Accelerated Regularized Estimation of MR Coil Sensitivities Using Augmented Lagrangian Methods: Supplementary Material

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This document contains additional information related to the algorithms presented in [1]. In particular, Section S-I presents a second ADMM algorithm for sensitivity profile estimation. Section S-II presents an alternate formulation that leads to additional AL estimation algorithms with similar performance. Section S-III illustrates the improved SENSE reconstruction quality resulting from using regularized sensitivity estimates over traditional ratio based estimates. Section S-IV demonstrates why, when performing masked sensitivity estimation on breast data, a convex hull of the object voxels should be used for the estimation mask. Section S-V illustrates the importance of using a finite differencing matrix for the case of non-periodic boundary conditions in our cost function (1).

S-I. ADMM–CG: ADMM SENSITIVITY ESTIMATION ALGORITHM WITH CONJUGATE GRADIENT SUBSTEPS

In this section we present and evaluate a second AL algorithm, ADMM–CG, which does not use the reformulation of the finite differencing matrix as discussed in Section II-B of the manuscript. We begin with the derivation of the algorithm which uses the same techniques as in the manuscript. We then compare this new algorithm to the methods presented in the manuscript using the same data sets and briefly discuss its properties.

A. Method Derivation

We begin our derivation by introducing two new variables, \(u_0 \in \mathbb{C}^M\) and \(u_1 \in \mathbb{C}^N\), to the initial cost function in (1). The purpose of these variables is to isolate the finite differencing matrix \(R\) from the diagonal matrix \(D\). The resulting constrained optimization problem is

\[
\hat{s} = \arg \min_{s,u_1} \frac{1}{2} \| z - Du_1 \|_W^2 + \frac{\lambda}{2} \| u_0 \|_2^2 \\
\text{s.t. } u_1 = s \text{ and } u_0 = Rs.
\]  

(S-1)

Solving this constrained optimization problem is exactly equivalent to solving the unconstrained problem (1).

As in (4), we can express (S-1) in the more concise notation:

\[
\hat{s} = \arg \min_{s,u} \frac{1}{2} \| h - Ku \|_2^2 \text{ s.t. } u = Gs,
\]  

(S-2)

where \(u\) and \(h\) were defined in (4),

\[
G \triangleq \begin{bmatrix} I \\ R \end{bmatrix}, \text{ and } K \triangleq \begin{bmatrix} W^\dagger D & 0 \\ 0 & \sqrt{\lambda}I \end{bmatrix}.
\]

We then tackle (S-2) using the previously described AL formalism and obtain the following AL function-based minimization problem:

\[
\arg \min_{s,u} \frac{1}{2} \| h - Ku \|_2^2 + \frac{1}{2} \| u - Gs - \eta \|_V^2,
\]  

(S-3)

where \(\eta\) and \(V\) were defined in (5).

Due to the complexity of jointly minimizing (S-3) over \(s\) and \(u\), we again consider an alternating minimization scheme. In particular, we sequentially solve the following set of equations:

\[
s^{(j+1)} = \arg \min_s \frac{1}{2} \| u^{(j)} - Gs - \eta^{(j)} \|_V^2, \]  

(S-4)

\[
\hat{u}^{(j+1)} = \arg \min_u \frac{1}{2} \| h - Ku \|_2^2 + \frac{1}{2} \| u - Gs^{(j+1)} - \eta^{(j)} \|_V^2.
\]  

(S-5)

As with ADMM–Circ, the update equation for \(\hat{u}\), (S-5), has a simple closed-form solution which can be decoupled into two parallel updates in terms of \(u_1\) and \(u_0\) due to the block diagonal structures of \(K\) and \(V\):

\[
u_1^{(j+1)} = D_2^{-1} \left[ D^H W z + \nu_1 (s^{(j+1)} + \eta_1^{(j)}) \right], \]  

(S-6)

\[
u_0^{(j+1)} = \frac{\nu_0}{\nu_0 + \lambda} \left( R s^{(j+1)} + \eta_0^{(j)} \right). \]  

(S-7)

where \(D_2 \triangleq D^H W D + \nu_0 I\) is a diagonal matrix.

