Epipolar Geometry

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http://web.eecs.umich.edu/~fouhey/teaching/EECS442_F19/
Multi-view geometry

Image Credit: S. Lazebnik
Multi-view geometry problems

Recovering structure: Given cameras and correspondences, find 3D.
Multi-view geometry problems

Stereo/Epipolar Geometry:
Given 2 cameras and find where a point could be
Multi-view geometry problems

Motion:
Figure out $R$, $t$ for a set of cameras given correspondences

Camera 1: $R_1, t_1$
Camera 2: $R_2, t_2$
Camera 3: $R_3, t_3$
Two-view geometry

Image Credit: Hartley & Zisserman
Camera Geometry Reminder

Have camera with pinhole at origin $0$

$p$ (2D point)
3 h. coordinates

Pretending image plane is in front

$X$ (3D point)
4 h. coordinates

$p^{-1}p$ (Ray)
3 h. coordinates

$K^{-1}p$ (Ray)
3 h. coordinates

$X$ (3D point)
4 h. coordinates

$p$ (2D point)
3 h. coordinates

Actual location

Have camera with pinhole at origin $0$
Epipolar Geometry

Suppose we have two cameras at origins o, o'

**Baseline** is the line connecting the origins
Epipolar Geometry

Now add a point $X$, which projects to $p$ and $p'$.
The plane formed by $X$, $o$, and $o'$ is called the epipolar plane. There is a family of planes per $o$, $o'$. 
Epipolar Geometry

- Epipoles $e$, $e'$ are where the baseline intersects the image planes
- Projection of other camera in the image plane
The Epipole

Photo by Frank Dellaert
• Epipolar lines go between the epipoles and the projections of the points.
• Intersection of epipolar plane with image plane
Example: Converging Cameras

Epipoles finite, maybe in image; epipolar lines converge
Example: Converging Cameras

Epipolar lines come in pairs: given a point $p$, we can construct the epipolar line for $p'$. 
Example 1: Converging Cameras

Image Credit: Hartley & Zisserman
Example: Parallel to Image Plane

Suppose the cameras are both facing outwards. Where are the epipoles (proj. of other camera)?
Example: Parallel to Image Plane

Epipoles *infinitely* far away, epipolar lines parallel
Example: Parallel to Image Plane

Image Credit: Hartley & Zisserman
Example: Forward Motion

Image Credit: Hartley & Zisserman
Example: Forward Motion

Image Credit: Hartley & Zisserman
Example: Forward Motion

Epipole is focus of expansion / principal point of the camera.

Epipolar lines go out from principal point.
Motion perpendicular to image plane

http://vimeo.com/48425421
So?
Epipolar Geometry

- Suppose we don’t know X and just have p
- Can construct the epipolar line in the other image
Suppose we don’t know X and just have p

Corresponding p’ is on corresponding epipolar line
Suppose we don’t know $X$ and just have $p'$.
Corresponding $p$ is on corresponding epipolar line.
Epipolar Geometry

- If I want to do stereo, I want to find a corresponding pixel for each pixel in the image:
  - Naïve search:
    - For each pixel, search every other pixel
  - With epipolar geometry:
    - For each pixel, search along each line (1D search)
Epipolar constraint example
Epipolar Constraint: One Note

• If you look around for other reading, you’ll find derivations with $p$, $p'$ flipped and constraints derived in a flipped way
• It all works the same
Epipolar Constraint: Calibrated Case

- If we know intrinsic and extrinsic parameters, set coordinate system to first camera.
- Projection matrices: $P_1 = K[I, 0]$ and $P_2 = K'[R, t]$.
- What are:

\[
P_1X \quad P_2X \quad K^{-1}p \quad K'^{-1}p'
\]
Epipolar Constraint: Calibrated Case

\[ \hat{p} = K^{-1} p \]
\[ \hat{p}' = K'^{-1} p' \]

- Given calibration, \( \hat{p} = K^{-1} p \) and \( \hat{p}' = K'^{-1} p' \) are “normalized coordinates”
- Note that \( \hat{p}' \) is actually translated and rotated
Epipolar Constraint: Calibrated Case

\[ \hat{p} = K^{-1} p \]

\[ \hat{p}' = K'^{-1} p' \]

