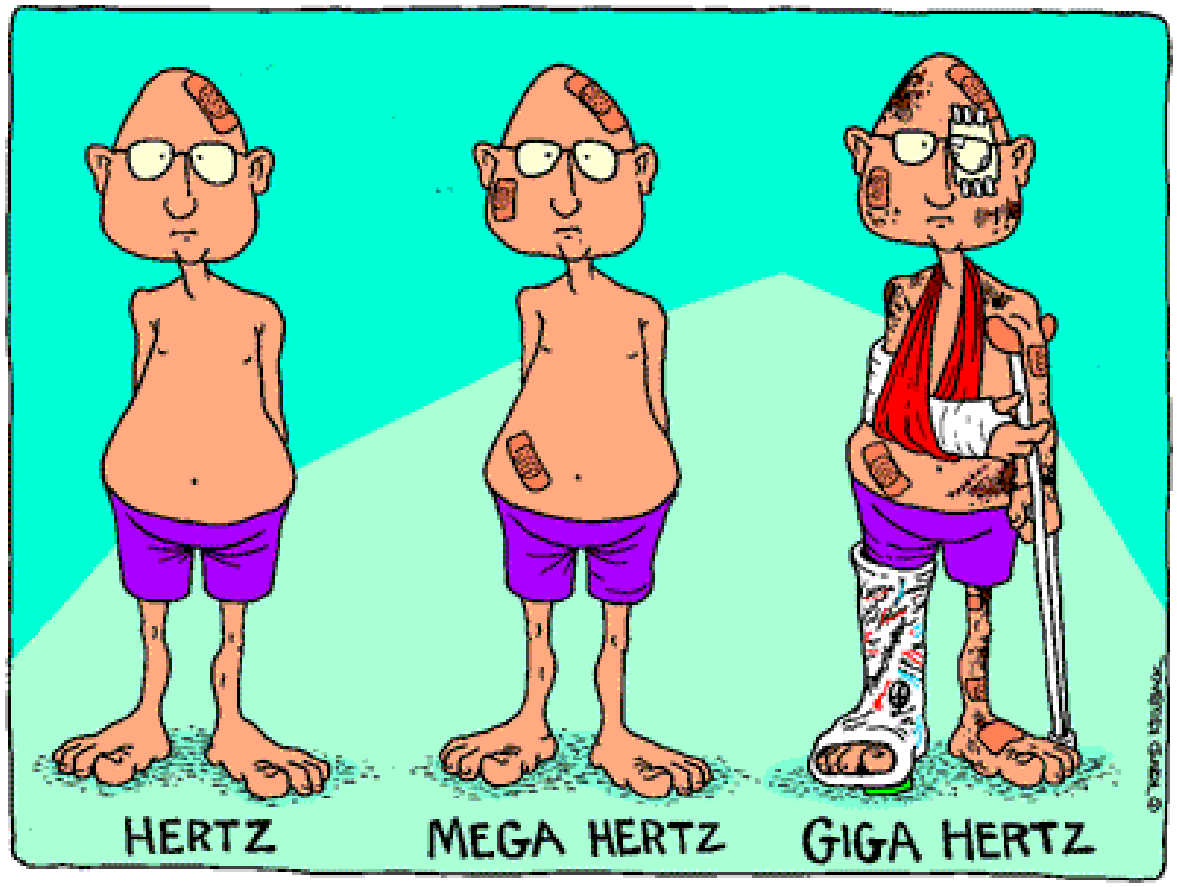

STEADY-STATE AC ANALYSIS I

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Fall, 2013

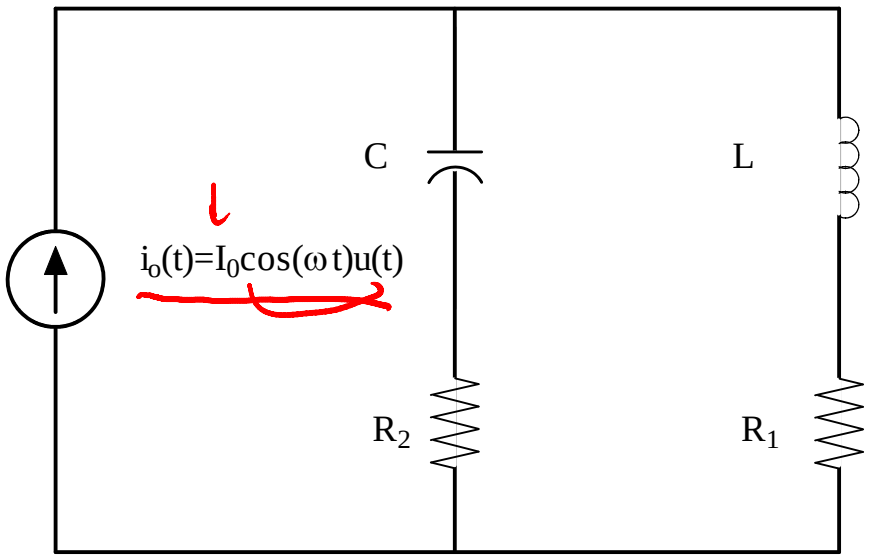
Material Order

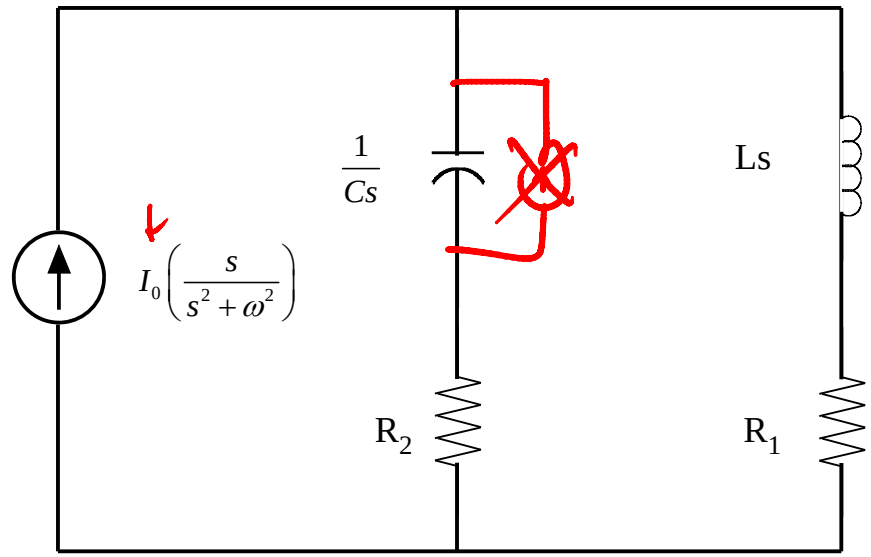
- Ulaby & Maharbiz Chapters 7, 8, 9
- Lecture: Somewhat Different Order than Text Chapters
- Today: Big Picture (Part of Chp 9)
  - Finding AC Steady State Response
  - Relationship to Transient Response/Differential Equations
- Following Lectures Will Return to ~Text Book Time-Line & Phasors

AC Circuit Analysis

- Objective: Find the Steady-State Response of a Linear Circuit to a Fixed AC Signal
  - $V_{in}(t)=V_0\cos(\omega t)$
- The Steady-State Response is the *Forced Solution* of the Differential Equation for the Circuit
- Why Do This?
  - Through Fourier Series, this gives us the response to ANY periodic signal
  - Directly Useful for All Analog Circuit Design, Understanding Clock Skew on Digital Lines, etc.

