On Weighted Least Squares Tomographic Reconstruction and the "Consistency Condition" of Chinn and Huang

Jeffrey A. Fessler
4240 EECS Bldg., University of Michigan, Ann Arbor, MI 48109-2122
734-763-1434, FAX: 734-763-8041, Email: fessler@umich.edu, EDICS: 2.3

Abstract

A recent letter in this journal (Chinn and Huang, Vol. 2, No. 3, 1995) analyzed the weighted least squares method for tomographic image reconstruction. The analysis hinges on a certain "consistency condition," namely that for any image $\mathbf{x}$, the vector $\mathbf{R}^{-1}\mathbf{P}\mathbf{x}$ must lie in the range of $\mathbf{P}$, where $\mathbf{R}$ is the measurement covariance matrix and $\mathbf{P}$ is the projection operator (e.g. discretized Radon transform). In this letter, we show that this condition rarely applies, except in the trivial case where $\mathbf{R}$ is a scaled identity matrix. Thus the conclusions of Chinn and Huang apply only to ordinary unweighted least squares, rather than weighted least squares.

I. Background

A typical model for tomography is:

$$\mathbf{y} = \mathbf{P}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^m$ is the measurement vector, $\mathbf{x} \in \mathbb{R}^n$ is the unknown image vector, $\mathbf{n} \in \mathbb{R}^m$ is noise noise, and $\mathbf{P}$ is a m by n system matrix (discretized Radon transform). In the context of studying least-squares solutions to (1), one generally assumes $m > n$. The weighted least squares WLS solution to (1) is well known to be

$$\hat{\mathbf{x}} = (\mathbf{P}^T\mathbf{W}\mathbf{P})^{-1}\mathbf{P}^T\mathbf{W}\mathbf{y}.$$

Typically one takes $\mathbf{W}$ to be $\mathbf{R}^{-1}$, the inverse of the noise covariance matrix, if it is known. Otherwise one can sometimes use approximations to $\mathbf{R}^{-1}$ [1]. In emission tomography $\mathbf{W}$ is virtually always taken to be a diagonal matrix. It is well known that when $\mathbf{W} = \mathbf{R}^{-1}$, the variance of $\mathbf{x}$ is less than (or equal to) the variance of the ordinary least-squares estimator.

II. The "Consistency Condition" of Chinn and Huang

In [2] and [3], Chinn and Huang attempt to analyze the WLS method using the following "consistency condition." The operator $\mathbf{W}\mathbf{P}$ is said to be consistent if and only if for any $\mathbf{x} \in \mathbb{R}^n$, there exists a $\mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{W}\mathbf{P}\mathbf{x} = \mathbf{P}\mathbf{z},$$

i.e. the operator $\mathbf{W}\mathbf{P}$ maps into the range of $\mathbf{P}$. Those authors show in [3] that *if* the operator $\mathbf{W}\mathbf{P}$ were consistent, then there would exist a $n \times n$ matrix $\tilde{\mathbf{W}}$ such that

$$\mathbf{W}\mathbf{P} = \mathbf{P}\tilde{\mathbf{W}}. \tag{2}$$

Furthermore, the matrix $\tilde{\mathbf{W}}$ would be given by $\tilde{\mathbf{W}} = (\mathbf{P}^T\mathbf{P})^{-1}\mathbf{P}^T\mathbf{W}\mathbf{P}$.

Chinn and Huang fail to discuss the existence of consistent operators. If $\mathbf{W}$ is a scaled identity matrix, i.e. $\mathbf{W} = \kappa\mathbf{I}$, then the consistency condition is satisfied trivially. But in this case WLS degenerates to ordinary least squares. Is the consistency condition satisfied more generally? Unfortunately, the answer is *no*.

Consider the simplest case $m = 2$ and $n = 1$. Then (2) implies

$$\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tilde{w}_1$$

Clearly if p_1 and p_2 are nonzero, then the only consistent solution is the case $w_1 = w_2 = \tilde{w}_1$, i.e. $\mathbf{W}$ is a scaled identity matrix.

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More generally, for any $m > n$, one can easily specify many operators $\mathbf{P}$ for which the only case where (2) has a solution for $\tilde{\mathbf{W}}$ is when $\mathbf{W}$ is a scaled identity matrix. An example is the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 \\ & \multicolumn{2}{c}{\text{anything}} & \end{bmatrix}.$$

Is there reason for optimism that the operators $\mathbf{P}$ in tomography are somehow different than the multitude of cases where the only solutions to (2) are scaled identity matrix? One can easily see that (2) corresponds to mn equations in n^2 unknowns of the form

$$\sum_k w_{ik} p_{kj} = \sum_k p_{ik} \tilde{w}_{kj}.$$

Since $m > n$, there is little hope of a consistent solution except when $\mathbf{W}$ is a scaled identity matrix.

III. Discussion

Since the existence of “consistent” operators $\mathbf{WP}$ is unlikely for realistic $\mathbf{W}$, the conclusions of [2], [3] are unsubstantiated. The “direct algorithm” for computing WLS images given in [2] is unlikely to produce correct results, and the suggestion in [3] that WLS and LS have equivalent mean square error is without basis (as Table I in [3] in fact shows). Our own simulation results consistently show significant differences in variance between weighted and unweighted least squares when resolution is matched properly.

References

[1] J A Fessler. Penalized weighted least-squares image reconstruction for positron emission tomography. *IEEE Tr. Med. Im.*, 13(2):290–300, June 1994.
[2] G Chinn and S C Huang. Weighted least-squares filtered backprojection tomographic reconstruction. *IEEE Signal Proc. Letters*, 2(3):49–50, March 1995.
[3] G Chinn and S C Huang. A comparison of WLS and LS reconstruction for PET. In *Proc. IEEE Nuc. Sci. Symp. Med. Im. Conf.*, volume 2, pages 1242–6, 1995.

IV. Notes

This letter was rejected based on the comments a single reviewer who stated that the results appear to be correct, but that the paper’s conclusions are overly strong. (Signal Processing Letters does not allow for revisions.)
Please draw your own conclusions.