Equation (S-4) does not have an efficient closed-form solution due to the size and complexity of \(R\). Instead, we approximately solve (S-4) using several iterations of the preconditioned conjugate gradient (PCG) method with warm starting, the optimal number of which is determined empirically. We design the specific preconditioner, \(P\), by considering the closed-form solution of (S-4):

\[
G^T s^{(j+1)} = G^T V (u^{(j)} - \eta^{(j)}), \]  

(S-8)

where \(G_2 \triangleq G^H V G = \nu_1 I + \nu_0 R^H R\). Our goal is to create an easily invertible \(P\) that preconditions \(G_2\) to obtain fast

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and in part by the National Institutes of Health (NIH) under Grant P01 CA87634. Asterisk indicates corresponding author.

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convergence for this subproblem. For typical finite differencing matrices with non-periodic boundaries, $R^H R$ has a near block circulant with circulant blocks structure. We therefore approximate $R^H R$ in our preconditioner as $Q^H \Omega Q$ where $Q$ is a (multidimensional) discrete Fourier transform (DFT) matrix and $\Omega$ is a diagonal matrix containing the spectrum of the convolution kernel of $R^H R$ [2]. Our resulting preconditioner is

$$ P = Q^H (\nu_1 I + \nu_0 \Omega) Q. $$  \hfill (S-9)

Fig. S-1 summarizes the resulting estimation algorithm composed of these update steps and the corresponding Lagrange multiplier updates, ADMM–CG. Note that the minimization in Step 1 is exact, requiring an iterative solution; however, the optimal number of iterations is typically small. Furthermore, it can be shown that this algorithm is equivalent to an ADMM algorithm with an approximate update step for which the errors at each outer iteration can be made absolutely summable by using enough PCG iterations. We can therefore conclude that this algorithm converges to the solution of (1) as per [3, Th. 8].

**ADMM–CG**

Initialize: $u_1^{(0)} = s^{(0)}$, $u_0^{(0)} = Rs^{(0)}$, $\eta_0^{(0)} = 0$, $\eta_1^{(0)} = 0$ and $j = 0$.

Set $D_2^{-1} = [D^{H} WD + \nu_1 I]^{-1}$ and $z_2 = D^{H} Wz$.

Repeat until stop criterion is achieved:

1) $s^{(j+1)}$ from PCG solution of (S-4) using (S-9).

2) $u_1^{(j+1)} = D_2^{-1} \left[ z_2 + \nu_1 \left( s^{(j+1)} + \eta_1^{(j)} \right) \right]$, $u_0^{(j+1)} = \frac{\nu_0}{\nu_0 + \lambda} \left( Rs^{(j+1)} + \eta_0^{(j)} \right)$.

3) $\eta_1^{(j+1)} = \eta_1^{(j)} - \left( u_1^{(j+1)} - s^{(j+1)} \right)$, $\eta_0^{(j+1)} = \eta_0^{(j)} - \left( u_0^{(j+1)} - Rs^{(j+1)} \right)$.

4) $j = j + 1$.

**B. Alternating Minimization with Intermediate Updating**

We also explored updating the Lagrange multipliers between each alternating minimization step. The resulting variation, ADMM–CG–IU, is presented in Fig. S-2. As with the ADMM–Circ–IU algorithm, this algorithm lacks a guarantee of convergence although such guarantees exist for similar intermediate updating algorithms [4].

**C. Parameter Selection**

The parameter selection strategy for our ADMM–CG based algorithms is similar to the strategy for ADMM–Circ because of the analogous structures of the alternating minimization steps. The major difference is that the update of $u_0$ in (S-7) does not require the inversion of a matrix but rather a scalar term. In fact, this scalar term has the same form as $\kappa(B_2)$ in the manuscript. Subsequently, we found that setting $\nu_0$ such that the scalar $\frac{\nu_0 + \lambda}{\nu_0}$ is in $[200, 400]$ and then setting $\nu_1$ such that $\kappa(G_2) \in [200, 1000]$ provided reasonable convergence rates.

**D. Results**

To evaluate our proposed ADMM–CG based algorithms, we performed the same experiments as in Section III of the manuscript. The cost function was set up as described in Section III-A and the same ratio based estimate was used to initialize the algorithms. We used a single PCG iteration for the update to $s$ in the ADMM–CG based algorithms as this provided the fastest convergence rates with respect to time. We selected the AL penalty parameters $\nu_0$ and $\nu_1$ for ADMM–CG such that $\frac{\nu_0 + \lambda}{\nu_0} = 225$ and $\kappa(G_2) = 600$. As further discussed in Section S-I-E, the optimal condition numbers for ADMM–CG–IU depended on the data and are therefore mentioned in the appropriate subsections.

