STEADY-STATE AC ANALYSIS II SINUSOIDS & PHASORS: U&M CHAPTER 7

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AC Steady State Analysis

- Real Sinusoidal Input
  - $V_{in}=V_0\cos(\omega t)$
- Assume Input is Complex

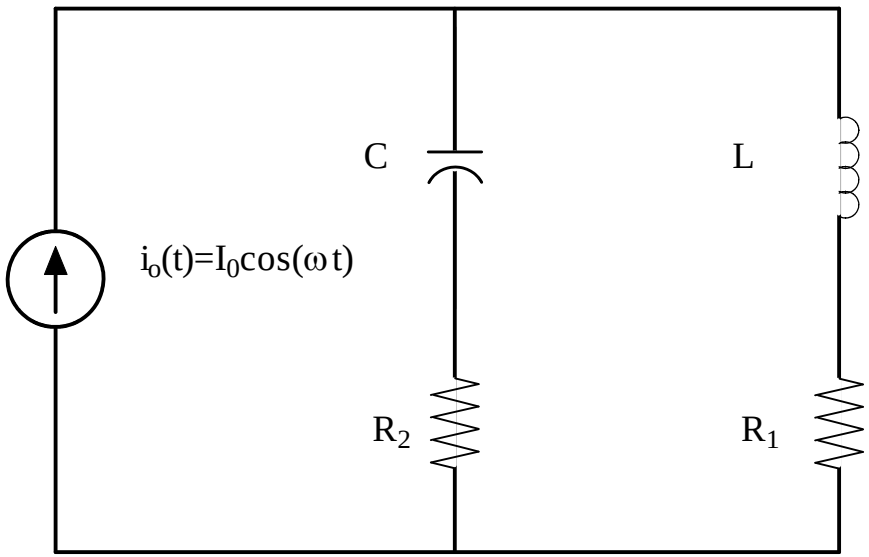
$$\widetilde{V_{in}}(t)=V_0e^{j\omega t}=V_0\left[\cos(\omega t)+j\sin(\omega t)\right]$$

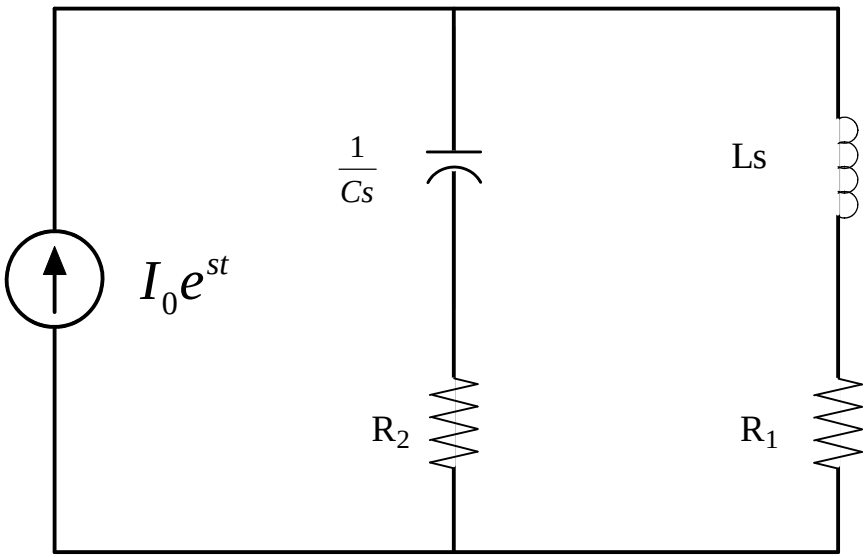
$$\mathrm{Re}\,al\ Input\ Signal$$

$$V_{in}(t)=\mathrm{Re}\left[\widetilde{V_{in}}(t)\right]$$

- By Superposition, ALL Circuit Quantities of Interest (voltages, currents) are given by taking the Real Part of the Response to the Complex Signal

Example: General RLC Problem: Get $v_c(t)$ steady state





Later, we will use the following numerical values for this circuit:

$L = 100nH \quad C = 0.01nF = 10\,pF$

$R_1 = 10\Omega \;\; R_2 = 10\Omega$

$I_0 = 1A \quad \omega = 10^9\, s^{-1}$

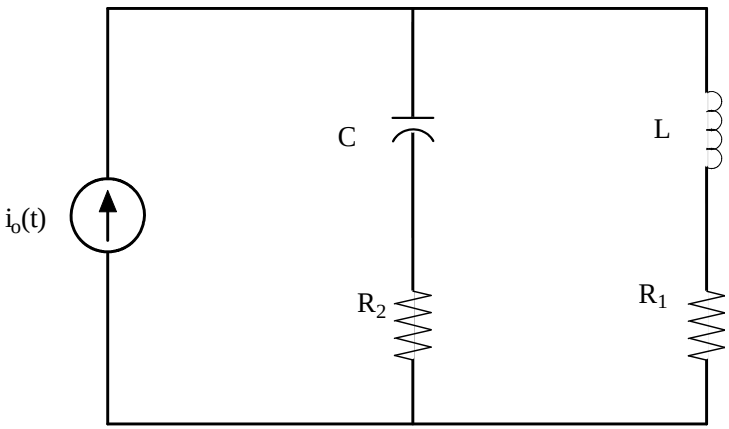
Final Result

$$\widetilde{V_C} = H(j\omega)\widetilde{i_0}(t)$$

$$H(s) = \frac{\widetilde{V_C}(t)}{\widetilde{i_0}(t)} = \left[\frac{sL + R_1}{\left(1 + s^2 LC\right) + s\left(R_1 + R_2\right)C}\right]$$

To get $H(j\omega)$ just take $s = j\omega$

$$H(j\omega) = H(\omega) = \left[\frac{j\omega L + R_1}{\left(1 - \omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C}\right]$$



H(**s**) is called a ***transfer function***

Numerical Example

For the example circuit, suppose:

$$L = 100nH \quad C = 0.01nF = 10\,pF \quad R_1 = 10\Omega \quad R_2 = 10\Omega \quad I_0 = 1A \quad \omega = 10^9\ s^{-1}$$

Find $V_c(t)$

$$H(\omega) = \left[\frac{j\omega L + R_1}{\left(1-\omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C}\right] = \left[\frac{j100\Omega + 10\Omega}{(1-1) + j0.2}\right] = \left(500 - 50j\right)\Omega$$

$$V_c(t) = \mathrm{Re}\left[\left(500-50j\right)\Omega \cdot 1e^{j10^9 t} A\right] = \mathrm{Re}\left[\left(500-50j\right)\left(\cos(10^9 t) + j\sin(10^9 t)\right)V\right]$$

$$V_c(t) = \left[500\cos(10^9 t) + 50\sin(10^9 t)\right]V$$

$$V_c(t) = |H(\omega)| \cdot |i_0| \cos\left(10^9 t + \measuredangle H(\omega)\right)$$

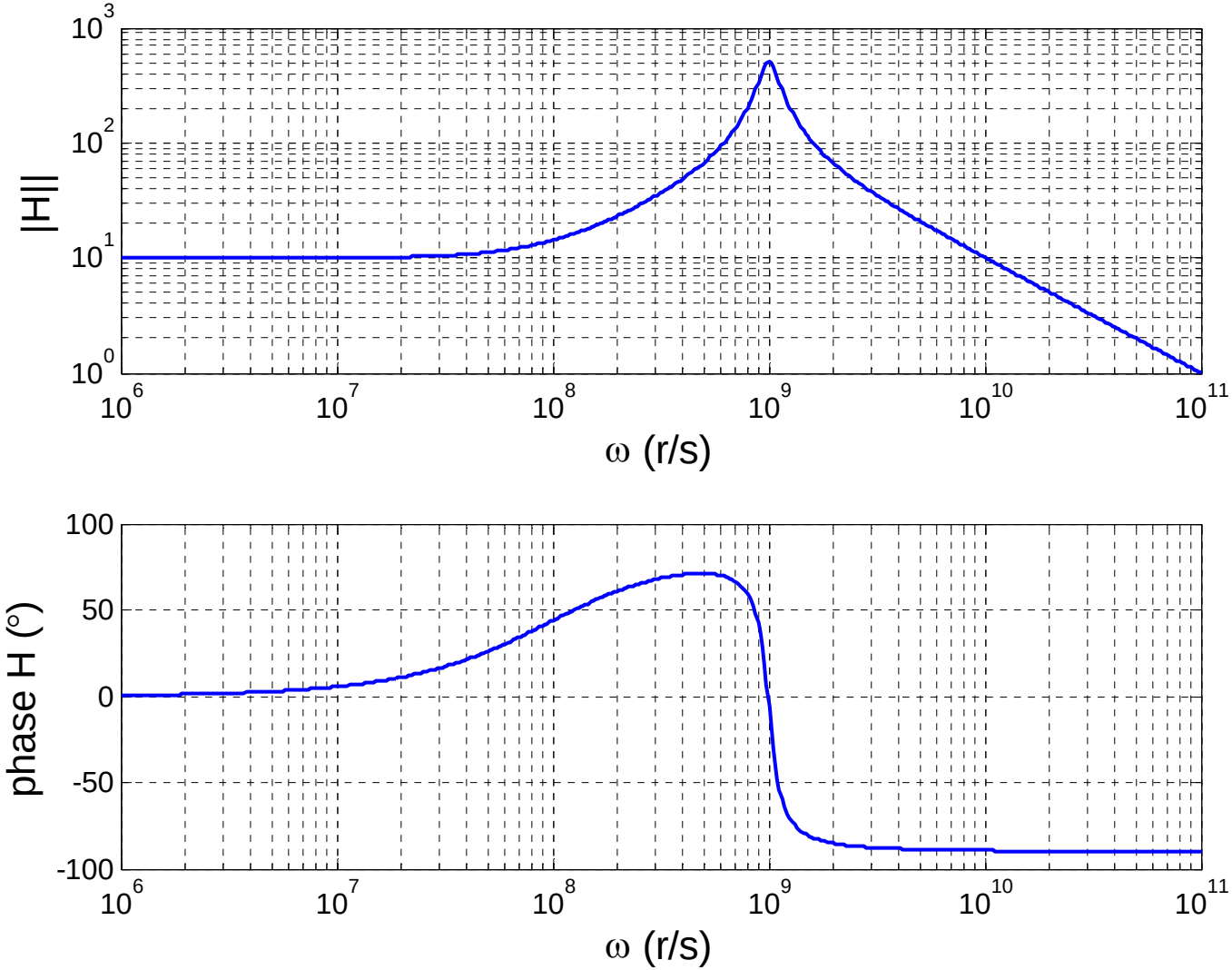
$$|H(\omega)| = \left(500^2 + 50^2\right)^{1/2} \approx 502.494\Omega$$

$$\measuredangle H(\omega) = \tan^{-1}\left(\frac{-50j}{500}\right) \approx -5.7106°$$