Example: General RLC Problem: Get $v_c(t)$ steady state





Later, we will use the following numerical values for this circuit:

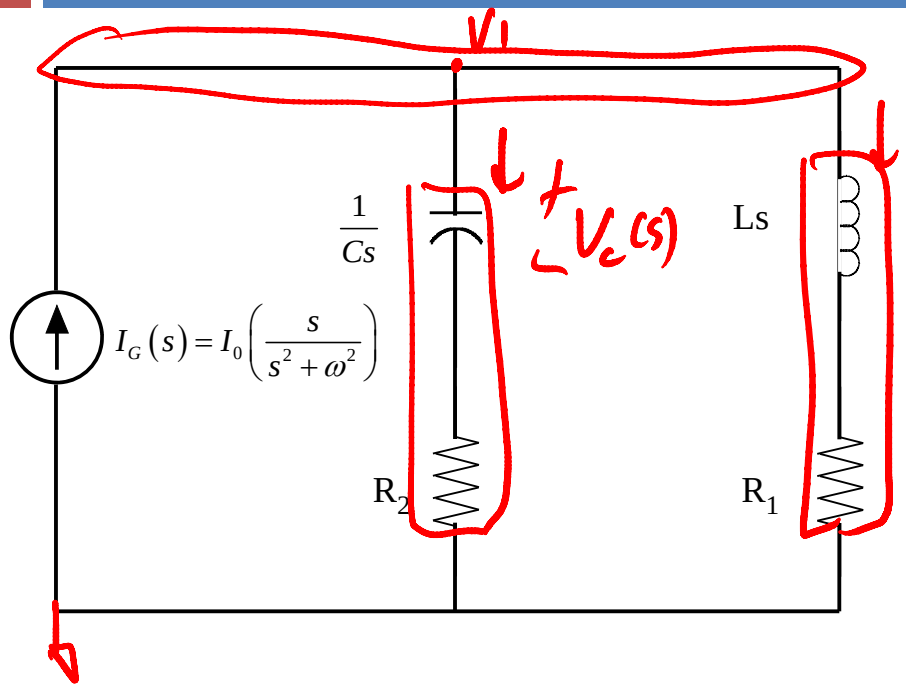
$L = 100nH \quad C = 0.01nF = 10\,pF$

$R_1 = 10\Omega \quad R_2 = 10\Omega$

$I_0 = 1A \quad \omega = 10^9\ s^{-1}$

$I_{L0} = 0A \quad V_{C0} = 0V$

Example: General RLC Problem: Get $v_c(t)$ steady state



$$I_G(s) = \frac{V_1}{\frac{1}{Cs}+R_2} + \frac{V_1}{Ls+R_1}$$

$$= V_1\left[\frac{Cs}{R_2Cs+1} + \frac{1}{Ls+R_1}\right]$$

$$= V_1\left[\frac{Cs(Ls+R_1)+R_2Cs+1}{(R_2Cs+1)(Ls+R_1)}\right]$$

$$= V_1\left[\frac{LCs^2+(R_1+R_2)Cs+1}{(R_2Cs+1)(Ls+R_1)}\right]$$

$$V_1 = \frac{(R_2 Cs + 1)(Ls + R_1)}{LCs^2 + (R_1 + R_2)Cs + 1} I_g$$

$$V_C(s) = \frac{1/Cs}{R_2 + 1/Cs} V_1 = \left(\frac{1}{R_2 Cs + 1}\right) V_1$$

$$= \boxed{\frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1}} I_g(s)$$

$$\left(\frac{s}{s^2 + \omega^2}\right) I_0$$

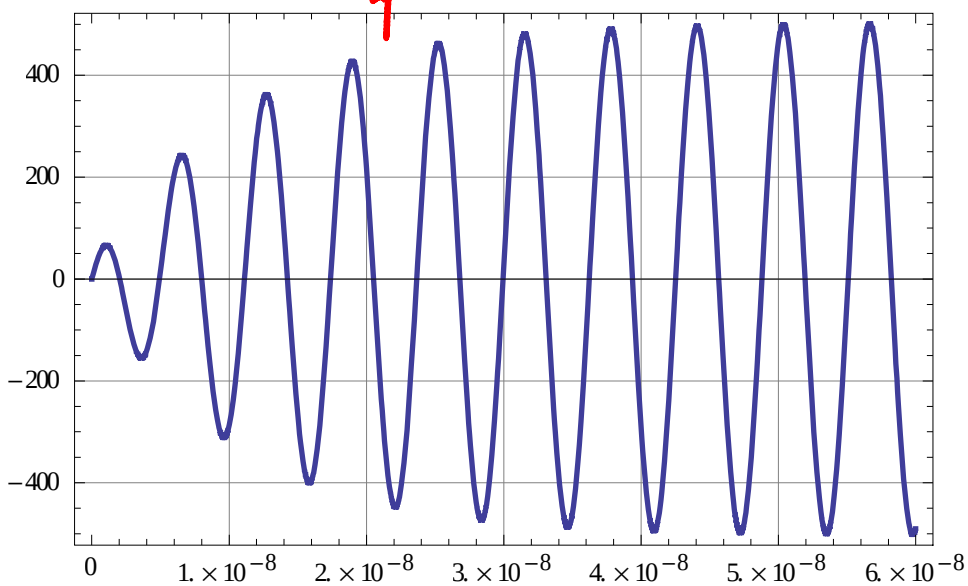
Laplace Solution

$$V_C(s) = \left[\frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1}\right] L\left[i_0(t)\right] = \left[\frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1}\right]\left[\frac{s}{s^2 + \omega^2}\right] I_0$$

$$L = 100nH \quad C = 0.01nF = 10\,pF \quad R_1 = 10\Omega \quad R_2 = 10\Omega \quad I_0 = 1A \quad \omega = 10^9\ s^{-1}$$

$$V_C(s) = \frac{10^{11} s(10^8 + s)}{(10^{18} + s^2)(10^{18} + 2x10^8 s + s^2)} = \frac{500(10^8 + s)}{10^{18} + s^2} - \frac{500(10^8 + s)}{(10^{18} + 2x10^8 s + s^2)}$$

$$v_c(t) = -500Cos[3x10^8\sqrt{11}t]e^{-10^8 t} + (500Cos[10^9 t] + 50Sin[10^9 t])$$



AC Steady State Response

Easy Way to Get Steady State?

- The Laplace Method Is Powerful, But Suppose We Want Only The AC Steady State Response – Is There An Easier Way?
- Yes, Use of Complex Numbers & Linearity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$j \equiv \sqrt{-1}$$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Taylor Expansions

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\Leftarrow e^{jx} = 1 + (jx) + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} \ldots$$

$$\cos(x) = \cos(0) - x\sin(0) - \frac{x^2}{2!}\cos(0) + \frac{x^3}{3!}\sin(0) + \frac{x^4}{4!}\cos(0) + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{\substack{k=0 \\ k\,even}}^{\infty} \frac{x^k}{k!}(-1)^{k/2}$$

$$\cos(x) = \cos(-x)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots =$$

$$\sum_{\substack{k=0 \\ k\,odd}}^{\infty} \frac{x^k}{k!}(-1)^{(k-1)/2}$$

$$\sin(x) = -\sin(-x)$$

Euler's Relation

$$j \equiv \sqrt{-1}$$

$j^2 = -1 \quad j^3 = -j \quad j^4 = 1 \quad j^5 = j$

$$e^{jx} = 1 + jx + j^2 \frac{x^2}{2!} + j^3 \frac{x^3}{3!} + j^4 \frac{x^4}{4!} + j^5 \frac{x^5}{5!} + j^6 \frac{x^6}{6!} + \ldots$$

$$= 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!} + \ldots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= \cos(x) + j\sin(x)$$