1) **Simulated Brain Data:** We ran $20000$ iterations of the ADMM–CG based algorithms on the simulated brain data described in Section III-B of the manuscript. For the ADMM–CG–IU algorithm, we selected $\nu_0$ and $\nu_1$ such that $\frac{\nu_0 + \lambda}{\nu_0} = 375$ and $\kappa(G_2) = 600$. Our proposed ADMM–CG and ADMM–CG–IU algorithms converged to a normalized $\ell_2$-distance of less than $-200$ dB from the Cholesky based solution to (1) and appeared nearly identical to Fig. 6. The convergence rates of the algorithms were similar for all four coils and thus we present the results for the same coil that was presented in Section III-B. Fig. S-3 plots $D(s^{(j)})$ with respect to both iteration and time for the ADMM–CG based algorithms as well as the algorithms evaluated in the manuscript. ADMM–CG–IU and ADMM–CG were both slower than PCG–Circ, but faster than conventional CG, reaching $D(s^{(j)}) = 0.1\%$ in approximately 145 and 185 seconds, respectively.

2) **Breast Phantom Data:** We also ran $20000$ iterations of the ADMM–CG based algorithms on the breast phantom data described in Section III-C of the manuscript. For the
ADMM–CG–IU algorithm, we selected $\nu_0$ and $\nu_1$ such that $\frac{\nu_0 + \nu_1}{\nu_0} = 250$ and $\kappa(G_d) = 600$. Again, both of our proposed algorithms converged to a normalized $\ell_2$-distance of less than -200 dB from the Cholesky based solution to (1) and appeared nearly identical to Fig. 9(b). The convergence rates of the algorithms were similar for all four coils and thus we present the results for the same coil that was presented in Section III-C. Fig. S-4 plots $D(s^{(j)})$ with respect to both iteration and time for the ADMM–CG–IU algorithms as well as the algorithms evaluated in the manuscript. ADMM–CG–IU was faster than both PCG–Circ and regular CG converging within $D(s^{(j)}) = 0.1\%$ in approximately 80 seconds. ADMM–CG was slower than its intermediate updating counterpart and PCG–Circ with a convergence time of nearly 120 seconds.

**E. Discussion**

The convergence rates with respect to iteration of the ADMM–CG based algorithms were close to their ADMM–Circ counterparts. However, the ADMM–CG based algorithms were much slower in time due to the added overhead of the PCG solution used to approximate Step 1. In fact, even when using only one iteration of PCG for this approximation, the per iteration costs of the ADMM–CG algorithms are much higher than those of the ADMM–Circ algorithms, Table S-1.

The convergence curves of the ADMM–CG based algorithms exhibit a higher rate of non-monotonic behavior than the ADMM–Circ algorithms. This is partly caused by the approximate update in Step 1. If we run several more PCG sub-iterations in Step 1, the convergence curves with respect to iteration of the ADMM–CG algorithms appear similar to their ADMM–Circ counterparts (although much slower with respect to time). As with ADMM–Circ, the parameter settings that provided the fastest convergence rates typically resulted in non-monotonicity in the $D(s^{(j)})$ plots.

The proposed ADMM–CG–IU algorithm was faster than PCG–Circ in the breast phantom experiment, but slower in the simulated brain experiment. As discussed in Section IV of the manuscript, the relative speed of the PCG–Circ algorithm depends on the accuracy of the preconditioner in (13) and thus on the data. Contrarily, the preconditioning used for the approximation of Step 1 in the ADMM–CG algorithms does not depend on the data; thus, these algorithms are less sensitive to such differences. The ADMM–CG algorithm, although initially faster, converged slower than PCG–Circ in both experiments. Therefore, using intermediate updating also significantly accelerated the convergence rates of this ADMM algorithm. All of our proposed algorithms were significantly faster than traditional CG.