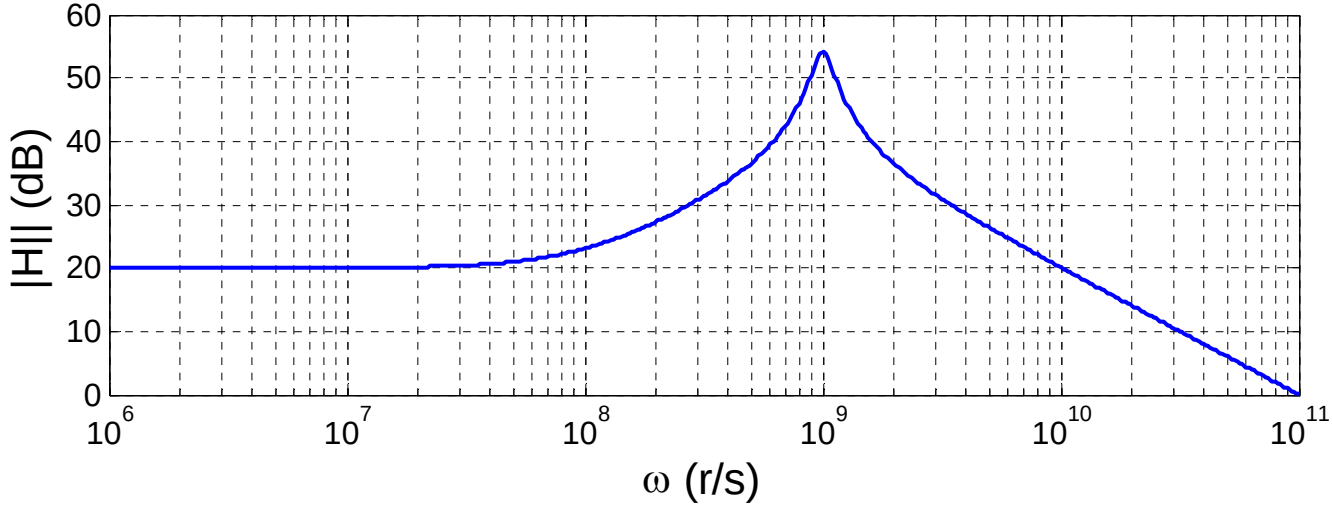
Key Result for Linear Circuits/Systems

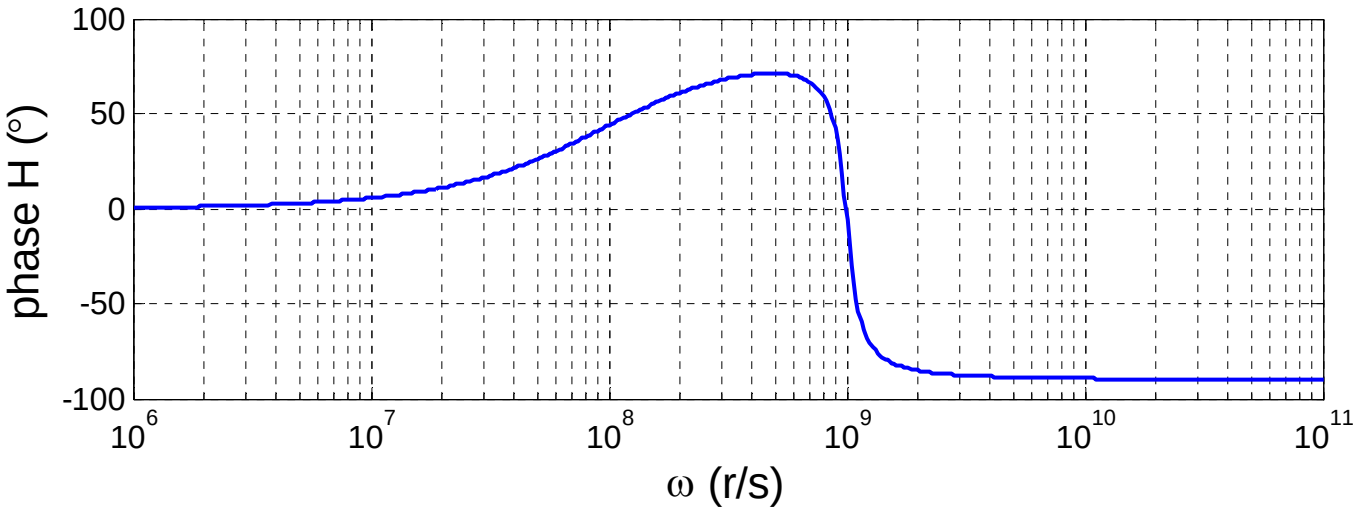
- Magnitude and Phase of Transfer Function Determine the AC Steady State Response
- Input and Output Frequencies are the same

$$\text{output(t)}=|H(\omega)|\cdot|input|\cos\left(\omega_{input}\,t+\measuredangle H(\omega)+\measuredangle input\right)$$


|H|
10^3
10^2
10^1
10^0
10^6
10^7
10^8
10^9
10^{10}
10^{11}
ω (r/s)
phase H (°)
100
50
0
-50
-100
10^6
10^7
10^8
10^9
10^{10}
10^{11}
ω (r/s)

$$h_{db} = 20\log_{10}\left(\left|H\right|\right)$$





AC Steady State Analysis II

- <u>For a Linear Circuit: If Input Signal is at frequency ω, All Circuit Responses are at the same frequency ω</u>
  - Outputs are Different only in Magnitude and Phase, not Frequency
  - Non-Linearities Produce Harmonics (2ω, 3ω, etc.)
- Multiple Frequency Problems Are Handled by Superposition
- When Working a Single Frequency Problem, Issue is Finding Magnitudes and Phases not Frequency
  - Concept of Phasor

Complex Numbers

We will find it is useful to alternatively represent sinusoids as a complex numbers

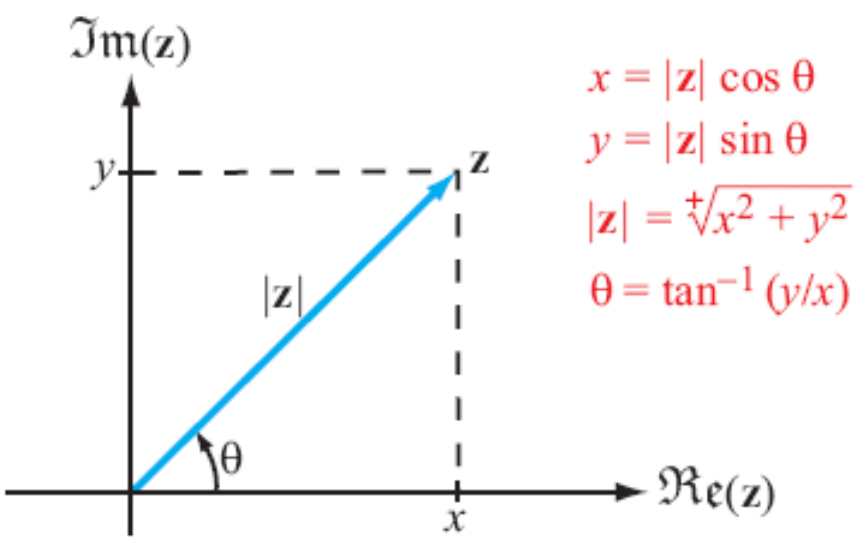
$$j = \sqrt{-1}$$

$$z = x + jy$$ **Rectangular coordinates**

$$z = |z|\angle\theta = |z|e^{j\theta}$$ **Polar coordinates**

$$\mathrm{Re}(z) = x$$

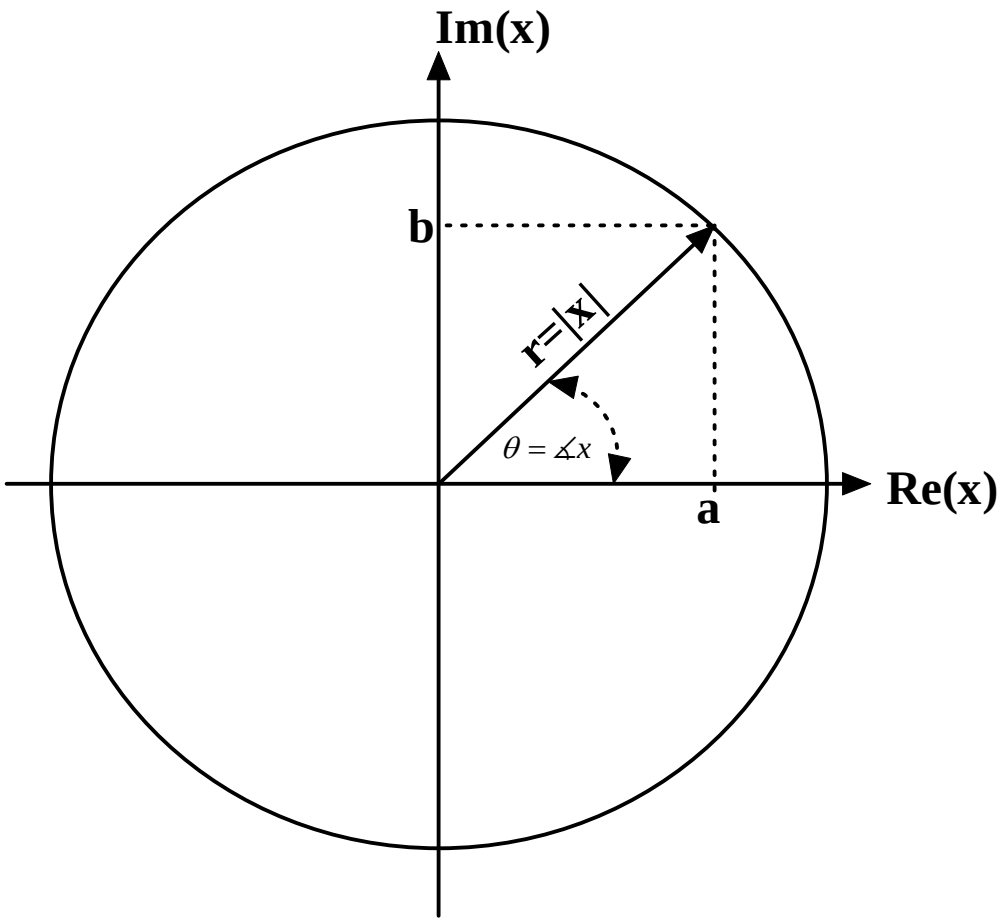
$$\mathrm{Im}(z) = y$$



Relations based on Euler Identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Basic Complex Numbers: Polar/Rectangular Representation



