Noniterative Tomographic Reconstruction Using Simultaneous Weighted Spline Smoothing and Deconvolution

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Abstract

In [1] we proposed a spatially-variant method for removing noise from tomographic projection data using weighted spline smoothing. The projection model in [1] was based on detectors with the same width as their spacing. A better approximation for the detector response of many tomographs is a rectangular function or Gaussian function whose width is twice the detector spacing. (Such systems are almost adequately sampled in a Nyquist sense.) In this note, we derive a new weighted spline smoothing algorithm that accommodates arbitrary detector models. The implementation described in this note is based on more numerically stable B-splines, rather than the piecewise polynomials of [1]. Simulations demonstrate that the bias/variance tradeoffs using the new algorithm with a more accurate system model are improved relative to the method of [1] and to conventional spatially-invariant smoothing.

I. Theory

A. Projection Model

Let $g(x_1, x_2)$ denote the object being imaged, restricted to two dimensions for simplicity. The ideal line-integral projection of this object at an angle ϕ and radial offset τ is given by

$$l_\phi(\tau) = \int g(\tau\cos\phi - t\sin\phi, \tau\sin\phi + t\cos\phi)\, dt.$$

Assume that the tomographic system has a detector response that is approximately depth independent, and for the remainder drop the dependence on ϕ. The mean response of the ith detector is approximately:

$$p_i = L_i l,\ i = 1, \ldots, n,$$

where

$$L_i l = \int h_i(\tau) l(\tau)\, d\tau \tag{1}$$

and $h_i(\tau)$ is the line response of the ith detector.

Actual detector measurements will fluctuate around the ideal value p_i according to a statistical model that depends on the imaging modality. We are only interested in the first two moments of the statistical model, so we assume that the following model is a reasonable approximation:

$$y_i \sim \mathcal{N}(p_i, \sigma_i^2),$$

where σ_i^2 may have to be estimated from the data and correction factors [1–3].

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B. Objective Function

We would like to recover $l(\tau)$ from $\{y_i\}$. Since direct deconvolution would amplify the noise, we include smoothness constraints. Such smoothness constraints were also used by Hutchins *et al.* [4,5], although without variance weighting.

Smoothing splines are naturally suited to problems with second-order statistics and smoothness constraints. Therefore, we propose to estimate $l(\tau)$ using the the following penalized least-squares objective function:

$$\hat{l} = \arg \min_{l \in C_m} \sum_{i=1}^{n} \left(\frac{y_i - L_i l}{\sigma_i} \right)^2 + \beta \int |l^{(m)}(\tau)|^2 \, d\tau, \tag{2}$$

where $l^{(m)}$ is the mth derivative of l, and C_m is the class of functions whose mth derivatives are square integrable. This objective trades off smoothness (measured by a squared derivative) and a weighted agreement with the measurements. The smoothing parameter β controls that tradeoff, and is analogous to the cutoff frequency of a CBP filter. In fact, one can show that if the variance is homoscedastic and the system is radially-invariant, then generalized spline smoothing corresponds to a Wiener filter with Butterworth regularization of order $2m$ and half-power frequency that is a function of β [6]. In the tomographic applications of interest, the data is very heteroscedastic, so the "impulse response" of the smoothing spline will be spatially variant, as illustrated in [1].

C. System Modeling

Let the centers of the detectors be $\{\tau_i\}_{i=1}^n$. For a uniformly spaced detector array with spacing Δ, we have

$$\tau_i = \Delta \left(i - 1 - \frac{n-1}{2} \right).$$

For ring PET systems, the τ_i's are nonuniformly spaced. Most spline smoothing is based on the "evaluation" functionals:

$$L_i l = l(\tau_i).$$

These functionals ignore the finite width of tomographic detectors. In [1], we used the functional

$$L_i l = \int_{(\tau_i + \tau_{i-1})/2}^{(\tau_i + \tau_{i+1})/2} l(\tau) d\tau, \tag{3}$$

which corresponds to strips with widths equal to their spacing. In this note, we focus on the following functionals:

$$L_i l = \frac{1}{2} \int_{\tau_{i-1}}^{\tau_{i+1}} l(\tau) d\tau, \tag{4}$$

which correspond to overlapping strips with width about twice their spacing.