The convergence rates of our proposed ADMM–CG algorithms were robust to the choice of condition numbers used to determine the AL penalty parameters $\nu_0$ and $\nu_1$. We found that the convergence rates remained similar for condition numbers that differed from the optimal values by up to fifty percent. The chosen condition numbers also worked well for a wide range of regularization parameter values $\lambda$. However, we found that
the optimal condition numbers for ADMM–CG–IU depended on the data unlike for ADMM–CG and the ADMM–Circ based algorithms. Still, like the ADMM–Circ algorithms, the choice of the optimal condition numbers does not depend on the surface coil image. Therefore, if one wanted to fine-tune the convergence rate of the algorithms, a single coil of a multi-coil array would suffice.

S–II. Similar Splitting

In formulating our proposed ADMM–Circ algorithm, we originally explored a different variable splitting strategy involving a double splitting within the regularization term of (1) [5], [6]:

$$\arg \min_{s, M_0} \frac{1}{2}\|s - Ds\|_W^2 + \frac{\lambda}{2}\|Bu_0\|_2^2 \quad \text{s.t. } u_0 = Cu_1 \quad \text{and } u_1 = s.$$  \hspace{1cm} (S–10)

This variable splitting led to update equations with nearly identical structures to those of ADMM–Circ. Furthermore, the resulting AL algorithm and its intermediate updating variation had similar convergence rates to their ADMM counterparts. However, analyzing the convergence properties of these algorithms was more complicated as they did not have ADMM structures. Thus, we focused on the ADMM formulations.

S–III. Improved SENSE Reconstruction Quality

The advantages and accuracy of similar regularized sensitivity profile estimators have been discussed in previous papers [7], [8]; however, there has been limited investigation into their effects on SENSE reconstruction quality. We therefore compare the quality of the SENSE reconstructions created with the coil sensitivities estimated using the regularized method in (1) to those estimated using the commonplace ratio and ratio of low resolution images methods.

A. Simulated Brain Data

Our first experiment was performed using the simulated brain data outlined in Section III–B of the manuscript. We began by simulating a full resolution calibration scan using the same parameters as Figs. 4 and 5. Next, we estimated the coil sensitivities from the resulting body and surface coil images using our regularized method, the ratio of low resolution images method, and the ratio method.

We implemented the regularized method using our ADMM–Circ–IU algorithm with the same parameters as in Section III of the manuscript. The ratio of low resolution images method was implemented by taking a set number of samples from the center of k-space of each coil, zero padding to get $256 \times 192$ element matrices (corresponding to a $256 \text{ mm} \times 192 \text{ mm FOV}$), and reconstructing low resolution body coil and surface coil images using inverse DFTs. Smooth sensitivity estimates were then obtained by taking the ratio of these low resolution images. We present the results for two different amounts of sampling. The first uses the center $13 \times 9$ k-space samples resulting in sensitivity estimates that extend smoothly to the image edges. The second uses the center $51 \times 38$ samples which was found to provide the best SENSE reconstruction quality for this method. In both cases we applied a Hamming window to the selected k-space data to reduce any Gibbs ringing artifacts. The conventional ratio estimate ($\hat{s}_i = z_i/y_i$) was masked to remove the highly corrupt estimates of the non-object pixels using a binary mask created by thresholding the body coil image. The resulting sensitivity profile estimates for a single, representative coil are presented in Fig. S–5.

As seen in Fig. 6 of the manuscript, the regularized estimate is very close to the true sensitivity, differing only at the corners of the image. The minor discrepancies at the corners of the estimates are in part due to selecting a regularization parameter that emphasized accuracy over the object pixels and their immediate surrounding area as well as from the fact that there is no information about the true sensitivity in this region of the image. The conventional ratio estimate is much noisier over the object pixels and has no extrapolation. Both low resolution ratio estimates are smooth over the object support with varying degrees of extrapolation into the surrounding

![Image](https://example.com/image1.png)

![Image](https://example.com/image2.png)

![Image](https://example.com/image3.png)

![Image](https://example.com/image4.png)
regions. However, the implicit smoothing of these methods creates inaccuracies in the estimates near object edges and in areas with predominantly low signal. Furthermore, any voxels significantly beyond the extrapolated regions exhibit large estimation errors. The typical errors that result from Gibbs ringing artifacts [7] have been reduced by the additional windowing.