Relations for Complex Numbers

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$	
$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$	$\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$
$x = \Re\mathfrak{e}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$	$\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$
$y = \Im\mathfrak{m}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$	$\theta = \tan^{-1}(y/x)$
$\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$
$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
$\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$
$-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm 180^\circ$	
$j = e^{j\pi/2} = 1\angle 90^\circ$	$-j = e^{-j\pi/2} = 1\angle -90^\circ$
$\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$

Learn how to perform these with *your* calculator/computer

Basic Complex Math

$$e^{j(\omega t+\phi)} = \cos(\omega t+\phi) + j\sin(\omega t+\phi) \qquad \left|e^{j(\omega t+\phi)}\right| = 1$$

If $\boldsymbol{z_1}, \boldsymbol{z_2}$ *are* ***complex numbers*** *with*

$$\boldsymbol{z_1} = r_1\measuredangle\phi_1 = x_1 + jy_1 = r_1 e^{j\phi_1} \qquad \boldsymbol{z_2} = r_2\measuredangle\phi_2 = x_2 + jy_2 = r_2 e^{j\phi_2}$$

Addition:

$$(\boldsymbol{z_1}+\boldsymbol{z_2}) = (x_1+x_2) + j(y_1+y_2) \neq \textit{Any Easy Combination of } r_1\measuredangle\phi_1, r_2\measuredangle\phi_2$$

Subtraction:

$$(\boldsymbol{z_1}-\boldsymbol{z_2}) = (x_1-x_2) + j(y_1-y_2) \neq \textit{Any Easy Combination of } r_1\measuredangle\phi_1, r_2\measuredangle\phi_2$$

Multiplication:

$$\boldsymbol{z_1}\boldsymbol{z_2} = r_1 r_2\measuredangle(\phi_1+\phi_2)$$

Division:

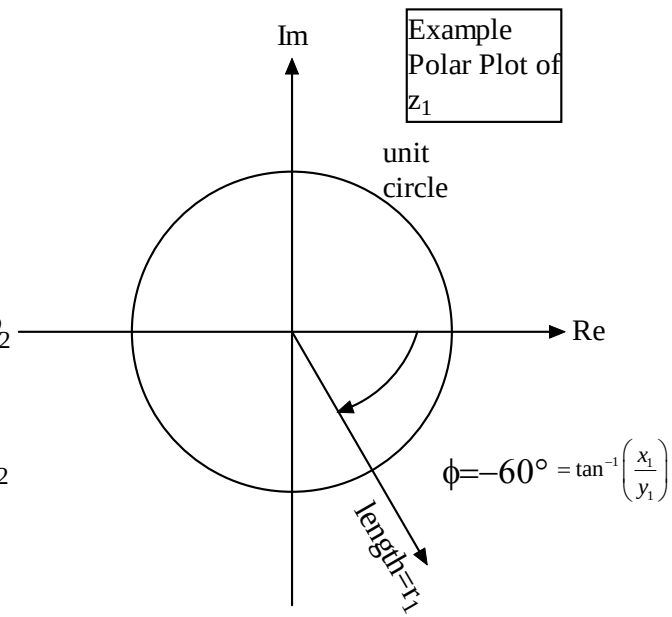
$$(\boldsymbol{z_1}/\boldsymbol{z_2}) = (r_1/r_2)\measuredangle(\phi_1-\phi_2)$$

Powers:

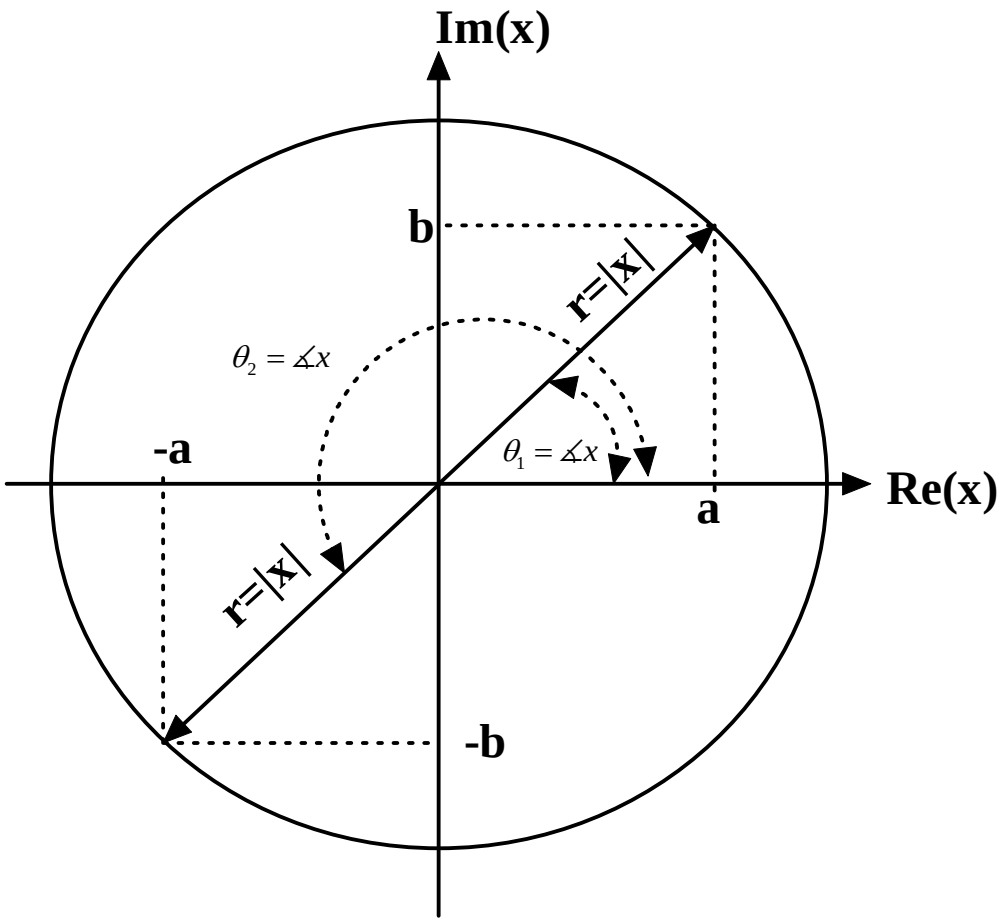
$$(\boldsymbol{z_1})^m = r_1^m\measuredangle(m\phi_1)$$

Complex Conjugate:

$$\boldsymbol{z_1}^* = r_1 e^{-j\phi_1} = r_1\measuredangle(-\phi_1) = x_1 - jy_1$$



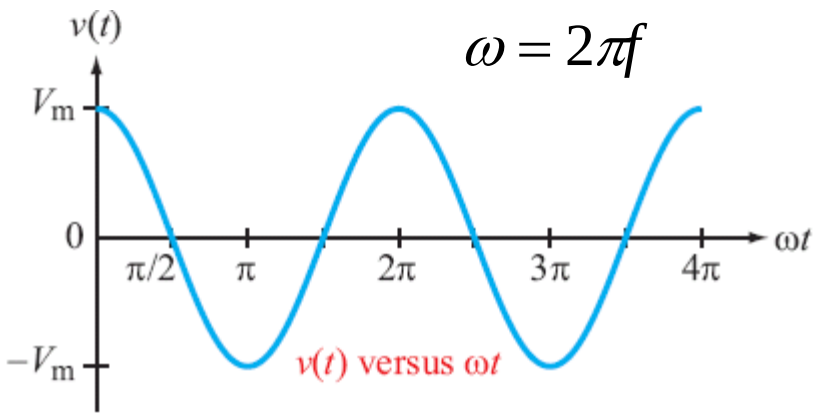
Principle Value of $\tan^{-1}$ Issue



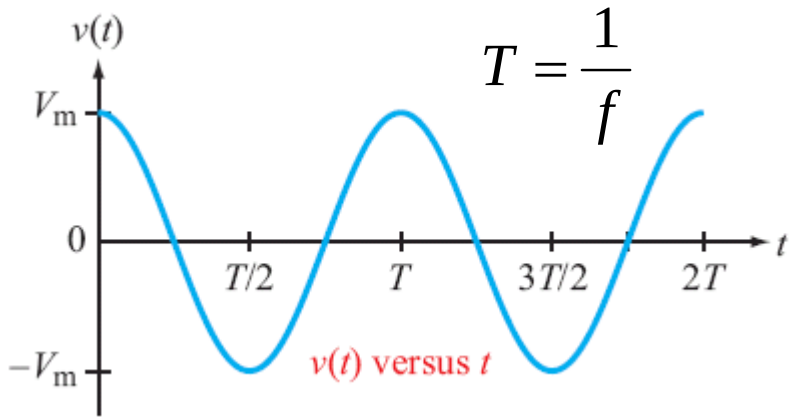
Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \phi)$$

$$\omega = 2\pi f$$



$$T = \frac{1}{f}$$



Useful relations: All Straightforward to Derive from Euler's relation

$$\sin x = \pm\cos(x \mp 90°)$$
$$\cos x = \pm\sin(x \pm 90°)$$
$$\sin x = -\sin(x \pm 180°)$$
$$\cos x = -\cos(x \pm 180°)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

Example: sin(x)sin(y)

$$\sin(x)\sin(y) = \left(\frac{e^{jx} - e^{-jx}}{2j}\right)\left(\frac{e^{jy} - e^{-jy}}{2j}\right)$$

$$= -\frac{1}{4}\left[e^{j(x+y)} - e^{j(x-y)} - e^{j(y-x)} + e^{-j(x+y)}\right]$$

$$= -\frac{1}{4}\left[\left(e^{j(x+y)} + e^{-j(x+y)}\right) - \left(e^{j(x-y)} + e^{j(y-x)}\right)\right]$$

$$= -\frac{1}{4}\left[2\cos(x+y) - 2\cos(x-y)\right]$$

$$= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

Phasors

Input of Form:

$$\widetilde{V_{in}} = \left|V_{in}\right| e^{j\omega t}$$

Output(s) will be of form:

$$\widetilde{V_{out}} = \left|V_{out}\right| e^{j(\omega t+\phi)} = \underbrace{\left(\left|V_{out}\right| e^{j\phi}\right)}_{\substack{Phasor\ of\ V_{out}, \\ Magnitude \& Phase}} e^{j\omega t} = \boldsymbol{V_{out}} e^{j\omega t}$$

$$V_{out} = \mathrm{Re}\left[\widetilde{V_{out}}\right] = \left|V_{out}\right| \cos\left(\omega t + \phi\right)$$