AC Steady State Analysis

- Real Sinusoidal Input
  - $V_{in}=V_0\cos(\omega t)$

$\cos(\omega t - \frac{\pi}{2}) = \sin(\omega t)$

- Assume Input is Complex

$$\widetilde{V_{in}}(t) = V_0 e^{j\omega t} = V_0\left[\cos(\omega t) + j\sin(\omega t)\right]$$

$$\text{Re}\,al\ Input\ Signal$$

$$V_{in}(t) = \text{Re}\left[\widetilde{V_{in}}(t)\right]$$

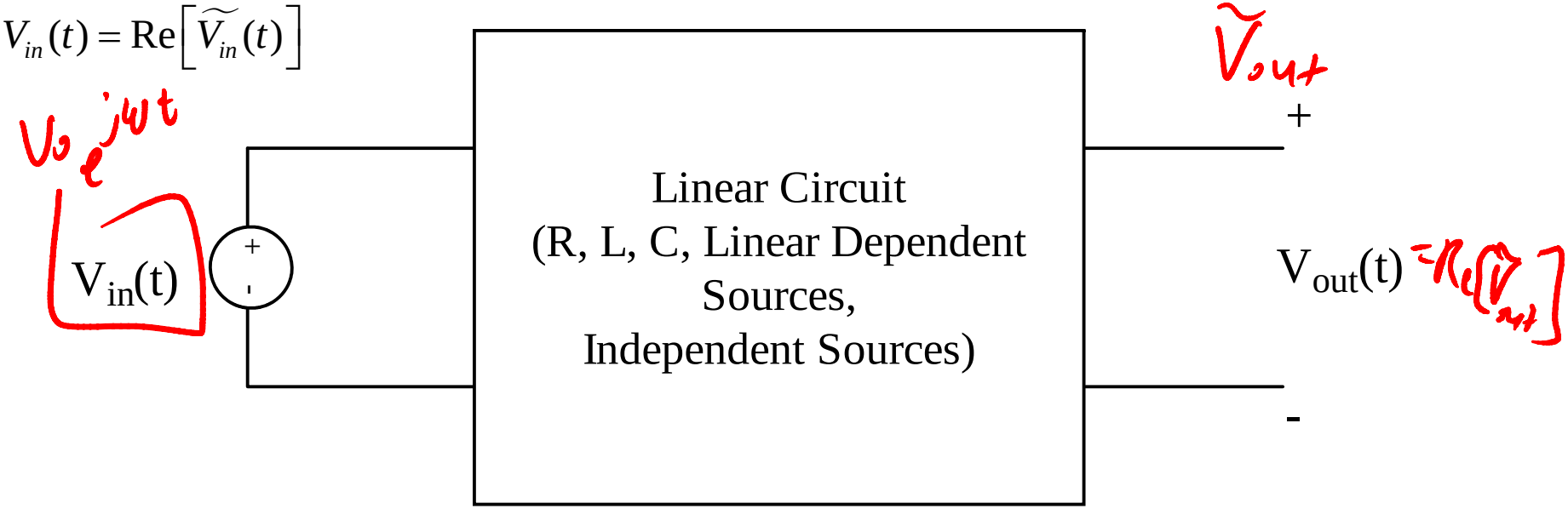
- By Superposition, ALL Circuit Quantities of Interest (voltages, currents) are given by taking the Real Part of the Response to the Complex Signal

Concept

$$\widetilde{V_{in}}(t) = V_0 e^{j\omega t} = V_0\left[\cos(\omega t) + j\sin(\omega t)\right]$$

Real Input Signal

$$V_{in}(t) = \mathrm{Re}\left[\widetilde{V_{in}}(t)\right]$$



$$\widetilde{V_{out}}(t) = \text{circuit response to } V_0 e^{j\omega t} = V_0\left[\cos(\omega t) + j\sin(\omega t)\right]$$

Real Output Signal

$$V_{out}(t) = \mathrm{Re}\left[\widetilde{V_{out}}(t)\right]$$

AC Steady State Analysis Outline

- Treat Sinusoidal Signals as Complex Exponentials
- Analyze Circuit Response Using Nodal, Mesh, *Whatever* Using Complex Impedances
- Find Desired Responses in Complex Form
- Take the Real Part to Get Physical Result
- *Wonderful Thing Is That This Approach Is Easier*

Capacitor in jω-Domain

$\tilde{v}_C = v_0 e^{st}$

$\tilde{i}_C = sC v_0 e^{st}$

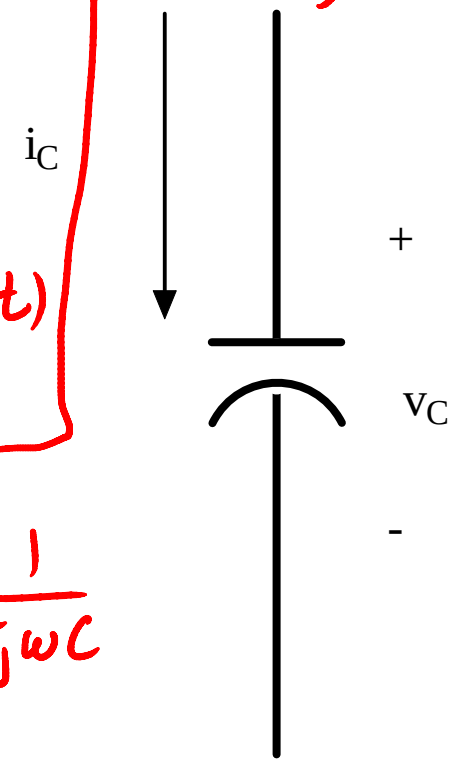
$Z_C = \frac{1}{sC}$

$i_C = C \frac{dv_C}{dt}$

$v_C(t) = v_0 \cos(\omega t)$

$i_C(t) = -C\omega v_0 \sin(\omega t)$

$\frac{v_C}{i_C} = -\frac{1}{C\omega} \frac{\cos(\omega t)}{\sin(\omega t)} = -\frac{1}{C\omega} \cot(\omega t)$