Data from the four surface coils were then simulated for every other vertical line in k-space. SENSE reconstructions [9] were performed using this undersampled data set and the various sensitivity profile estimates. We restricted the reconstruction to a masked region found by dilating the thresholded body coil image by two pixels. These reconstructions and their differences to the truth are presented in Fig. S-6.

The resulting normalized root-mean-square errors (NRMSE) between the SENSE reconstructions and the truth are presented in Table S-2. The regularized method led to the most accurate SENSE reconstruction in terms of NRMSE as well as the one with the fewest structural artifacts (beyond the amplified noise inherent to SENSE reconstruction). The low resolution ratio method with the center $51 \times 38$ samples led to the second most accurate SENSE reconstruction; however, the inaccuracies in the sensitivity estimates at the object edges resulted in larger artifacts due to the shift, particularly at the far right side of the brain. The low resolution ratio method with the center $13 \times 9$ samples again resulted in the worst SENSE reconstruction. The shift of two pixels to the right emphasized the inaccuracies in the estimates near the object edges by introducing even larger artifacts (not visible with the current contrast windowing). The SENSE reconstruction based on the conventional ratio method was significantly affected by the shift. In particular, the lack of any extrapolation in the estimated sensitivity profiles resulted in large artifacts within the object support.

### Table S-2

<table>
<thead>
<tr>
<th>Shift</th>
<th>Regularized $51 \times 38$</th>
<th>Low Res. Ratio $51 \times 38$</th>
<th>Ratio $13 \times 9$</th>
<th>Ratio $13 \times 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.06</td>
<td>0.07</td>
<td>0.16</td>
<td>0.12</td>
</tr>
<tr>
<td>2 pixels</td>
<td>0.06</td>
<td>0.07</td>
<td>0.16</td>
<td>0.16</td>
</tr>
</tbody>
</table>

The resulting NRMSEs between the SENSE reconstructions and the true shifted brain are presented in Table S-2. The regularized method again led to the most accurate SENSE reconstruction with similar NRMSE and a lack of structural artifacts. The low resolution ratio method with the center $51 \times 38$ samples led to the second most accurate SENSE reconstruction; however, the inaccuracies in the sensitivity estimates at the object edges resulted in larger artifacts due to the shift, particularly at the far right side of the brain. The low resolution ratio method with the center $13 \times 9$ samples again resulted in the worst SENSE reconstruction. The shift of two pixels to the right emphasized the inaccuracies in the estimates near the object edges by introducing even larger artifacts (not visible with the current contrast windowing). The SENSE reconstruction based on the conventional ratio method was significantly affected by the shift. In particular, the lack of any extrapolation in the estimated sensitivity profiles resulted in large artifacts within the object support.

### C. High SNR Simulated Brain Data

To better illustrate the typical inaccuracies produced by the ratio of low resolution images estimation method, we repeated the previous SENSE reconstruction experiments using simulated brain data with a higher SNR of 20. The specific body coil and four surface coil images are presented in Fig. S-8. We performed sensitivity estimation using the regularized method and the ratio of low resolution images method with $51 \times 38$ samples. The resulting estimates for a representative coil are presented in Fig. S-9.

The sensitivity profile estimates are similar to those for the case of lower SNR brain data found in Fig. S-5. The regularized estimate is again very close to the true sensitivity differing only at the corners of the image. The ratio of low resolution images estimate is smooth over the object support and exhibits some extrapolation. However, there are noticeable inaccuracies in areas corresponding to regions of low signal within the brain.

We performed two-fold accelerated SENSE reconstructions with the higher SNR brain data and these sensitivity profile estimates. The results for both the case of no shift between calibration and scan, as well as a two pixel shift, are presented in Fig. S-10. As with the case of low SNR brain data, the reconstructions created using the regularized sensitivity estimate have very low error and no major structural artifacts. Furthermore, the two pixel shift had little effect on the reconstruction quality indicating accurate extrapolation within the estimate. In contrast, the reconstruction created using the low resolution ratio estimates had several large structural artifacts (indicated with a yellow arrow) that were a result of the inaccurate sensitivity profile estimates in regions of low signal. The two pixel shift increased these artifacts indicating inaccurate extrapolation within the sensitivity estimates.