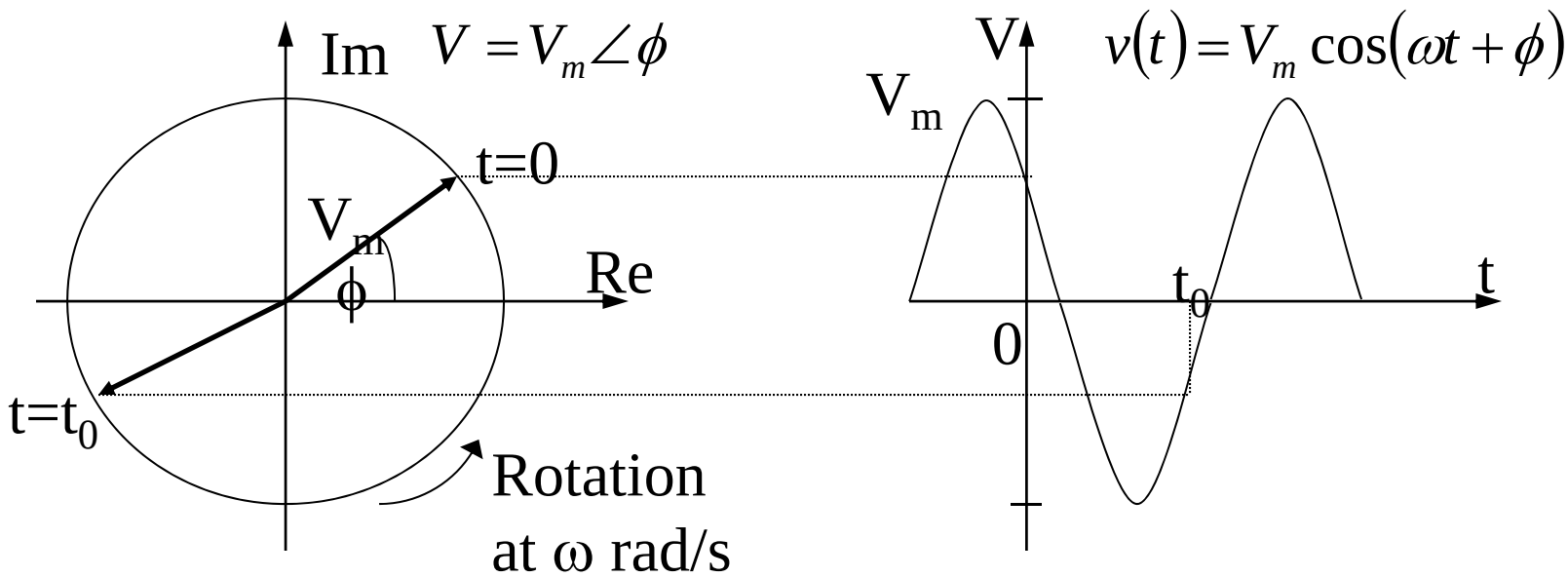
Phasor written as:

$$\boldsymbol{V_{out}} = \left|V_{out}\right| \measuredangle \phi$$

Phasor Domain and Time Domain

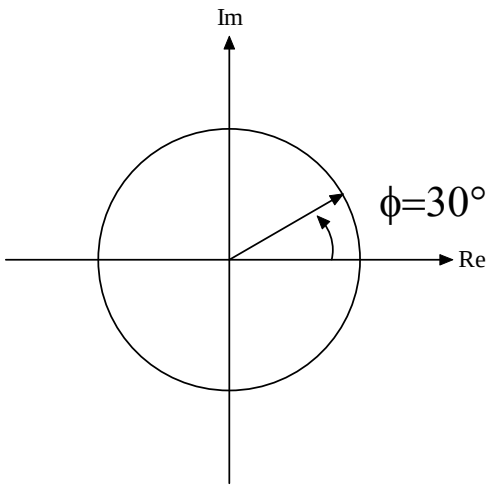
Phasor Domain

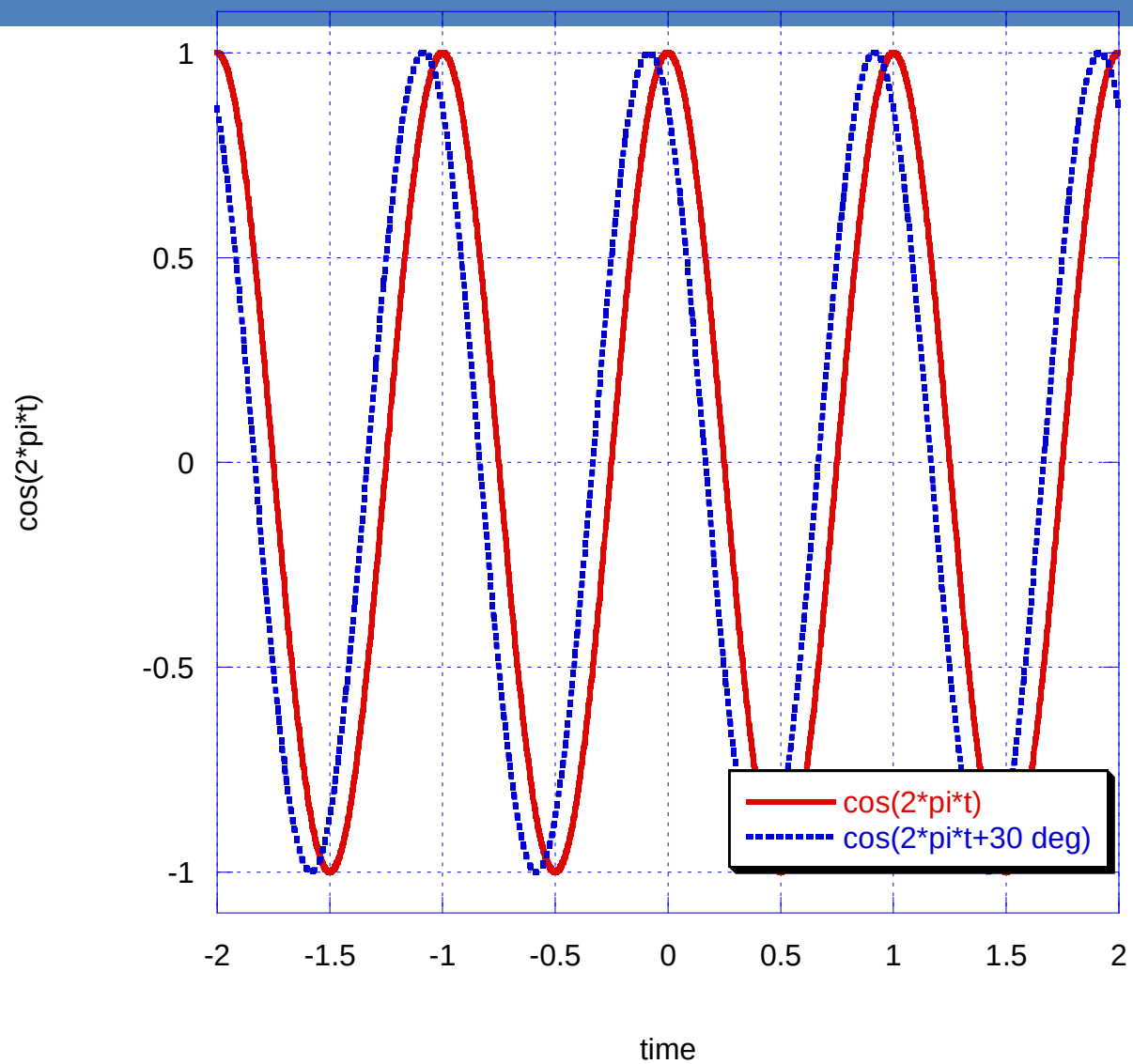
Time Domain



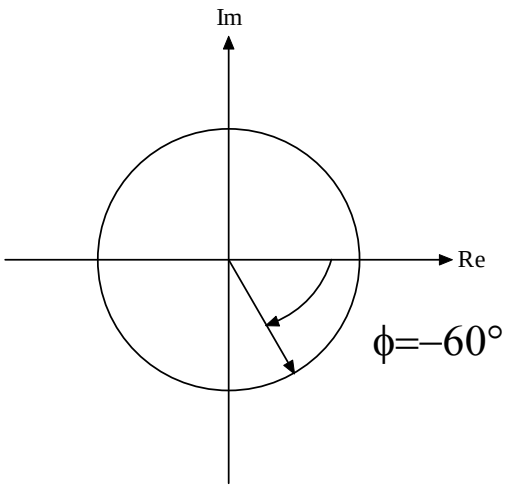
- For input frequency at ω, all circuit response will be at ω
- Only thing we need is magnitude and phase
- Phasors conveniently track magnitude and phase info

