L1\,R2\,s^2 + C2\,L1\,L2\,s^3 + C1\,C2\,L1\,R1\,R2\,s^3 + C1\,C2\,L1\,L2\,R1\,s^4),$
$v1 \to -(-R2\,vg - L1\,s\,vg - L2\,s\,vg - C2\,L1\,R2\,s^2\,vg - C2\,L1\,L2\,s^3\,vg) / (R1 + R2 + L1\,s + L2\,s + C1\,R1\,R2\,s + C2\,R1\,R2\,s + C1\,L1\,R1\,s^2 + C1\,L2\,R1\,s^2 + C2\,L2\,R1\,s^2 + C2\,L1\,R2\,s^2 + C2\,L1\,L2\,s^3 + C1\,C2\,L1\,R1\,R2\,s^3 + C1\,C2\,L1\,L2\,R1\,s^4),$
$v2 \to ((R2 + L2\,s)\,vg) / (R1 + R2 + L1\,s + L2\,s + C1\,R1\,R2\,s + C2\,R1\,R2\,s + C1\,L1\,R1\,s^2 + C1\,L2\,R1\,s^2 + C2\,L2\,R1\,s^2 + C2\,L1\,R2\,s^2 + C2\,L1\,L2\,s^3 + C1\,C2\,L1\,R1\,R2\,s^3 + C1\,C2\,L1\,L2\,R1\,s^4)\}\}$

4 Order Butterworth Low Pass Filter: Nodal Analysis



Answer:

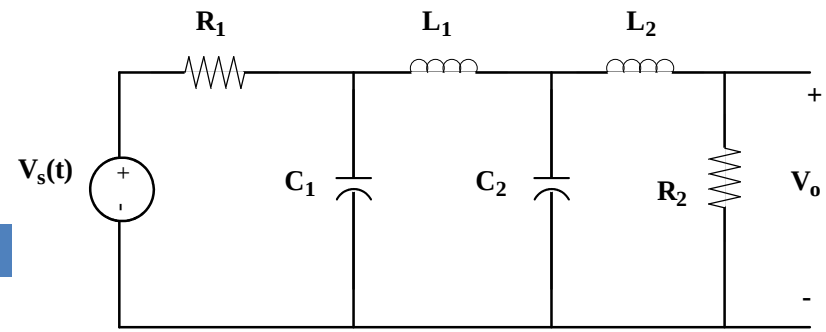
$$H(s)=\frac{R_2}{\left\{\begin{array}{l}\left[C_1C_2L_1L_2R_1\right]s^4+\left[C_2L_1L_2+C_1C_2L_1R_1R_2\right]s^3+\\ \left[C_1R_1\left(L_1+L_2\right)+C_2\left(R_1L_2+R_2L_1\right)\right]s^2+\left[L_1+L_2+\left(C_1+C_2\right)R_1R_2\right]s+\\ \left(R_1+R_2\right)\end{array}\right\}}$$

$$H(s)=\frac{R_2/\left(R_1+R_2\right)}{\left\{\begin{array}{l}\left[\dfrac{C_1C_2L_1L_2R_1}{\left(R_1+R_2\right)}\right]s^4+\left[\dfrac{C_2L_1L_2+C_1C_2L_1R_1R_2}{\left(R_1+R_2\right)}\right]s^3+\left[\dfrac{C_1R_1\left(L_1+L_2\right)+C_2\left(R_1L_2+R_2L_1\right)}{\left(R_1+R_2\right)}\right]s^2+\\ \left[\dfrac{L_1+L_2+\left(C_1+C_2\right)R_1R_2}{\left(R_1+R_2\right)}\right]s+1\end{array}\right\}}$$

$$H(s)=\frac{\left|H(0)\right|}{\left(1-s/p_1\right)\left(1-s/p_2\right)\left(1-s/p_3\right)\left(1-s/p_4\right)}$$

$$\left|H(0)\right|=R_2/\left(R_1+R_2\right)$$

4 Order Butterworth Low Pass Filter: Nodal Analysis



Plug in values: Choose $R_1=R_2=10K\Omega$

Solve Remaining 4 Equations for 4 Unknowns:

$C_1 \approx 243.5pF$ $C_2 \approx 588.2pF$ $L_1 \approx 58.82mH$ $L_2 \approx 24.35mH$ yields

Result using rounded component values:

$$H(\omega) = \frac{1}{(1 - j\omega/p_1)(1 - j\omega/p_2)(1 - j\omega/p_3)(1 - j\omega/p_4)}$$

$$p_{1,2} \approx -120285 \pm j(290226)\, r/s \rightarrow \omega_{n1} = p_1 p_2 \approx 3.14165x10^5\, r/s \approx 50KHz$$

$$p_{3,4} \approx -290393 \pm j(120215)\, r/s \rightarrow \omega_{n2} = p_3 p_4 \approx 50.02KHz$$

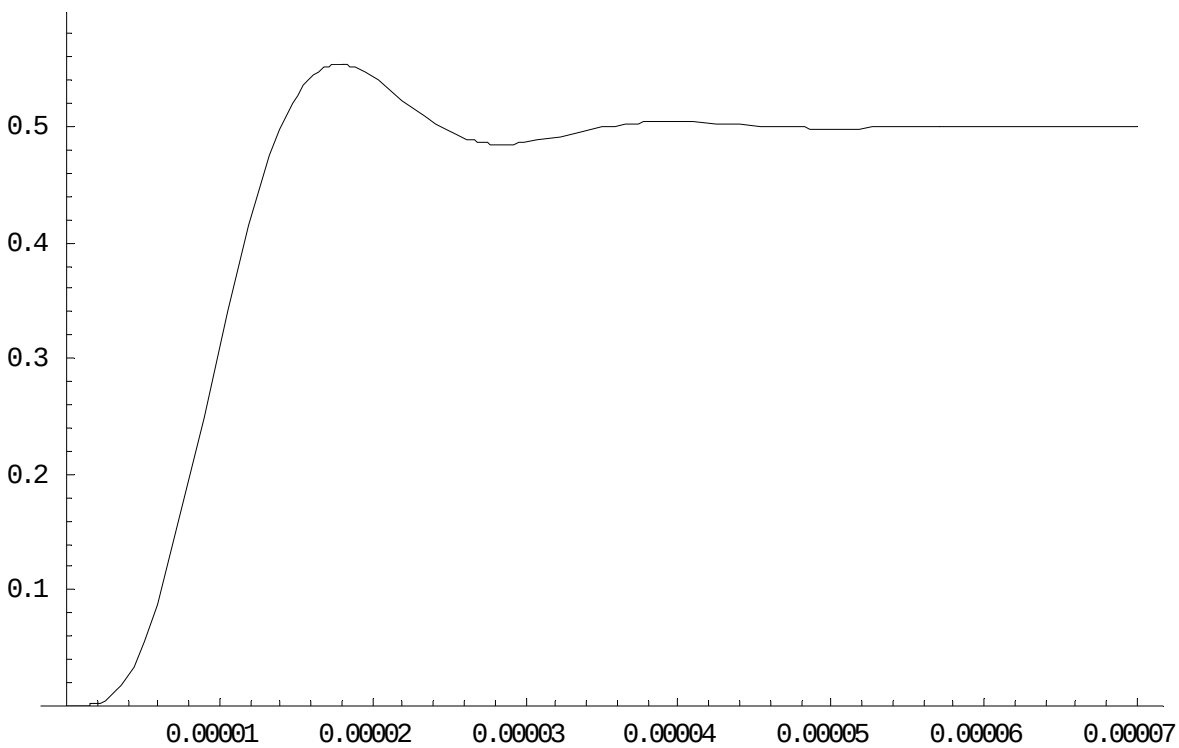
Unit Step Response

In[208]:= **VOS = VGS * F**

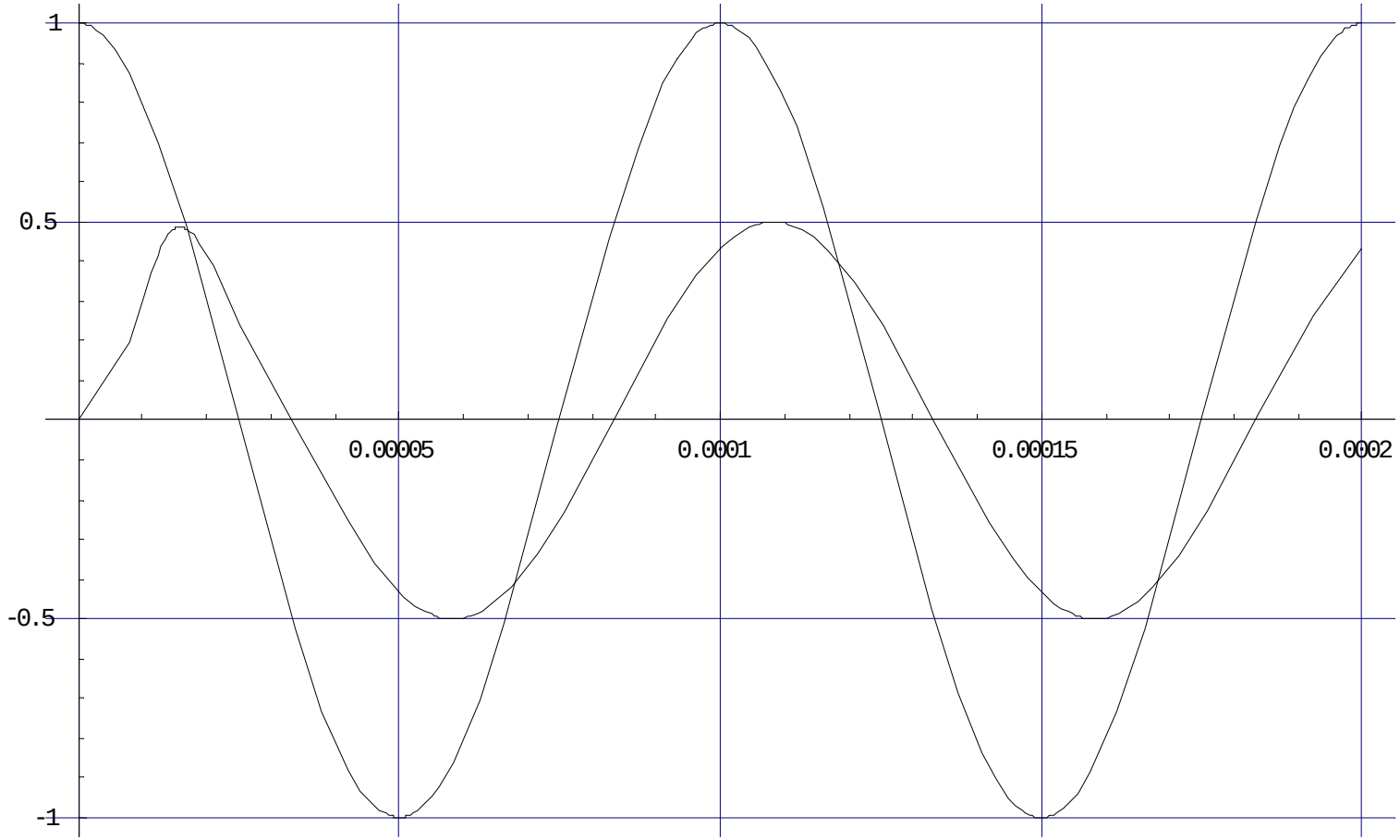
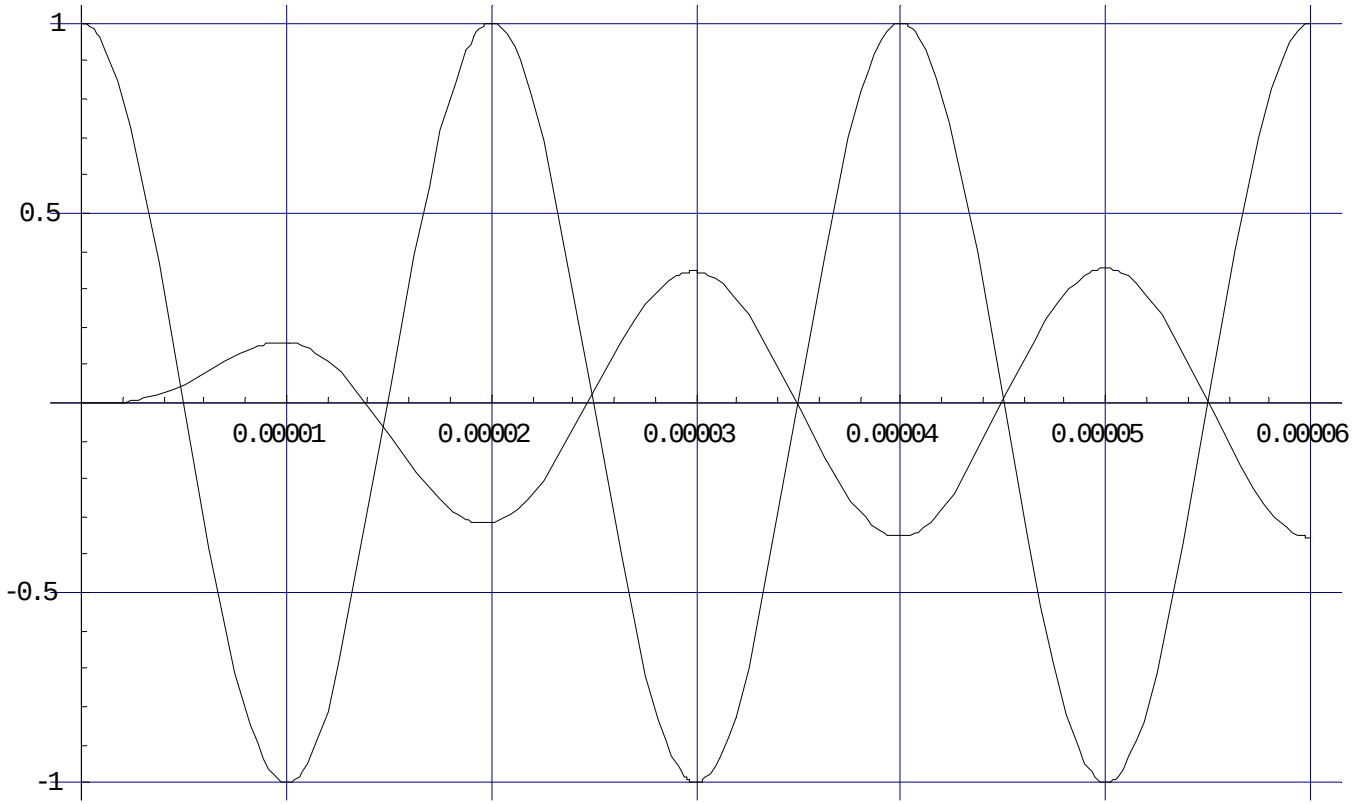
Out[208]=
$$\frac{0.5}{s\left(1-\frac{e^{-\frac{5 i \pi}{8}} s}{100000\pi}\right)\left(1-\frac{e^{\frac{5 i \pi}{8}} s}{100000\pi}\right)\left(1-\frac{e^{-\frac{7 i \pi}{8}} s}{100000\pi}\right)\left(1-\frac{e^{\frac{7 i \pi}{8}} s}{100000\pi}\right)}$$

In[233]:= **VOS1 = Apart[VOS]**

Out[233]=
$$\frac{0.5}{s}+\frac{0.176777+0.176777 i}{(120224.-290245. i)+s}-\frac{0.426777-0.426777 i}{(290245.-120224. i)+s}+\frac{0.0956709+0.23097 i}{-314159.+(-1)^{5/8} s}+\frac{0.557611+0.23097 i}{-314159.+(-1)^{7/8} s}$$



$$v_g(t) = 1\cos(2\pi\, x\, 10{,}000t)\,u(t)$$



$$v_g(t) = 1\cos(2\pi\, x\, 50{,}000t)\,u(t)$$