### D. Breast Phantom Data

We also compared the sensitivity estimation methods using our breast phantom data from Section III-C of the manuscript. In this case, we estimated the sensitivities of the breast...
phantom images presented in Fig. 9 using the same four methods as before: the regularized method with $\lambda = 2^7$, the ratio method, and the low resolution ratio method with both the center $77 \times 19$ and center $19 \times 5$ samples zero padded to $384 \times 96$ elements. The resulting estimates for a representative coil are presented in Fig. S-11.

The regularized estimate is smooth over the entire field-of-view and closely matches the general trend in the ratio estimate. The low resolution ratio estimate with the center $77 \times 19$ samples is reasonably smooth over the object support with some extrapolation into the surrounding pixels. There are inaccuracies in the estimate near regions of low signal such as at the object edges and over the far right breast. The low resolution ratio estimate with the center $19 \times 5$ samples is smoother than the case of $77 \times 19$ samples and exhibits greater extrapolation. However, this estimate suffers from oversmoothing and is highly inaccurate at the object edges. Both of the low resolution ratio methods benefited from using a Hamming window to reduce the Gibbs ringing artifacts. The ratio estimate is very noisy over the object pixels and has no extrapolation.

To simulate the minor changes in the data that would occur between a calibration scan and an acquisition scan, we performed a SENSE reconstruction on a neighboring two-dimensional slice of our breast phantom data. The fully sampled body and surface coil images of this slice are presented in Fig. S-12. First, we undersampled the surface coil images by selecting every other vertical line in $k$-space. As was done for the simulated brain data, we then performed SENSE reconstructions over a masked region using the previously estimated coil sensitivities. These reconstructions are presented in Fig. S-13.

The SENSE reconstruction resulting from the regularized estimate has very high quality and few visible artifacts when compared to the body coil image in Fig. S-12(a). The reconstruction resulting from the low resolution ratio estimate with the center $77 \times 19$ samples appears similar to that of the regularized estimate; however, the inner parts of the breasts are darker than in the body coil and surface coil images. This is largely a result of inaccurate sensitivity estimation in these low signal regions. In addition to the darkening artifact in the low signal regions of the image, the reconstruction resulting from the low resolution ratio estimate with the center $19 \times 5$ samples also has aliased edges of the breasts within the object support (indicated by a yellow arrow). These are a result of inaccurate sensitivity estimates at the object edges caused by oversmoothing. The reconstruction resulting from the conventional ratio method is very noisy and has several bright artifacts. This is due to inaccurate sensitivity estimation over the low signal pixels within the object support and a lack of extrapolation.

E. Discussion

From these experiments, we conclude that the regularized sensitivity estimation method outlined in (1), although more computationally expensive, provides improved sensitivity estimates for use in SENSE reconstructions compared to other commonly used non-parametric methods. Using a ratio of low
resolution images provides reasonable estimates if the correct number of samples is selected. However, even after windowing to reduce the Gibbs ringing artifacts, these estimates are typically inaccurate at object edges and in areas of low signal. This results in artifacts in the SENSE reconstructions. The lack of smoothing and extrapolation in the conventional ratio method results in SENSE reconstructions that are very noisy and prone to large artifacts due to motion.

S-IV. Estimation Over a Convex Hull Mask

If estimating over a masked region in order to reduce computation time, the mask must be carefully selected to ensure accurate estimates. For data sets with spatially-contiguous support, such as the simulated brain in our manuscript, this is relatively trivial; however, this is not the case for more complicated data sets such as our breast phantom data whose field-of-view (FOV) contains several spatially distinct objects. Due to the underlying physics, the typical coil sensitivity profile should smoothly vary across the entire FOV and generally decrease with distance from the coil. However, using a tight mask isolates the estimate over each object and this can result in large errors for objects that have low signal or only a few pixels. This can be avoided by using a mask consisting of a convex hull containing the spatially distinct objects.

To illustrate this phenomenon, we considered another slice of our breast phantom data, Fig. S-14. This image has a small object to the left of the right breast (indicated by an arrow). We perform regularized estimation over a masked region consisting of spatially distinct objects as well as a masked region consisting of a convex hull of these points. Fig. S-15 contains the two different masks and their corresponding sensitivity estimates. Fig. S-16 presents line profiles of the absolute value of the sensitivity estimates taken horizontally through the center of the FOV for both estimates.