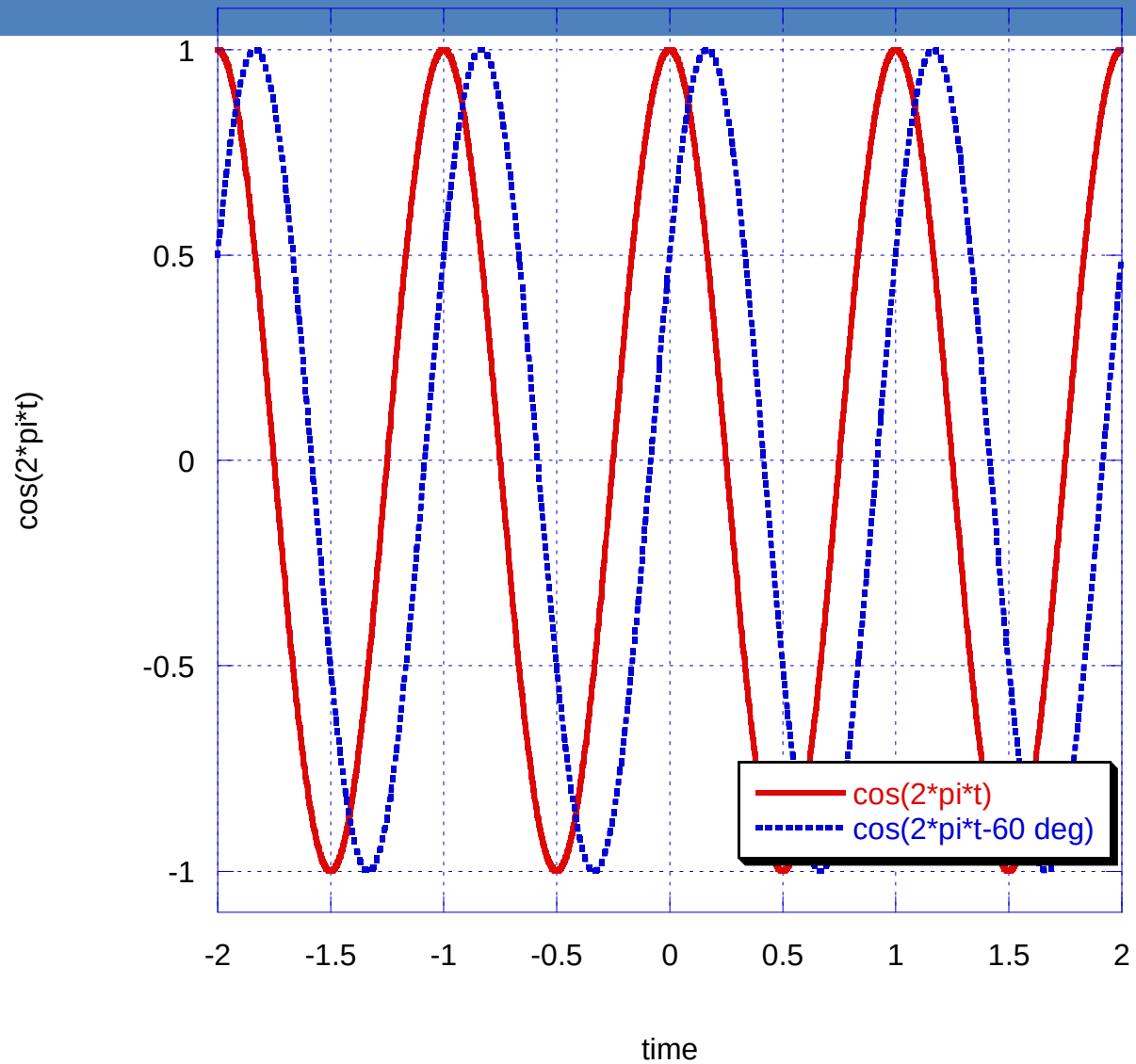
Phase/Phasor Plot: $\phi=30°=\pi/6$ Signal Advanced



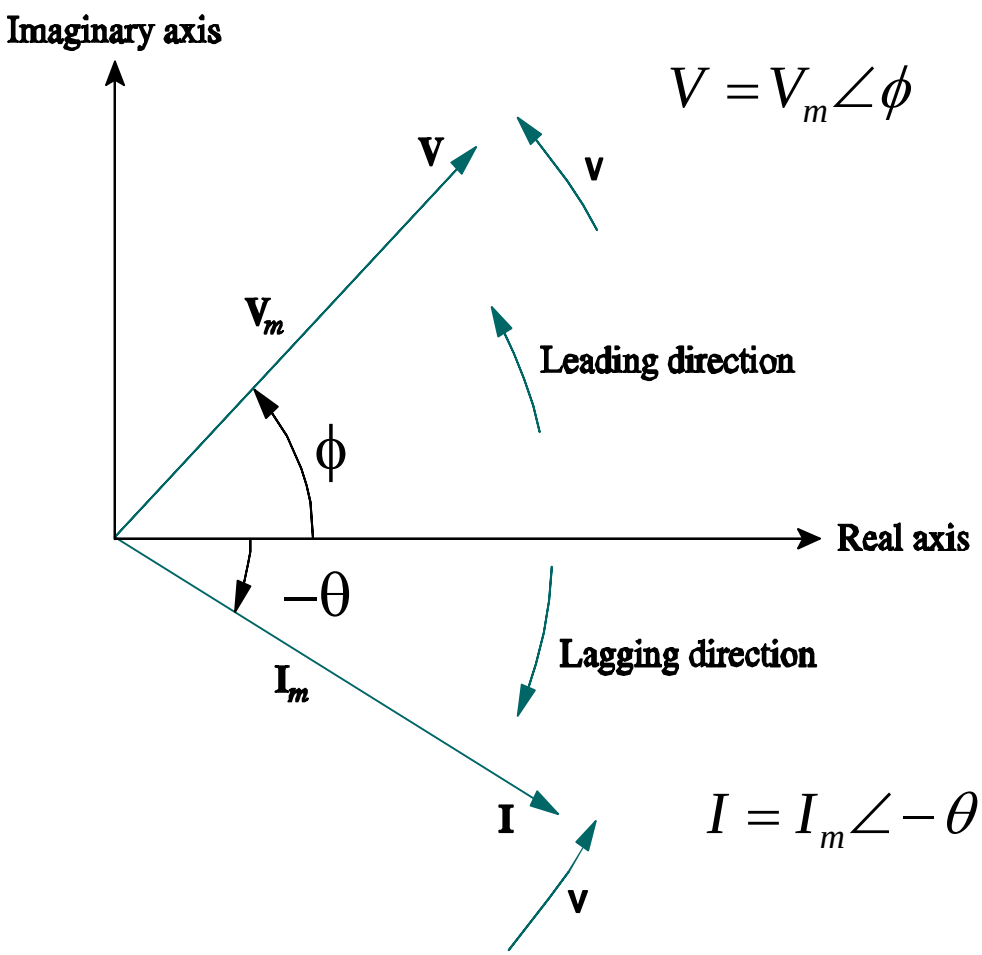


Phase/Phasor Plot: ϕ=-60°=-π/3 Signal Delayed or Retarded



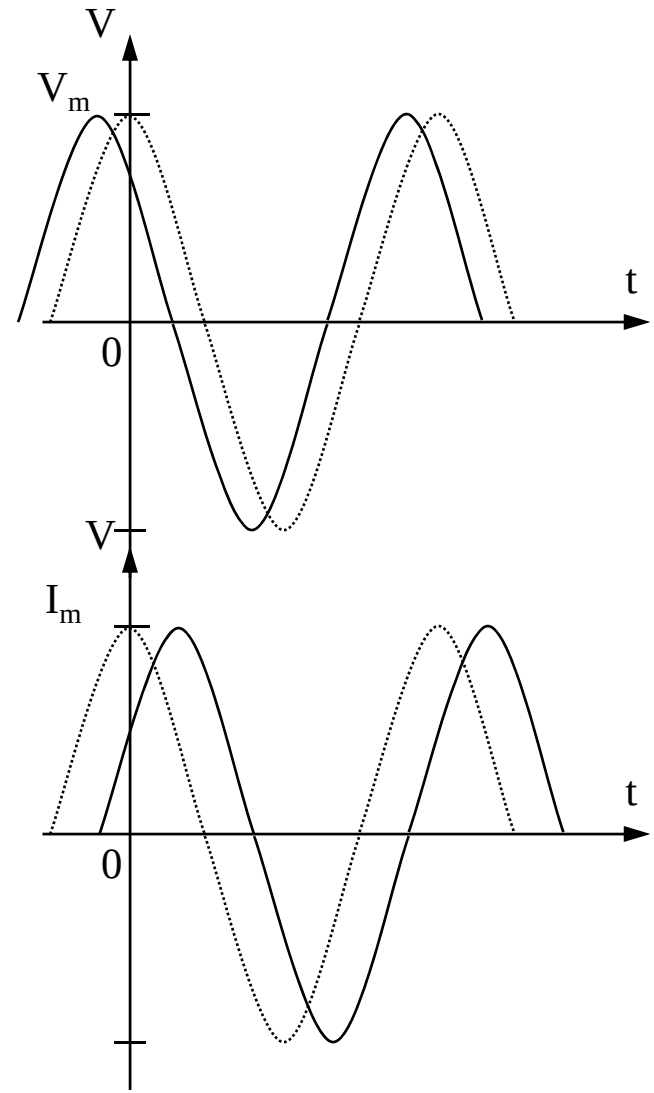


Leading and Lagging



$$V = V_m \angle \phi$$

$$I = I_m \angle -\theta$$



Capacitor in jω-Domain

$$i_c = C\frac{dV_c}{dt}$$

$$\widetilde{V_c} = |V_c| e^{j\omega t}$$

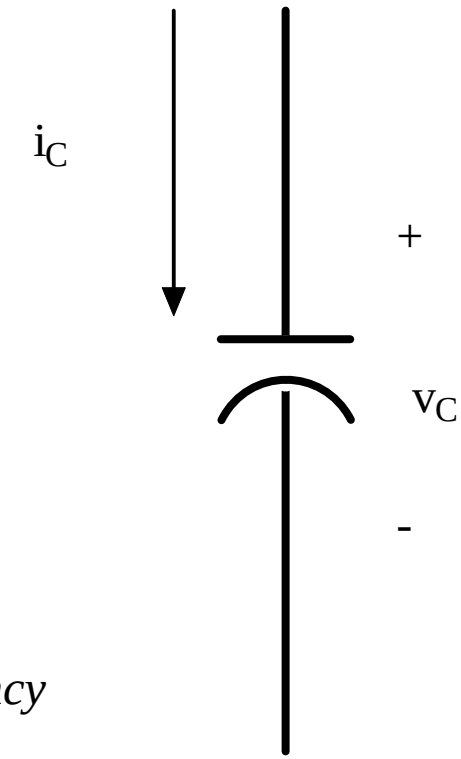
$$\widetilde{i_c} = (j\omega C)\widetilde{V_c} = \left(\omega C e^{j\pi/2}\right)\widetilde{V_c}$$ Current leads voltage by $\pi/2$

$$Z_c = \left(\frac{1}{j\omega C}\right)$$ ***Impedance*** (Ω)

$$Y_c = (j\omega C)$$ ***Admittance*** $(\Omega^{-1} = S)$

$$Z_c \to \infty \text{ as } \omega \to 0 \quad \textit{Capacitor acts like open at DC}$$

$$Z_c \to 0 \text{ as } \omega \to \infty \quad \textit{Capacitor acts like short at High Frequency}$$