$\tilde{v}_C = v_0 e^{j\omega t}$

$\tilde{i}_C = j\omega C v_0 e^{j\omega t}$

$\frac{\tilde{v}_C}{\tilde{i}_C} = \frac{1}{j\omega C}$

$$\boxed{Z_C = \frac{1}{j\omega C}}$$

Inductor in jω -Domain

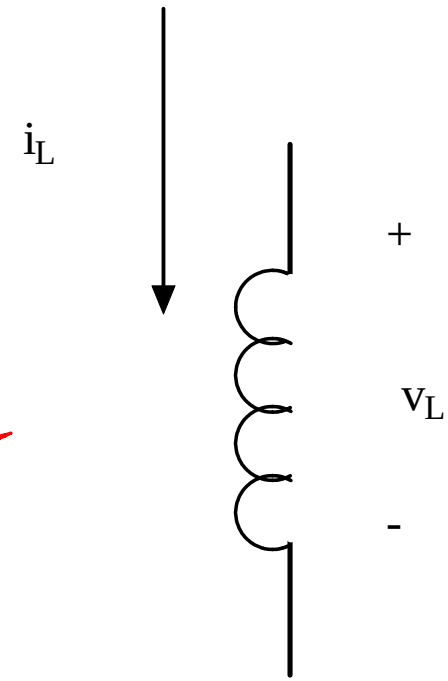
$$v_L = L\frac{di_L}{dt}$$

$$\boxed{s = j\omega}$$

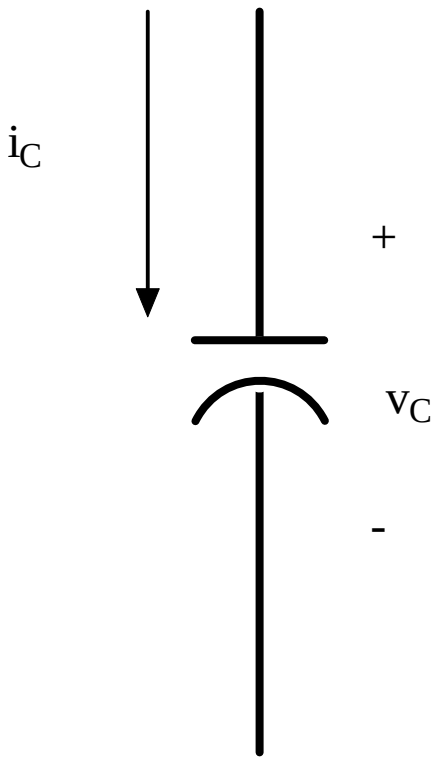
$$\tilde{i}_L = i_0 e^{st}$$

$$\tilde{v}_L = L s i_0 e^{st}$$

$$Z_L = \frac{\tilde{v}_L}{\tilde{i}_L} = Ls = j\omega L$$



Capacitors/Inductors in jω -Domain



Suppose $V_C(t) = Ae^{j\omega t}$

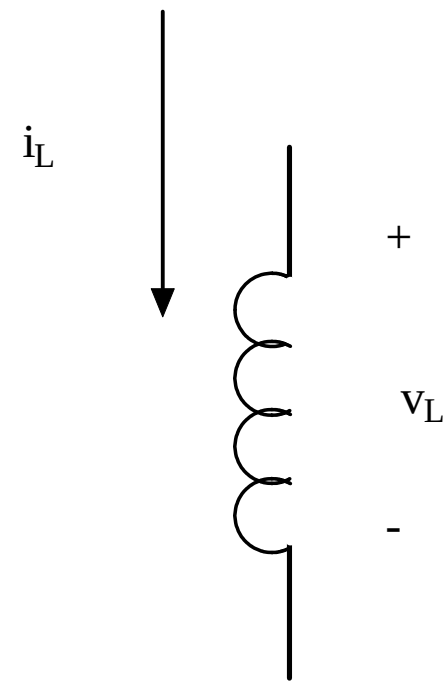
$$i_C = C\frac{\partial V_C}{\partial t} = j\omega CAe^{j\omega t}$$

$$V_C = Z_C i_C \rightarrow Z_C = \frac{1}{j\omega C}$$

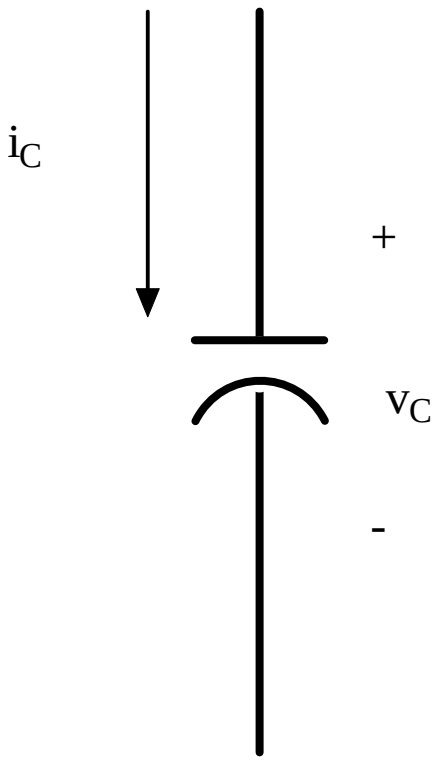
Suppose $i_L(t) = Ae^{j\omega t}$

$$V_L = L\frac{\partial i_L}{\partial t} = j\omega LAe^{j\omega t}$$

$$V_L = Z_L i_L \rightarrow Z_L = j\omega L$$



Capacitors/Inductors in s-Domain



Suppose $V_C(t) = Ae^{st}$

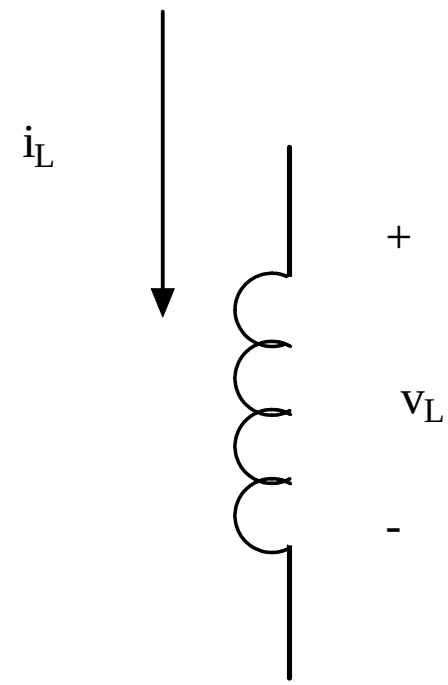
$$i_C = C\frac{\partial V_C}{\partial t} = sCAe^{st}$$

$$V_C = Z_C i_C \rightarrow Z_C = \frac{1}{sC}$$

Suppose $i_L(t) = Ae^{st}$

$$V_L = L\frac{\partial i_L}{\partial t} = sLAe^{st}$$

$$V_L = Z_L i_L \rightarrow Z_L = sL$$



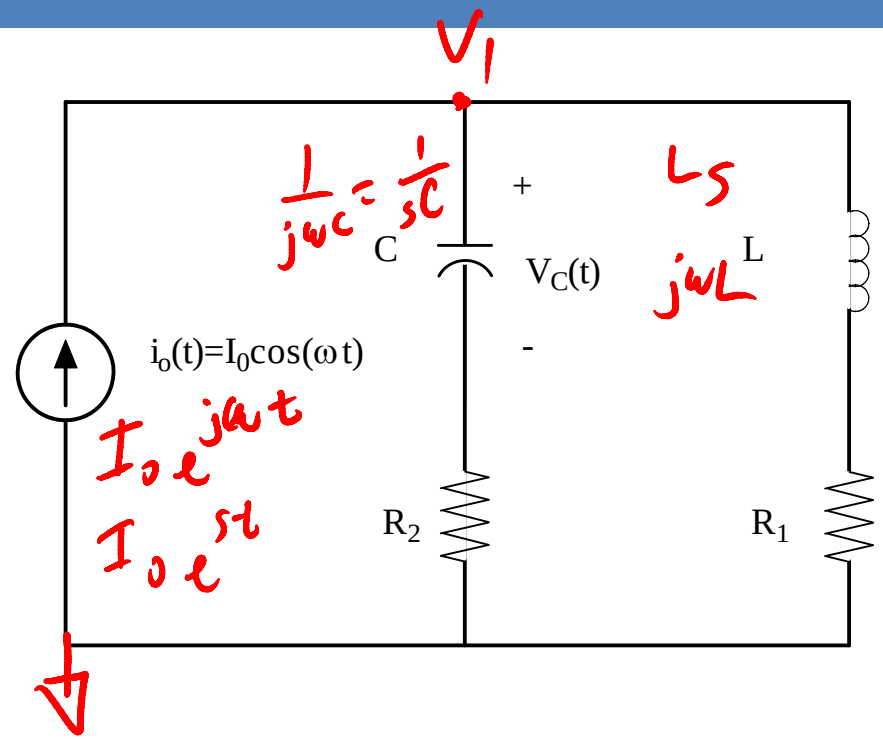
Get all the previous results simply by taking $s = j\omega$