- The following are all co-planar: \( \hat{p}, t, R\hat{p}' \) (can ignore translation for co-planarity here)
- One way to check co-planarity (triple product):
  \[ \hat{p}^T (t \times R\hat{p}) = 0 \]
Epipolar Constraint: Calibrated Case

\[ \hat{p} = K^{-1}p \]

\[ \hat{p}' = K'^{-1}p' \]

\[ \hat{p}^T (t \times R \hat{p}') = 0 \]

Want something like \( x^T M y = 0 \). What's \( M \)?
Epipolar Constraint: Calibrated Case

\[ \hat{\mathbf{p}} = K^{-1} \mathbf{p} \]
\[ \hat{\mathbf{p}}' = K'^{-1} \mathbf{p}' \]

Essential matrix (Longuet-Higgins, 1981): \( E = [t_x]R \)
If you have a normalized point \( \hat{\mathbf{p}} \), its correspondence \( \hat{\mathbf{p}}' \) must satisfy \( \hat{\mathbf{p}}^T E \hat{\mathbf{p}}' = 0 \)
Essential Essential Matrix Facts

• Suppose we know $E$ and $\hat{p}^T E \hat{p}' = 0$. What is the set $\{x: x^T E \hat{p}' = 0\}$?
• $E \hat{p}$ gives equation of the epipolar line (in $ax+by+c=0$ form) in image for $o$.
• What’s $E^T \hat{p}$?
Essential Essential Matrix Facts

- $\mathbf{E} \mathbf{e}' = 0$ and $\mathbf{E}^T \mathbf{e} = 0$ (epipoles are the nullspace of $\mathbf{E}$ – note all epipolar lines pass through epipoles)
- **Degrees of freedom (Recall $\mathbf{E} = [t_x] \mathbf{R}$)?**
  - $5 - 3 \text{ (R)} + 3 \text{ (t)} - 1$ due to scale ambiguity
- $\mathbf{E}$ is singular (rank 2); it has two non-zero and *identical* singular values

\[
\begin{align*}
\hat{\mathbf{p}} &= \mathbf{K}^{-1} \mathbf{p} \\
\hat{\mathbf{p}'} &= \mathbf{K}'^{-1} \mathbf{p}'
\end{align*}
\]
Essential Essential Matrix Facts

• One nice thing: if I estimate $E$ from two images (more on this later), it’s unique up to easy symmetries

\[
\hat{p} = K^{-1}p
\]

\[
\hat{p}' = K'^{-1}p'
\]
What if we don’t know $K$?

Have: \( \hat{p} = K^{-1}p, \hat{p}' = K'^{-1}p', \hat{p}^T E \hat{p}' = 0 \)

\[
(K^{-1}p)^T E (K'^{-1}p') = 0 \rightarrow p^T K^{-T} E K'^{-1} p' = 0
\]

Set: \( F = K^{-T} E K'^{-1} \) \hspace{2cm} \text{Then:} \quad p^T F p' = 0

Fundamental Matrix (Faugeras and Luong, 1992)
Fundamental Matrix Fundamentals

- $Fp', F^T p$ are epipolar lines for $p'$, $p$
- $Fe' = 0, F^T e = 0$
- $F$ is singular (rank 2)
- $F$ has seven degrees of freedom
- $F$ definitely not unique
Estimating the fundamental matrix
Estimating the fundamental matrix

• F has 7 degrees of freedom so it’s in principle possible to fit F with seven correspondences, but it’s a slightly more complex and typically not taught in regular vision classes
Estimating the fundamental matrix

Given correspondences \( p = [u, v, 1] \) and \( p' = [u', v', 1] \) (e.g., via SIFT) we know: \( p^T F p' = 0 \)

\[
\begin{bmatrix}
  f_{11} & f_{12} & f_{13} \\
  f_{21} & f_{22} & f_{23} \\
  f_{31} & f_{32} & f_{33}
\end{bmatrix}
\begin{bmatrix}
  u' \\
  v' \\
  1
\end{bmatrix} = 0
\]

\[
[u, v, 1] \cdot [uu', uv', u, vu', vv', v, u', v', 1] \cdot [f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}] = 0
\]

How do we solve for \( f \)?
How many correspondences do we need?
Leads to the eight point algorithm
Eight Point Algorithm

Each point gives an equation:
\[
[uu', uv', u, vu', vv', v, u', v', 1] \cdot [f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}] = 0
\]