Inductor in jω -Domain

$$V_L = L\frac{di_L}{dt}$$

$$\tilde{i}_L = |i_L| e^{j\omega t}$$

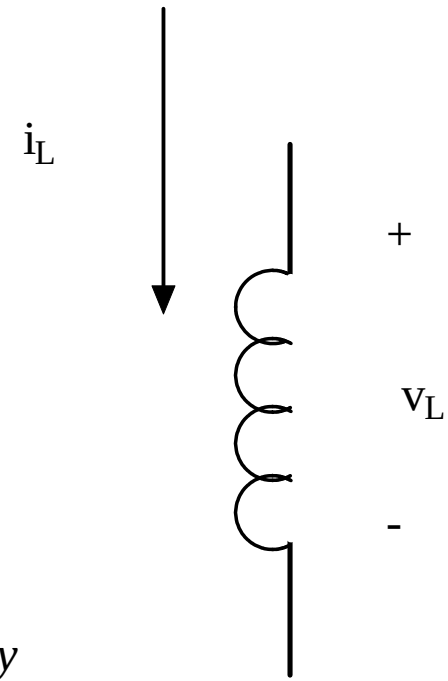
$$\tilde{V}_L = (j\omega L)\tilde{i}_L = \left(\omega L e^{j\pi/2}\right)\tilde{i}_L$$ Voltage leads current by $\pi/2$

$$Z_L = (j\omega L)\ \textbf{\textit{Impedance}}\ (\Omega)$$

$$Y_L = \left(\frac{1}{j\omega L}\right)\ \textbf{\textit{Admittance}}\ (\Omega^{-1} = S)$$

$$Z_L \to 0\ as\ \omega \to 0\quad Inductor\ acts\ like\ short\ at\ DC$$

$$Z_L \to \infty\ as\ \omega \to \infty\quad Inductor\ acts\ like\ open\ at\ High\ Frequency$$



AC Steady State Analysis Outline

- Treat Sinusoidal Signals as Complex Exponentials
- Analyze Circuit Response Using Nodal, Mesh, *Whatever* Using Complex Impedances
- Find Desired Responses in Complex Form
- Take the Real Part to Get Physical Result
- *Wonderful Thing Is That This Approach Is Easier*
- KCL, KVL, and Derived Impedance Combinations Rules (Series, Parallel, Wye-Delta) Work with Complex Impedances

Phasor Domain

Convert time domain sinusoidal signal to a complex number in frequency domain – simplifies math!

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}\left(V_m e^{j\omega t} e^{j\phi}\right) = \mathrm{Re}\left(V e^{j\omega t}\right)$$

$$V = V_m e^{j\phi} = V_m \angle \phi = \textit{phasor representation}$$

Time Domain		Phasor Domain
$v(t) = V_0 \cos \omega t$	$\Longleftrightarrow$	$\mathbf{V} = V_0$
$v(t) = V_0 \cos(\omega t + \phi)$	$\Longleftrightarrow$	$\mathbf{V} = V_0 e^{j\phi}$.

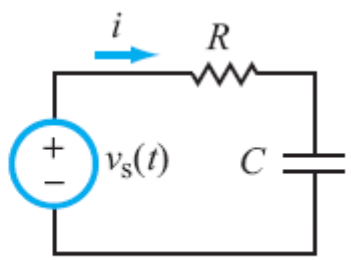
If $\phi = -\pi/2$,

$$v(t) = V_0 \cos(\omega t - \pi/2) \Longleftrightarrow \mathbf{V} = V_0 e^{-j\pi/2}.$$

AC Phasor Analysis General Procedure

Using this procedure, we can apply our techniques from DC Analysis

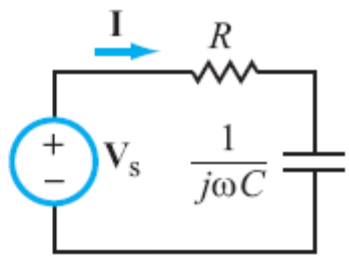
Step 1
Adopt Cosine Reference
(Time Domain)



$v_s(t) = 12\sin(\omega t - 45°)$

Step 2
Transfer to Phasor Domain

$i \rightarrow \mathbf{I}$
$v \rightarrow \mathbf{V}$
$R \rightarrow \mathbf{Z}_R = R$
$L \rightarrow \mathbf{Z}_L = j\omega L$
$C \rightarrow \mathbf{Z}_C = 1/j\omega C$



$\mathbf{V}_s = 12e^{-j135°}$ (V)

Step 3
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

Step 4
Solve for Unknown Variable
(Phasor Domain)

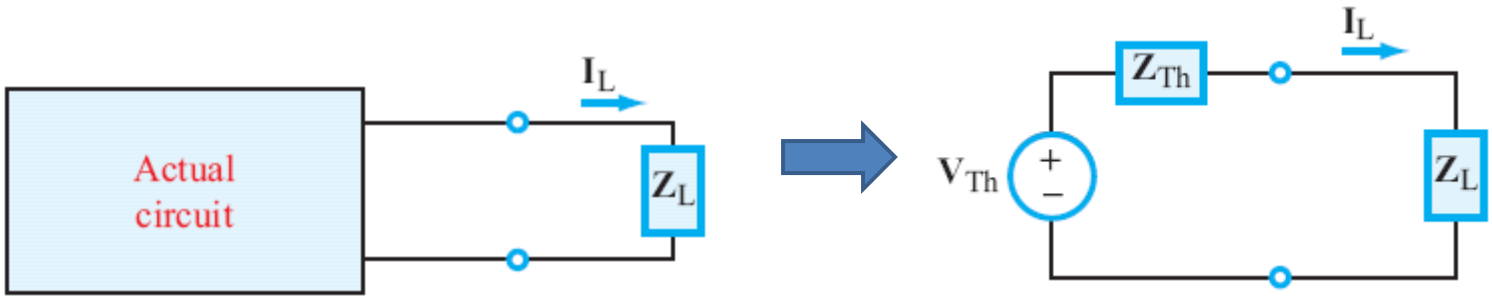
$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

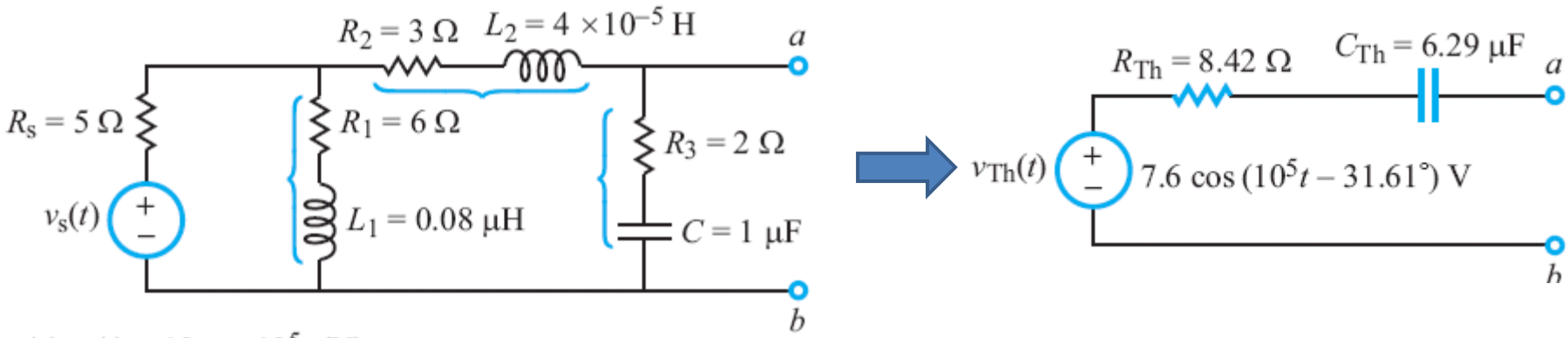
Step 5
Transform Solution
Back to Time Domain

$$i(t) = \Re\mathfrak{e}[\mathbf{I}e^{j\omega t}] = 6\cos(\omega t - 105°) \text{ (mA)}$$

Linear Circuit Properties

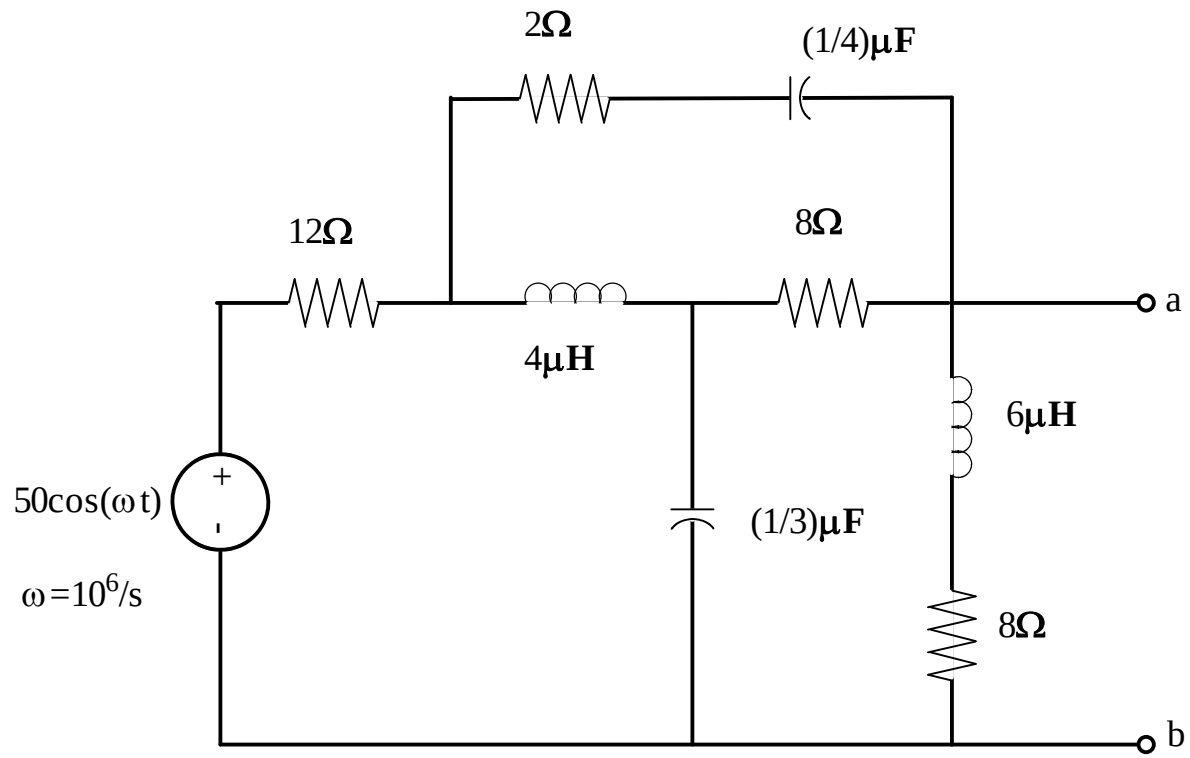
Thevenin/Norton, Source Transformation Also Valid