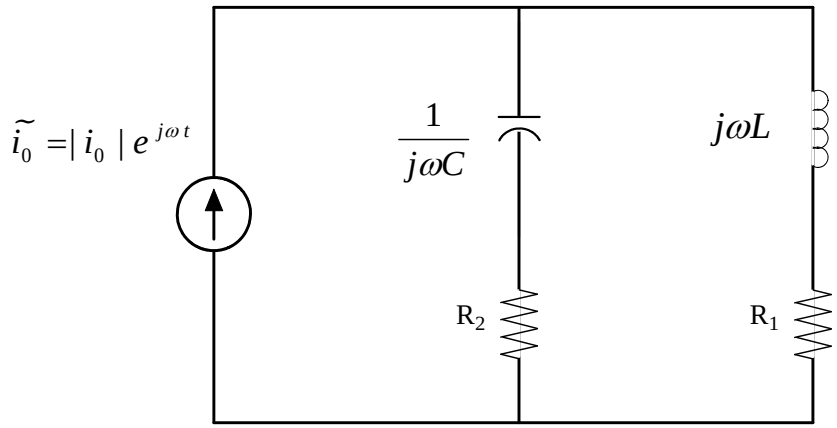
General RLC Example 1: Current Divider in $j\omega$- Domain

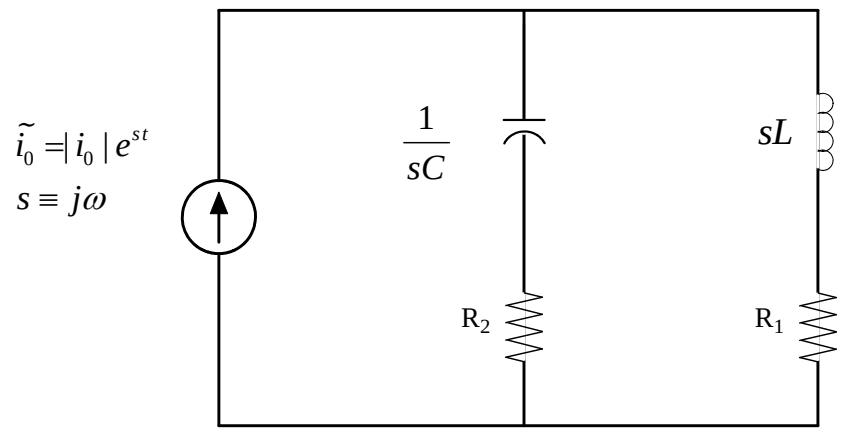
Find $V_c(t)$

$$I_0 e^{st} = \frac{V_1}{\frac{1}{sC} + R_2} + \frac{V_1}{Ls + R_1}$$

$$V_c = \frac{\frac{1}{sC}}{\frac{1}{sC} + R_2} V_1$$




$\tilde{i}_0 = |i_0| e^{j\omega t}$
$\frac{1}{j\omega C}$
$j\omega L$
R_2
R_1


$\tilde{i}_0 = |i_0| e^{st}$
$s \equiv j\omega$
$\frac{1}{sC}$
sL
R_2
R_1

$$Z_1 = Ls + R_1$$

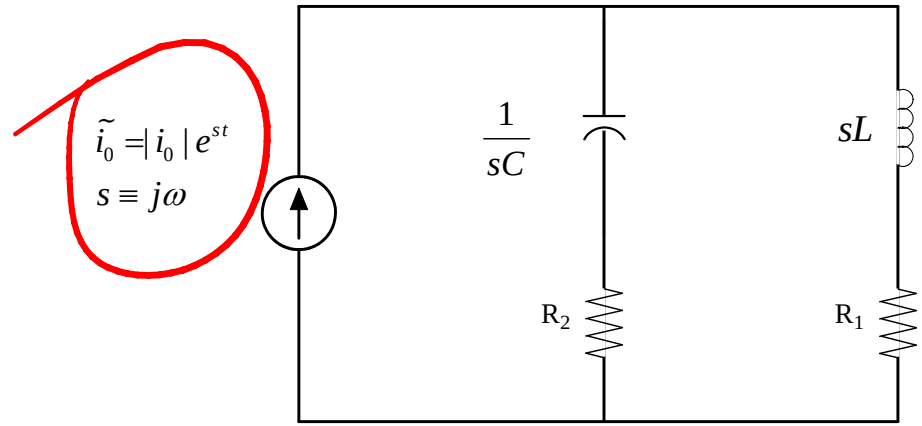
$$Z_2 = \frac{1}{Cs} + R_2$$

$$i_c = \left[\frac{1/Z_2}{1/Z_2 + 1/Z_1}\right] i_0 = \left[\frac{Z_1}{Z_1 + Z_2}\right] i_0$$

$$\tilde{V}_c = i_c / Cs = \left[\frac{Z_1}{Z_1 + Z_2}\right] \frac{1}{Cs} i_0$$

$$= \frac{Ls + R_1}{Ls + R_1 + \frac{1}{Cs} + R_2} \left(\frac{1}{Cs}\right) i_0$$

$$= \left[\frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1}\right] \tilde{i}_0$$



$$V_c = \frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1} i_0$$

$$\left[LCs^2 + (R_1 + R_2)Cs + 1\right] V_c = \left[Ls + R_1\right] i_0$$

$$\left[s^2 + \frac{(R_1 + R_2)}{L} s + \frac{1}{LC}\right] V_c = \left[\frac{1}{C} s + \frac{R_1}{L}\right] i_0$$

which maps back to:

$$\frac{d^2 V_c}{dt^2} + \frac{(R_1 + R_2)}{L} \frac{dV_c}{dt} + \frac{1}{LC} V_c = \frac{1}{C} \frac{di_0}{dt} + \frac{R_1}{L} i_0$$

If we want, we can get the governing differential equation from this s-domain expression (see above box) but our main point here will be to find the AC steady state response

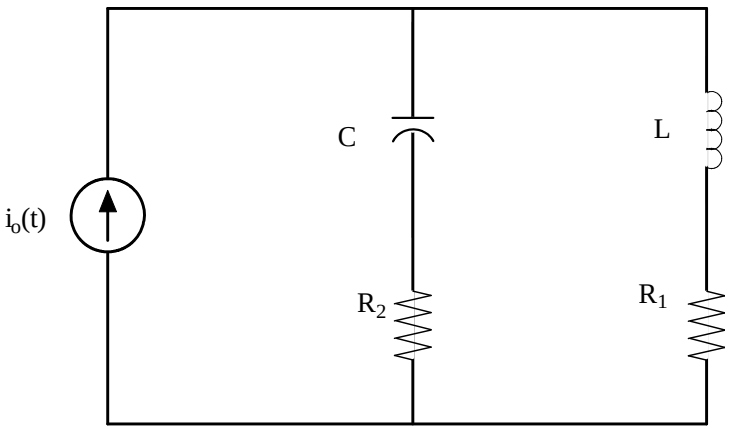
Final Result

$$\widetilde{V_C} = H(j\omega)\widetilde{i_0}(t)$$

$$H(s) = \frac{\widetilde{V_C}(t)}{\widetilde{i_0}(t)} = \left[\frac{sL + R_1}{\left(1 + s^2 LC\right) + s\left(R_1 + R_2\right)C}\right]$$