Comparing the two estimates, it is clear that they are similar for regions with relatively high SNR; however, they differ greatly over the small object next to the right breast. When using a tight mask, the estimated sensitivity in this region is very high in comparison to the nearby breast which does not match the underlying physics. This inaccuracy is a result of the estimate in this region being based on only a few low signal pixels. In contrast, the convex hull estimate is smooth over the entire masked region and the estimate over the small object is more realistic. This is because such a mask enforces smoothness both within and between all of the objects in the FOV. Thus, a convex hull should be used for the estimation mask to avoid inaccuracies in the final estimates.

S-V. Circulant versus Non-Circulant Finite Difference Matrices

In this section we demonstrate the importance of using a finite differencing matrix for the case of non-periodic boundary conditions (R or BC) rather than a finite differencing matrix for the case of periodic boundary conditions (C) in our cost function. Since $C^H C$ is block circulant with circulant blocks, we will refer to the matrix for the case of periodic boundary conditions as the “circulant matrix”. In contrast, we will refer
Fig. S-8. The magnitude of the fully sampled (a) body coil and (b) surface coil images for our high SNR simulated brain data.

Fig. S-9. Example sensitivity profile estimates found for the high SNR brain data using (a) the regularized method and (b) the ratio of low resolution images method with the center 51 × 38 samples.

to the matrix for the case of non-periodic boundary conditions as the “non-circulant matrix”.

The receive coil is usually placed at or just beyond the boundary of the field-of-view. Since coil sensitivity is a physical phenomenon, its intensity will typically decrease with increased distance from the coil. However, if we use a circulant finite differencing matrix, we will be penalizing differences in the estimated sensitivities at opposing boundaries of the volume. Since there is often little information about the sensitivity at the edges of the volume, this penalization will result in a sensitivity estimate that dips near the coil and rises at the opposite side of the field-of-view. This is a clear mismatch with the underlying physics of the problem. Furthermore, because of the lack of meaningful information outside of the object voxels, this error will propagate to the estimate at the edges of the object. These errors within (and just outside) the object support can generate significant artifacts in SENSE reconstructions (see Section S-III). Padding the image with zeros will not sufficiently remove this propagated estimation error. Thus, one must use a more realistic modeling assumption and select a non-circulant finite differencing matrix that avoids penalizing between opposite boundaries at the expense of increased complexity. To illustrate these claims, we recreated the estimates found in Section III of our manuscript using both the existing non-circulant finite differencing matrix (\(R\)) and a circulant finite differencing matrix (\(C\)).

A. Simulated Brain Data

We present the results for one coil of the simulated brain data. Fig. S-17 presents the body coil image, true sensitivity, and resulting surface coil image used in this experiment. Fig. S-18 presents the resulting estimates using both the non-circulant and circulant finite differencing matrices, as well as the percentage difference image for each estimate compared to the truth.

As stated before, the estimate using the circulant finite differencing matrix dips before the boundary near the coil and rises at the opposite edge of the field-of-view. This results in significant inaccuracies in the estimate at the image boundaries. In contrast, the estimate using the non-circulant matrix increases smoothly towards the image boundary closest to the coil. The overall estimation error is therefore much smaller and is confined to the outer corners of the image.

Fig. S-19 presents the same estimates as Fig. S-18, but masked in the spatial domain to highlight the error over the object support. In these images, we see that the error in the estimates from the non-circulant matrix has propagated to within the object support. This highly structured inaccuracy will cause large artifacts in SENSE reconstructions. In contrast, the error in the estimate from the circulant matrix is much lower over the entire object support and contains significantly less structure.

B. Padded Simulated Brain Data

We also padded the brain data in Fig. S-17 with zeros to get a 256 × 256 image (an addition of 32 pixels to both the left and right sides of the image). Fig. S-20 presents the resulting estimates, masked to highlight the error over the object support. In these images, we see that the error in the estimates from the non-circulant matrix has propagated to within the object support. This highly structured inaccuracy will cause large artifacts in SENSE reconstructions. In contrast, the error in the estimate from the circulant matrix is much lower over the entire object support.
to within the object support. Furthermore, padding the images to within the object support. Thus, the need for a non-circulant matrix is also evident for the case of non-periodic boundary conditions avoided these errors. Thus, these experiments illustrate the need to use a non-circulant finite differencing matrix in the regularized estimator of (1).