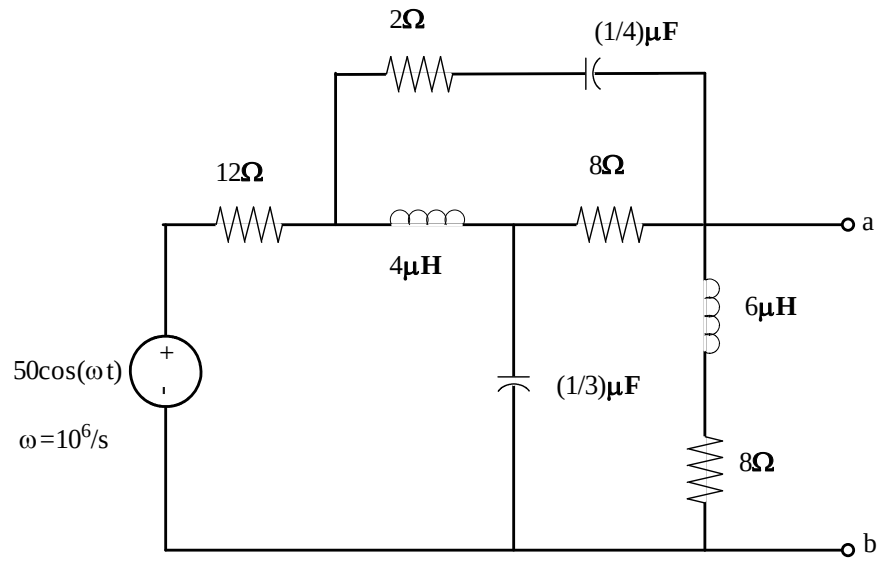


(a) $v_s(t) = 10 \cos 10^5 t$ (V)