To get $H(j\omega)$ just take $s = j\omega$

$$H(j\omega) = H(\omega) = \left[\frac{j\omega L + R_1}{\left(1 - \omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C}\right]$$



H(**s**) is called a ***transfer function***

Final Result

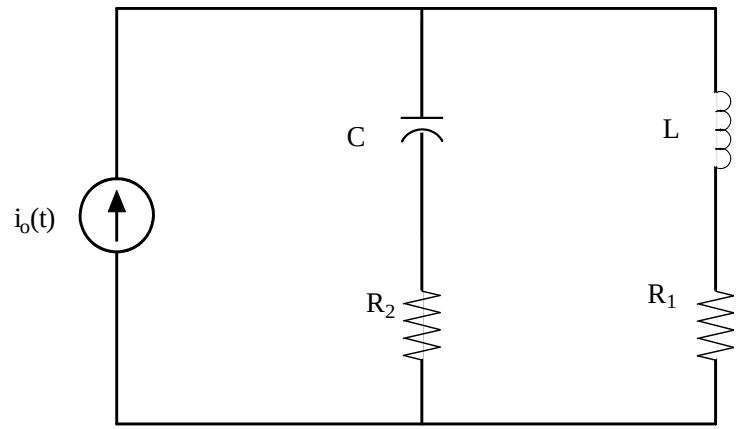
$v_c(t) = Re[\tilde{v}_c] = Re[\widetilde{H(\omega)|i_0|e^{j\omega t}}]$

$$\tilde{V}_C(t) = \left[\frac{j\omega L + R_1}{\left(1-\omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C}\right]\tilde{i}_0(t)$$

$$= H(\omega)\tilde{i}_0(t)$$

$$H(\omega) = \frac{\tilde{V}_C(t)}{\tilde{i}_0(t)}$$

$$= \left[\frac{j\omega L + R_1}{\left(1-\omega^2 LC\right) + j\omega\left(R_1 + R_2\right)C}\right]$$



$A e^{j\omega t} = H(\omega)|i_0|e^{j\omega t}$

$A = H(\omega)|i_0|$

H(ω) is called a ***transfer function***

Differential Equation View

$$\frac{d^2 v_c}{dt^2} + \frac{(R_1 + R_2)}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{1}{C}\frac{di_0}{dt} + \frac{R_1}{L}i_0$$

$$i_0 = \mathrm{Re}[Be^{j\omega t}] = \mathrm{Re}[Be^{st}]$$

$$try\, v_c = \mathrm{Re}[Ae^{j\omega t}] = \mathrm{Re}[Ae^{st}]$$

$$s^2 Ae^{st} + \frac{(R_1 + R_2)}{L} sAe^{st} + \frac{1}{LC}Ae^{st} = \frac{1}{C}Be^{st}i_0 + \frac{R_1}{L}Be^{st}$$

$$\left(s^2 + \frac{(R_1 + R_2)}{L}s + \frac{1}{LC}\right)Ae^{st} = \left(\frac{1}{C}s + \frac{R_1}{L}\right)Be^{st}$$

$$\left(-\omega^2 + \frac{(R_1 + R_2)}{L}j\omega + \frac{1}{LC}\right)A = \left(\frac{1}{C}j\omega + \frac{R_1}{L}\right)B$$

$$A = \frac{\left(\frac{1}{C}j\omega + \frac{R_1}{L}\right)}{\left(-\omega^2 + \frac{(R_1 + R_2)}{L}j\omega + \frac{1}{LC}\right)}B$$

$$v_c = \mathrm{Re}\left[Ae^{j\omega t}\right]$$

$V_c(t) = |H(\omega)|\, i_0 \cos(\omega t + \measuredangle H(\omega))$

Numerical Example

For the example circuit, suppose:

$L = 100nH \quad C = 0.01nF = 10\,pF \quad R_1 = 10\Omega \quad R_2 = 10\Omega \quad I_0 = 1A \quad \boxed{\omega = 10^9\ s^{-1}}$

Find $V_c(t)$

$$H(\omega) = \left[\frac{Ls + R_1}{LCs^2 + (R_1 + R_2)Cs + 1}\right] = \left[\frac{j\omega L + R_1}{(1 - \omega^2 LC) + j\omega(R_1 + R_2)C}\right] = \left[\frac{j100\Omega + 10\Omega}{(1-1) + j0.2}\right] = (500 - 50j)\Omega$$

$$V_c(t) = \mathrm{Re}\left[(500 - 50j)\Omega \cdot 1e^{j10^9 t} A\right] = \mathrm{Re}\left[(500 - 50j)\left(\cos(10^9 t) + j\sin(10^9 t)\right)V\right]$$

$$V_c(t) = \left[500\cos(10^9 t) + 50\sin(10^9 t)\right]V$$

$$V_c(t) = |H(\omega)| \cdot |i_0| \cos\left(10^9 t + \measuredangle H(\omega)\right)$$

$$|H(\omega)| = \left(500^2 + 50^2\right)^{1/2} \approx 502.494\Omega$$

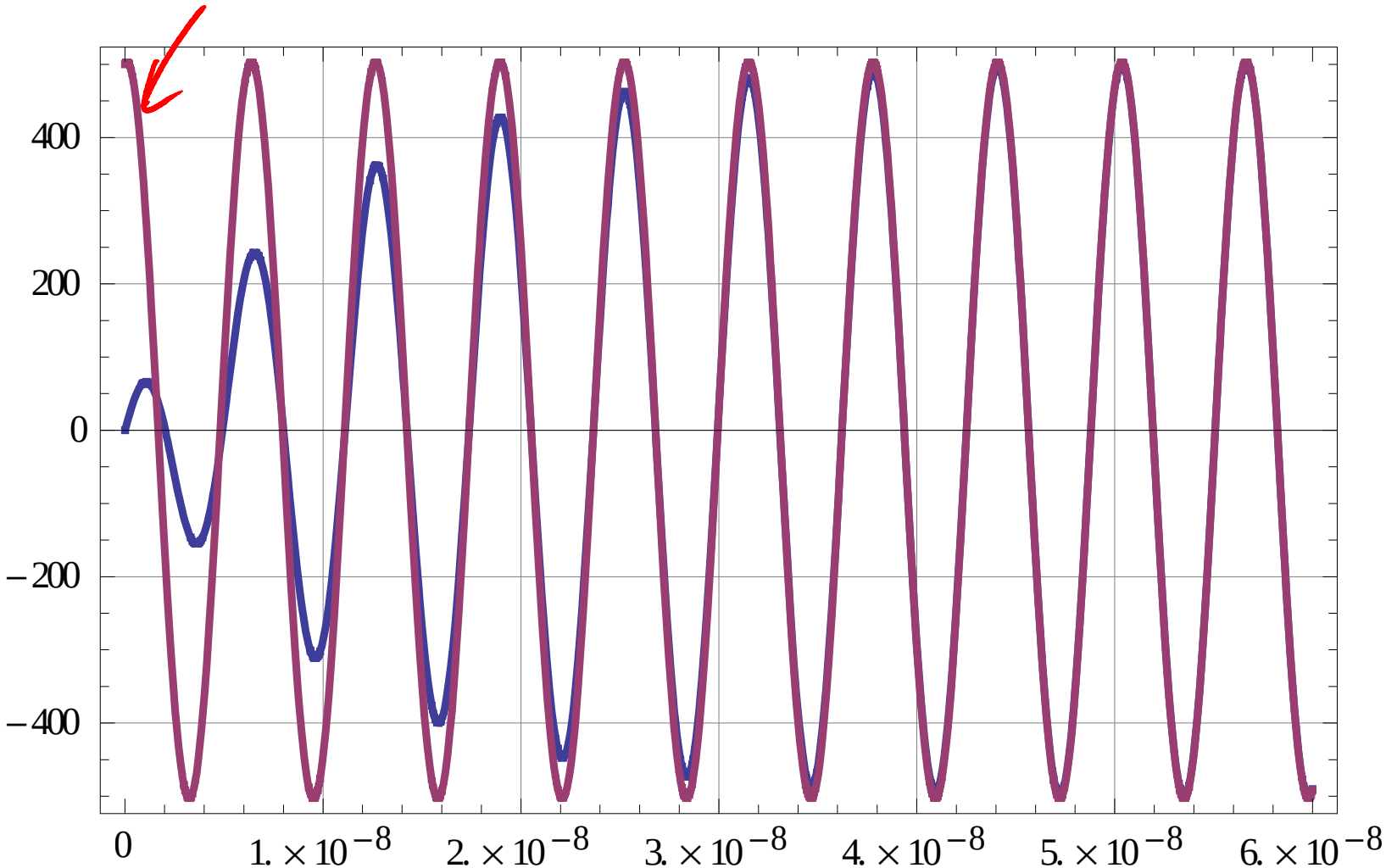
$$\measuredangle H(\omega) = \tan^{-1}\left(\frac{-50}{500}\right) \approx -5.7106°$$