Stack equations to yield \( \mathbf{U} \):
\[
\mathbf{U} = \begin{bmatrix}
    u_i u'_i & u_i v'_i & u_i & v_i u'_i & v_i v'_i & v_i & u'_i & v'_i & 1
\end{bmatrix}
\]

Usual eigenvalue stuff to find \( \mathbf{f} \) (\( \mathbf{F} \) unrolled):
\[
\arg \min_{\|\mathbf{f}\|=1} \|\mathbf{U}\mathbf{f}\|_2^2 \quad \text{Eigenvector of } \mathbf{U}^T\mathbf{U} \text{ with smallest eigenvalue}
\]
If we estimate $F$, we get some $3 \times 3$ matrix $F$. We know $F$ needs to be singular/rank 2. How do we force $F$ to be singular?

$$U \Sigma V^T = F_{init}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Sigma' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Open it up with SVD, mess with singular values, put it back together.

$$F = U \Sigma' V^T$$

See Eckart–Young–Mirsky theorem if you’re interested.
Eight Point Algorithm – Difficulty 1

Estimated F (Wrong)  Estimated + SVD’d F (Correct)

Slide Credit: S. Lazebnik
Eight Point Algorithm – Difficulty 2

\[
\begin{bmatrix}
    u, u', v, v', v', 1
\end{bmatrix} \cdot \begin{bmatrix}
    f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}
\end{bmatrix}^T = 0
\]

Recall: \( u, u' \) are in pixels. Suppose image is 1Kx1K

How big might \( uu' \) be? How big might \( u \) be?

Each row looks like:

\[
U = \begin{bmatrix}
    10^6 & 10^6 & 10^3 & 10^6 & \vdots & 10^6 & 10^3 & 10^3 & 10^3 & 1
\end{bmatrix}
\]

Then: \( U^T U_{1,1} \) is \( \sim 10^{12} \), \( U^T U_{2,9} \) is \( \sim 10^3 \)
Eight Point Algorithm – Difficulty 2

Numbers of varying magnitude $\rightarrow$ instability

Remember: a floating point number (float/double) isn’t a “real” number: for sign, coefficient, exponent integers

$$(-1)^{\text{sign}} \times \text{coefficient} \times 2^{\text{exponent}}$$

Exercise to see how this screws up: add up Gaussian noise (mean=100, std=10), divide by number you added up
Remember Numerical Instability?

Code:

\[ x += N(100,10) \]
\[ i += 1 \]
\[ \text{mean} = x/i \]

Only change is the # of bits in accumulator x

Note: 50M is 50 1Kx1K images
Solution: Normalized 8-point

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute $F$ from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of $F$ and throw out the smallest singular value)
- Transform fundamental matrix back to original units: if $T$ and $T'$ are the normalizing transformations in the two images, than the fundamental matrix in original coordinates is $T'^T F T$

R. Hartley

In defense of the eight-point algorithm TPAMI 1997
Last Trick

Minimizing via $U^TU$ minimizes sum of squared algebraic distances between points $p_i$ and epipolar lines $Fp'_i$ (or points $p'_i$ and epipolar lines $F^Tp_i$):

$$\sum_i (p_i^T F p'_i)^2$$

May want to minimize geometric distance:

$$\sum_i d(p_i, Fp'_i)^2 + \sum_i d(p'_i, F^Tp_i)^2$$

$Fp'_i = 0$

Slide Credit: S. Lazebnik
## Comparison

<table>
<thead>
<tr>
<th></th>
<th>8-point</th>
<th>Normalized 8-point</th>
<th>Nonlinear least squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>Av. Dist. 1</td>
<td>2.33 pixels</td>
<td>0.92 pixel</td>
<td>0.86 pixel</td>
</tr>
<tr>
<td>Av. Dist. 2</td>
<td>2.18 pixels</td>
<td>0.85 pixel</td>
<td>0.80 pixel</td>
</tr>
</tbody>
</table>

Slide Credit: S. Lazebnik
The Fundamental Matrix Song

http://danielwedge.com/fmatrix/
Estimating the fundamental matrix is known as “weak calibration”.

If we know the calibration matrices of the two cameras, we can estimate the essential matrix:

\[ E = K' K \]

The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters.

Alternatively, if the calibration matrices are known, the **five-point algorithm** can be used to estimate relative camera pose.