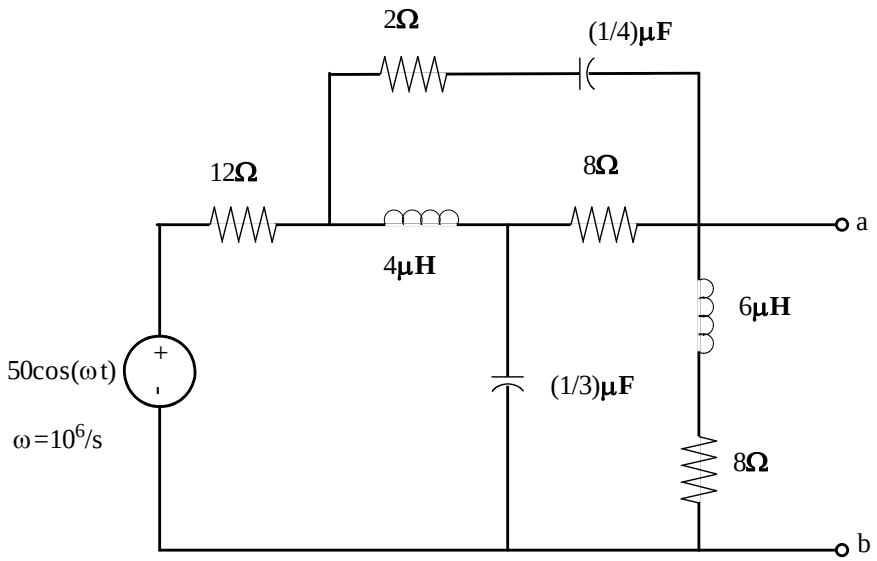
Steady State AC Analysis II



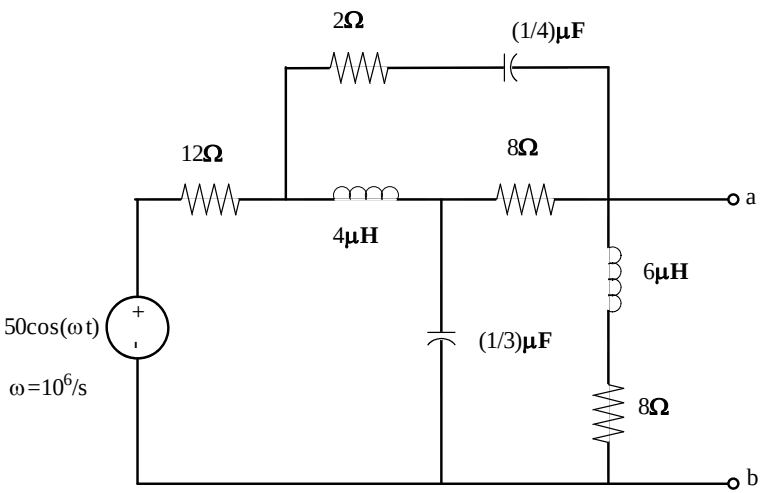
Steady State AC Analysis II



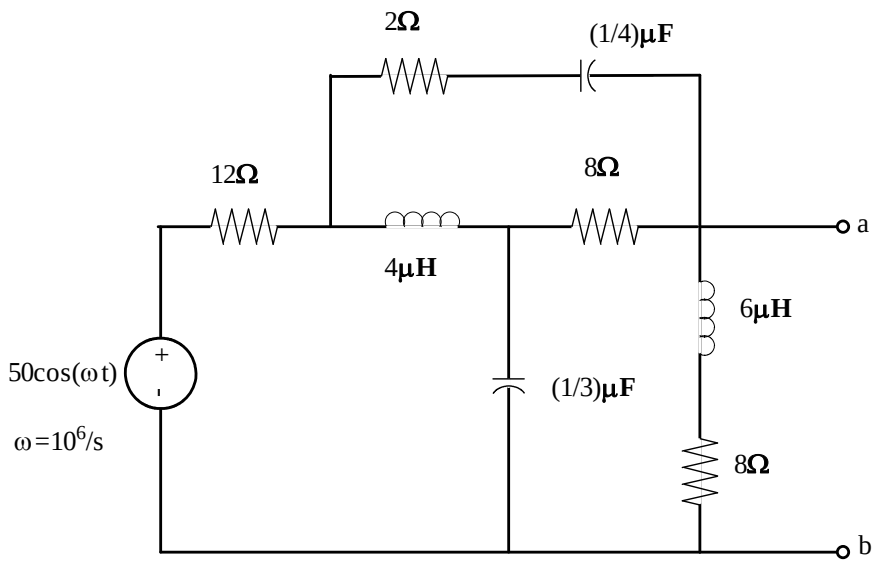
Steady State AC Analysis II



Steady State AC Analysis II



Steady State AC Analysis II



Answers

v1oc =6.1311 + 3.2209i

v2oc =-1.0086 -10.8869i

v3oc =-0.1925 + 0.7030i

VTH =-0.1925 + 0.7030i

Magnitude	Phase(r)	Phase(degrees)
0.7289	1.8381	105.3167

v1sc =6.1422 + 3.2570i

v2sc =-1.1600 -11.5110i

ISC =-0.1822 + 0.1153i

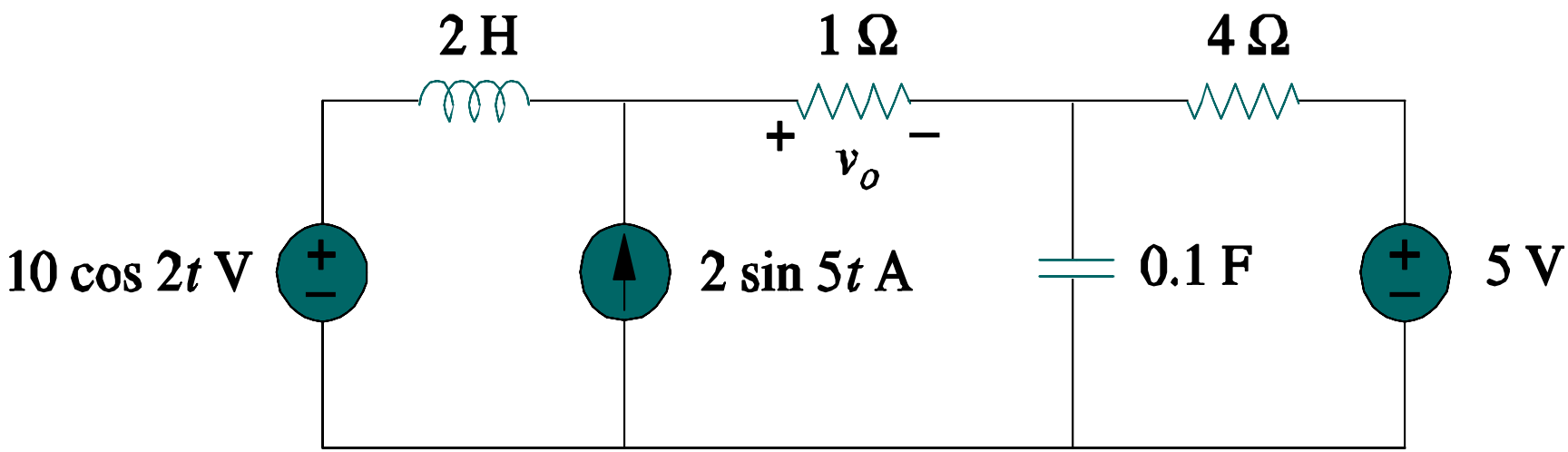
RTH =2.4982 - 2.2783i

Magnitude	Phase(r)	Phase(degrees)
3.3811	-0.7394	-42.3637

Equivalent Circuit Model at $\omega=10^6 s^{-1}$

AC Example: Superposition

Find v_0 for circuit below, using superposition

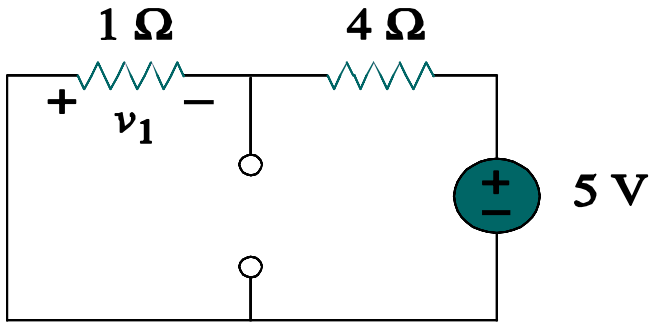


Allows us to analyze multiple sources, different frequency

- DC (5V) source
- cos and sin sources at 2 different frequencies

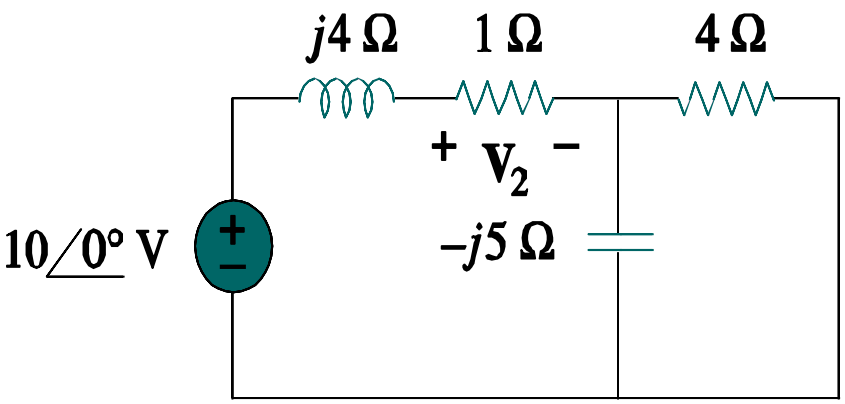
AC Example: Superposition

Response from 5V source



$$-v_1 = 1V$$

Response from AC voltage source



$$Z = -j5 \,||\, 4 = 2.439 - j1.951$$

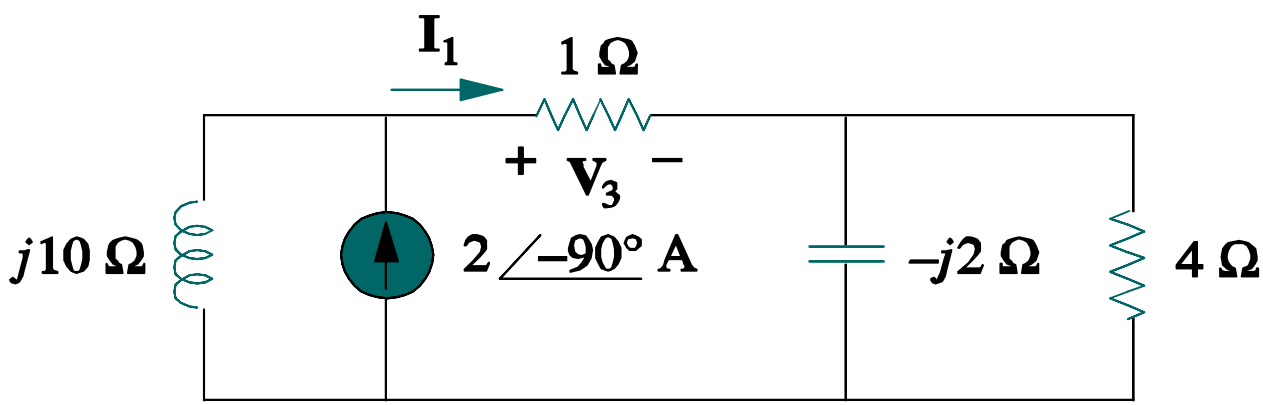
$$V_2 = \frac{1}{1 + j4 + Z}\left(10\angle 0^o\right)$$

$$V_2 = 2.498\angle -30.79^o$$

$$v_2 = 2.498\cos\left(2t - 30.79^o\right)$$

AC Example: Superposition

Response from AC current source



$$Z_1 = -j2 \,\|\, 4 = 0.8 - j1.6$$

$$I_1 = \frac{j10}{j10 + 1 + Z_1}\left(2\angle -90^o\right)$$

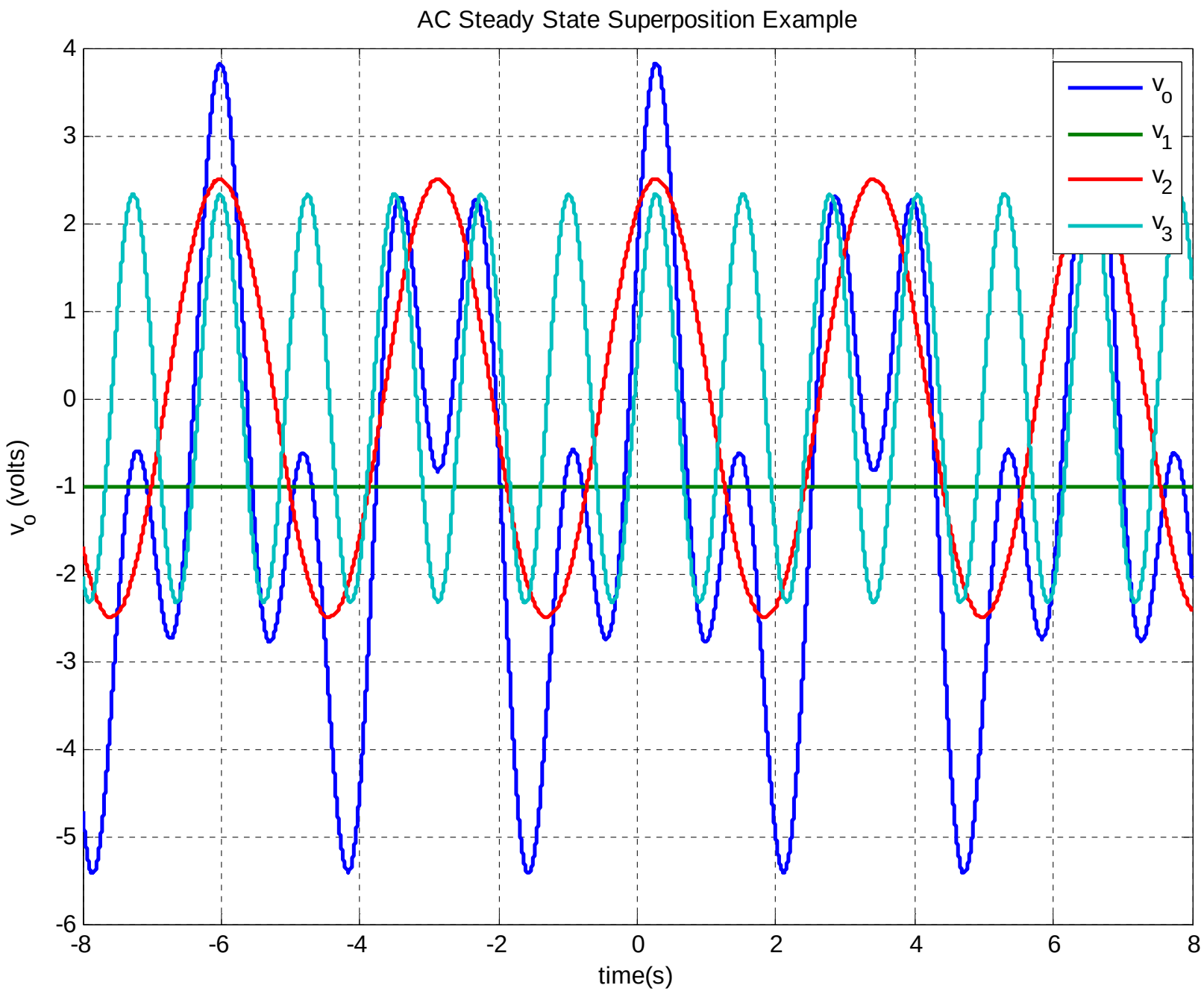
$$V_3 = 1I_1 = 2.328\angle -80^o$$

$$v_3 = 2.33\cos\left(5t - 80^o\right)$$

Total Response

$$v_0 = v_1 + v_2 + v_3$$

$$v_0 = -1 + 2.498\cos\left(2t - 30.79^o\right) + 2.33\cos\left(5t - 80^o\right)$$


AC Steady State Superposition Example
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-8
-6
-4
-2
0
2
4
6
8
time(s)
v_o (volts)
v_o
v_1
v_2
v_3

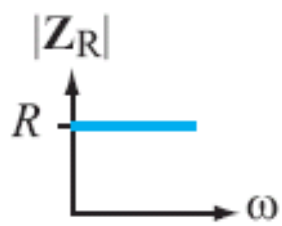
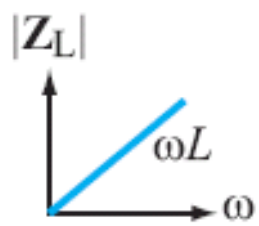
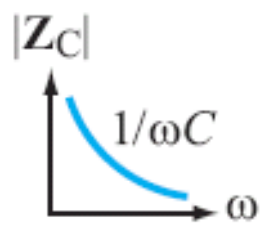
Time and Phasor Domain

$x(t)$		$\mathbf{X}$
$A\cos\omega t$	$\leftrightarrow$	A
$A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi\pm\pi)}$
$A\sin\omega t$	$\leftrightarrow$	$Ae^{-j\pi/2}=-jA$
$A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi-\pi/2)}$
$-A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi+\pi/2)}$
$\frac{d}{dt}(x(t))$	$\leftrightarrow$	$j\omega\mathbf{X}$
$\frac{d}{dt}[A\cos(\omega t+\phi)]$	$\leftrightarrow$	$j\omega Ae^{j\phi}$
$\int x(t)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}\mathbf{X}$
$\int A\cos(\omega t+\phi)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}Ae^{j\phi}$

It is much easier to deal with the exponentials in phasor domain than sinusoidal relations in time domain

Just need to track magnitude/phase, knowing that everything is at frequency ω

Summary of R, L, C

Property	R	L	C
v–i	$v = Ri$	$v = L\dfrac{di}{dt}$	$i = C\dfrac{dv}{dt}$
V–I	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
Z	R	$j\omega L$	$\dfrac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response			

Impedance and Admittance

Impedance is voltage/current

$$Z = \frac{V}{I} = R + jX$$

R = resistance = Re(Z)

X = reactance = Im(Z)

Admittance is current/voltage

$$Y = \frac{I}{V} = G + jB$$

G = conductance = Re(Y)

B = susceptance = Im(Y)

Resistor	$Z = R$	$Y = 1/R$
Inductor	$Z = j\omega L$	$Y = 1/j\omega L$
Capacitor	$Z = 1/j\omega C$	$Y = j\omega C$

Note that this is equivalent to “s-domain” with s=jω!