C. Breast Phantom Data

We performed similar experiments on one coil of our breast phantom data found in Section III-C of the manuscript. Fig. S-21 presents the body coil and surface coil images used in this experiment.

Fig. S-22 presents the resulting estimates using both the non-circulant and circulant finite differencing matrices. As with the brain data, there is an unrealistic dip in the estimate near the coil and a rise at the opposing boundary when using the circulant finite differencing matrix, Fig. S-22(a). The estimate using the non-circulant matrix is more realistic, Fig. S-22(b).

Fig. S-23(a–b) presents the same estimates as Fig. S-22, but masked in the spatial domain to highlight the error over the object support. Fig. S-23(c) shows the difference between these two estimates. From these images, we see that the inaccuracies in the estimate at the boundaries of the image caused by the circulant finite differencing matrix propagated to within the object support. Thus, the need for a non-circulant finite differencing matrix is also evident for the case of real data.

D. Discussion

Using a finite differencing matrix for the case of periodic boundary conditions in our experiments caused substantial errors at the boundaries of the field-of-view and these propagated to within the object support. Furthermore, padding the images did not entirely mitigate the error. As seen in Section S-III, these types of errors can cause significant artifacts in SENSE reconstructions. However, using a finite differencing matrix for the case of non-periodic boundary conditions avoided these errors. Thus, these experiments illustrate the need to use a non-circulant finite differencing matrix in the regularized estimator of (1).

REFERENCES

Fig. S-11. Example sensitivity profile estimates found for the breast phantom data using (a) the regularized method (b) the ratio of low resolution images method with the center $77 \times 19$ samples, and (c) the ratio of low resolution images method with the center $19 \times 5$ samples, and (d) the conventional ratio method.

Fig. S-12. The magnitude of the fully sampled (a) body coil and (b) surface coil images for the neighboring two-dimensional slice of our breast phantom data.

Fig. S-13. Resulting two-fold accelerated SENSE reconstructions of the neighboring slice of breast phantom data (Fig. S-12) using the previous (a) regularized method (b) ratio of low resolution images method with the center $77 \times 19$ samples, (c) ratio of low resolution images method with the center $19 \times 5$ samples, and (d) conventional ratio method sensitivity profile estimates. The arrow in (b) points to a dark region in the reconstruction, while the arrow in (c) points to a reconstruction artifact caused by inaccurate sensitivity estimation at object edges.

Fig. S-14. The magnitude of the (a) body coil and (b) surface coil image for an additional slice of our breast phantom data. The yellow arrow points to a small object within the FOV.
Fig. S-15. The masks for the cases of (a) convex hull and (b) independent objects. The corresponding regularized estimates over the masked regions for the (c) convex hull and (d) independent objects.

Fig. S-16. Horizontal line profiles taken through the center of the sensitivity estimates presented in Fig. S-15.

Fig. S-17. The (a) body coil, (b) true coil sensitivity, and (c) resulting surface coil magnitude images for the simulated brain data.
Fig. S-18. The resulting sensitivity estimates for the brain data using a (a) circulant matrix and a (b) non-circulant matrix. The percentage difference between the truth and the estimates from the (c) circulant matrix and the (d) non-circulant matrix are shown below.

Fig. S-19. The same sensitivity estimates for the brain data as in Fig. S-18 but masked to highlight the error over the object support. (a) and (c) are the resulting estimate and percentage difference to the truth, respectively, resulting from a circulant matrix. (b) and (d) are the same but resulting from a non-circulant matrix.

Fig. S-20. The masked sensitivity estimates for padded brain data generated using a (a) circulant matrix and a (b) non-circulant matrix. The masked percentage difference between the truth and the estimates from the (c) circulant matrix and the (d) non-circulant matrix are shown below.

Fig. S-21. The (a) body coil and (c) surface coil magnitude images for the breast phantom data.

Fig. S-22. The sensitivity estimates for the breast phantom data generated using a (a) circulant matrix and a (b) non-circulant matrix.
Fig. S-23. The same sensitivity estimates for the breast data as in Fig. S-22 but masked to highlight the error over the object support. The difference between the estimates in (a) and (b) is presented in (c).