$\mathrm{Re}[H(\omega)\, i_0 e^{j\omega t}]$

$H(\omega) = |H(\omega)| e^{j\measuredangle H(\omega)}$

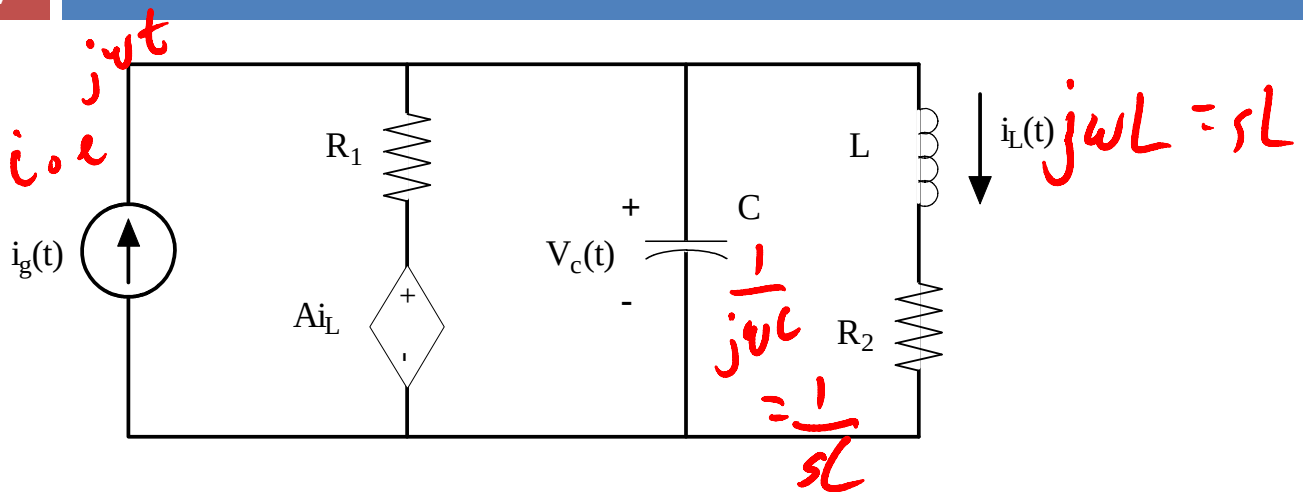
$\mathrm{Re}[|H| e^{j\measuredangle H} i_0 e^{j\omega t}] = |H| i_0 \mathrm{Re}[e^{j(\omega t + \measuredangle H)}]$

Transient and Steady State Solutions



Example

$\omega = \cancel{1000\ sec^{-1}}$ $\omega = 1000\ sec^{-1}$



$$i_g = \left[90\ \cos\left(\underline{1000}t\right) mA\right] u(t) \quad R_1 = 1K\Omega \quad R_2 = 5\Omega$$

$$C = 1\mu F \quad L = 1mH \quad A = 996\Omega$$

$$I_L(0) = 1A$$

$$V_C(0) = 5V$$

General Solution Using Mathematica

$$V_c(s)=\left\{\frac{(R_1R_2+LR_1s)\left[I_G(s)\right]+(A-R_1)LI_{L0}+C_1(R_1R_2+LR_1s)V_{C0}}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right\}$$

$$=\left\{\begin{aligned}&\left[\frac{R_1R_2+LR_1s}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]\left[I_G(s)\right]+\\&\left[\frac{(A-R_1)L}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]I_{L0}+\\&\left[\frac{C_1(R_1R_2+LR_1s)}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]V_{C0}\end{aligned}\right\}$$

$H(s)$

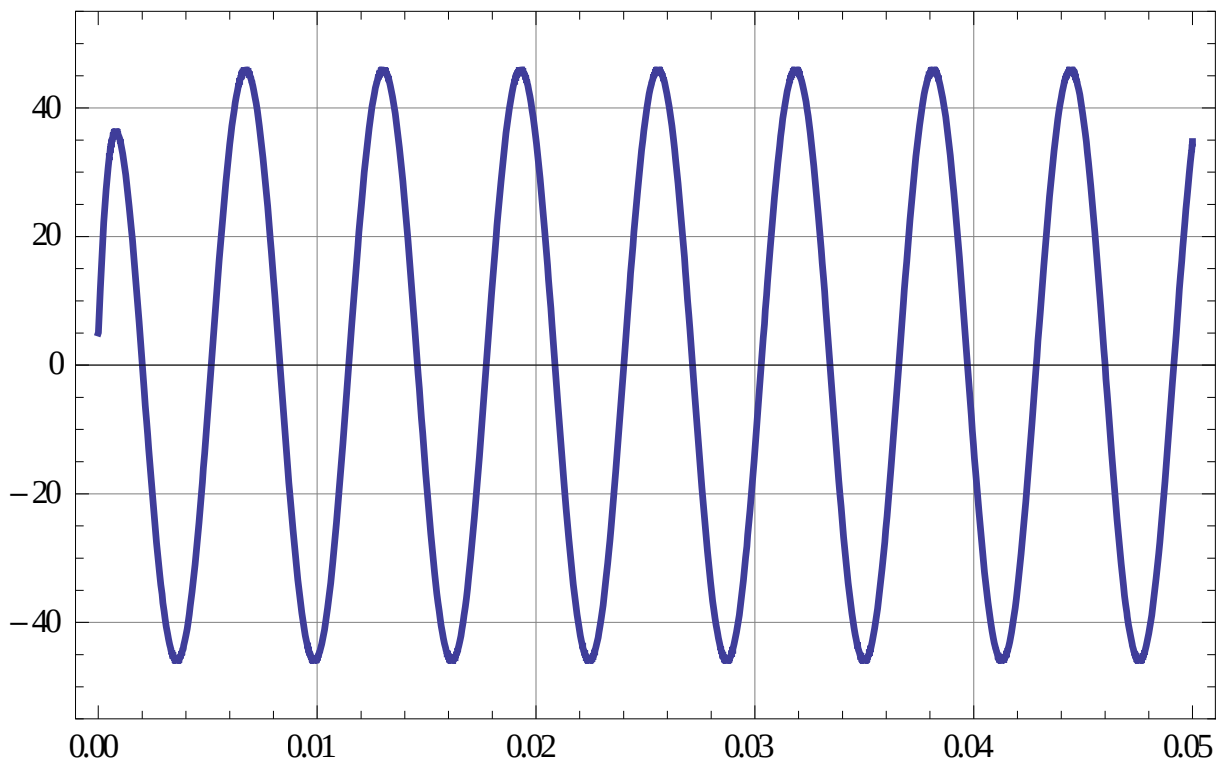
$v_C(t) = |H(\omega)|\cos(\omega t + \measuredangle H(\omega))$ $\quad s = j\omega \quad (\omega = 1000\ \text{sec}^{-1})$

Solution: Full Response

$$L[i_g] = L\left[90\cos(1000t)\,mA\,u(t)\right] = \left(\frac{s}{s^2+1000^2}\right)90mA$$

$$V_C(s) = -\frac{48000}{(3000+s)^2} - \frac{182}{5(3000+s)} + \frac{9(11000+23s)}{5(1000000+s^2)}$$

$$v_c(t) = -\frac{182}{5}e^{-3000t} - (48000)te^{-3000t} + \frac{9}{5}\left(23Cos[1000t] + 11Sin[1000t]\right)$$



AC Steady State Response 1

$$H(s)=\left[\frac{R_1R_2+LR_1s}{(R_1+R_2-A)+(L+C_1R_1R_2)s+C_1LR_1s^2}\right]$$

$$H(\omega)=\left[\frac{R_1R_2+LR_1j\omega}{(R_1+R_2-A)+(L+C_1R_1R_2)j\omega-C_1LR_1\omega^2}\right]$$

$$=\left[\frac{R_1R_2+LR_1j\omega}{(R_1+R_2-A-C_1LR_1\omega^2)+(L+C_1R_1R_2)j\omega}\right]$$

$$=\left[\frac{5000\Omega^2+j\omega(1\Omega^2\bullet\sec)}{(9\Omega-(10^{-6}\Omega\bullet\sec^2)\omega^2)+(6x10^{-3}\Omega\bullet\sec)j\omega}\right]$$

$$=\left[\frac{5000+j\omega(1\sec)}{(9-(10^{-6}\sec^2)\omega^2)+(6x10^{-3}\sec^2)j\omega}\right]\Omega$$

AC Steady State Response 1

$$H(\omega)=\left[\frac{5000+j\omega(1\sec)}{\left(9-\left(10^{-6}\sec^2\right)\omega^2\right)+\left(6x10^{-3}\sec\right)j\omega}\right]\Omega$$

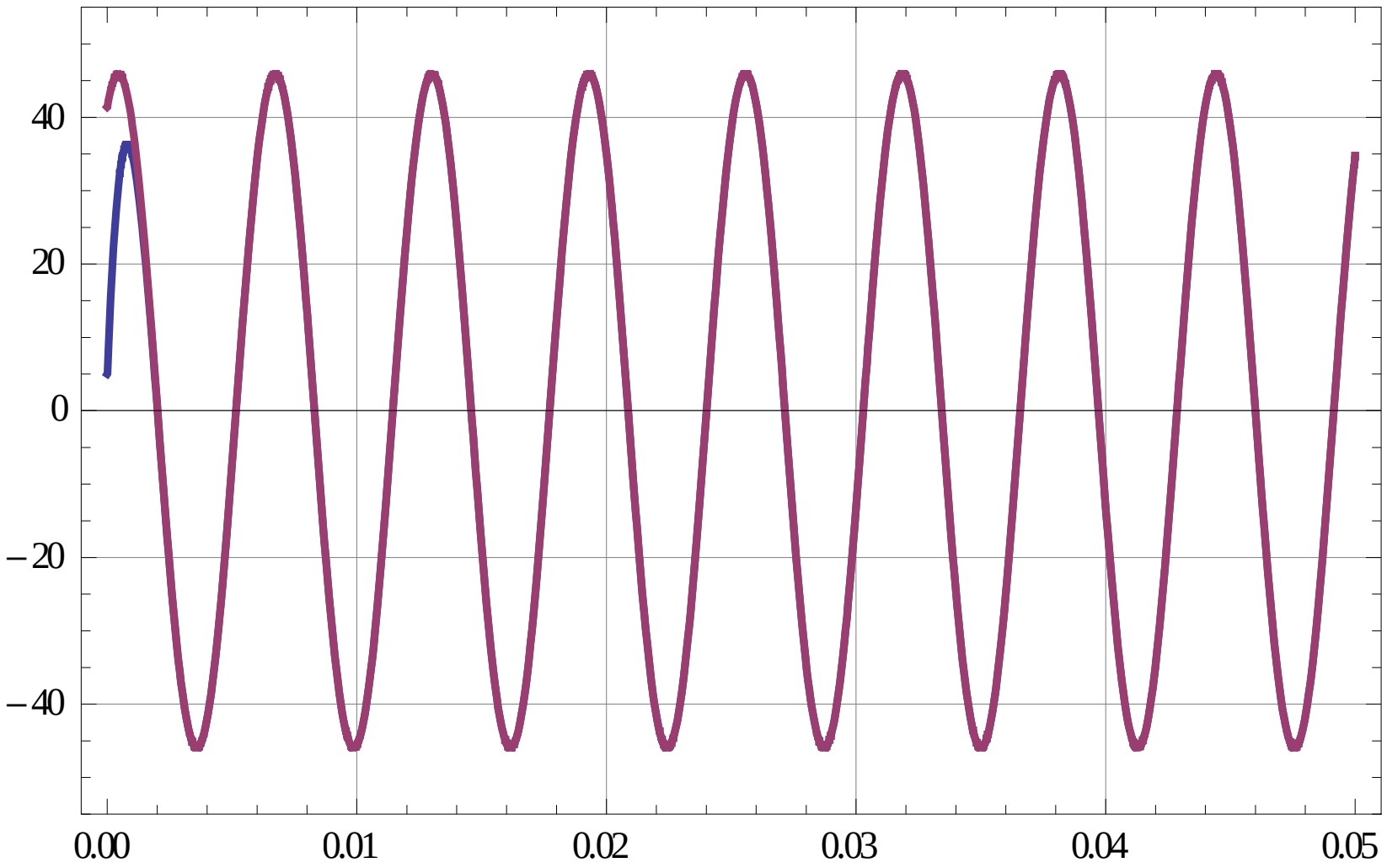
$$H(1000)=\left[\frac{5000+j1000}{(9-1)+j6}\right]\Omega=\left[\frac{5000+j1000}{8+j6}\right]\Omega=\left[\frac{2500+j500}{4+j3}\right]\Omega=500\left[\frac{5+j}{4+j3}\right]\Omega$$

$$=500\frac{(5+j)(4-j3)}{(4+j3)(4-j3)}\Omega=500\left(\frac{20+3+j4-j15}{16+9}\right)\Omega$$

$$=500\left(\frac{23-j11}{25}\right)\Omega=20(23+j11)\Omega=\underline{(460-j220)\Omega}$$

$$=\left(100\sqrt{26}\right)e^{-0.44610554894340365\,j}$$

$$v_c(t)=\mathrm{Re}\left[H(1000)90mAe^{j1000t}\right]\approx\left(9\sqrt{26}\right)\cos(1000t-0.4461rad)Volts$$

$$\approx\left(9\sqrt{26}\right)\cos(1000t-25.560°)Volts$$

Full Solution vs. AC Steady State