Under the system model (4), it follows from the Euler-Lagrange formulae for the variational problem (2) that its minimizer is a spline of order $2m$. The spline has an even order because of (4); evaluation functionals result in the more familiar odd-order splines (although see [7]).

As outlined in the Appendix, the coefficients of the smoothing spline are easily computed noniteratively using fast banded-matrix operations [8]. The computation increases with m, so we focus our attention on the case $m = 1$.

II. APPENDIX: SPLINE CALCULATIONS

In this appendix, we show that the solution to the minimization problem posed in (2) is a spline composed of polynomials of order at most $2m$, and then describe the banded matrices that are used to compute the polynomial coefficients.

The objective function of interest is:

$$\Phi(f) = \sum_{i=1}^{n} \left(\frac{y_i - \int h_i(\tau) f(\tau) \, d\tau}{\sigma_i} \right)^2 + \beta \int |f^{(m)}(\tau)|^2 \, d\tau. \tag{5}$$

By setting the Euler-Lagrange formula for the variational problem (5) to zero, one can show that:

$$\hat{f}^{(2m)}(\tau) = (-1)^m \beta^{-1} \sum_{i=1}^{n} \left(\frac{y_i - \int h_i(\tau) \hat{f}(t) \, dt}{\sigma_i} \right)^2 h_i(\tau). \tag{6}$$

From (4), $h_i(\tau)$ is $1/2$ for $\tau \in [\tau_{i-1}, \tau_{i+1})$ and zero elsewhere. Therefore, under (3) or (4) $\hat{f}$ is a polynomial of order $2m$ on each interval $[\tau_{i-1}, \tau_{i+1})$, and is a polynomial of order $2m - 1$ for $\tau \notin [\tau_0, \tau_{n+1}]$.

In the remainder, we focus our attention on the case $m = 1$, in which case the spline is quadratic on the measurement intervals and linear outside of $[\tau_0, \tau_{n+1}]$. One can show that under (3) or (4) the solution space of (6) (under appropriate end conditions) is spanned by the numerically stable B-spline basis:

$$\hat{f}(\tau) = \sum_{i=1}^{n} x_i b_i(\tau), \tag{7}$$

where

$$b_i(\tau) = \begin{cases} \frac{h_i + h_{i+1}}{4h_{i-1}} (\tau - \tau_{i-1})^2, & \tau \in [\tau_{i-1}, \tau_i) \\ \frac{(h_i + h_{i+1})(h_{i-1} + T_i)}{4} - \frac{h_{i-1} + 2h_i + h_{i+1}}{4h_i} (\tau - \tau_i - T_i)^2, & \tau \in [\tau_i, \tau_{i+1}) \\ \frac{h_{i-1} + h_i}{4h_{i+1}} (\tau - \tau_{i+2})^2, & \tau \in [\tau_{i+1}, \tau_{i+2}) \\ 0, & \tau \notin [\tau_{i-1}, \tau_{i+2}) \end{cases},$$

where $h_i = \tau_{i+1} - \tau_{i-1}$ is the interval width, and $T_i = h_i(h_i + h_{i+1})/(h_{i-1} + 2h_i + h_{i+1})$. One can verify that these basis functions are continuous, have continuous first derivatives, and satisfy $b_i(\tau_{i-1}) = b^{(1)}(\tau_{i-1}) = b_i(\tau_{i+2}) = b^{(1)}(\tau_{i+2}) = 0$. If one uses system models different from (3) or (4), then this basis does not necessarily span the space of solutions to (6). However, we expect that it should be adequate even in such cases.

Let $a_{ij} = \int h_i(\tau) b_j(\tau) \, d\tau$, so that

$$\int h_i(\tau) \hat{f}(\tau) \, d\tau = \sum_j a_{ij} x_j,$$

and let $\boldsymbol{A} = \{a_{ij}\}$. Note that

$$\int |\hat{f}^{(m)}(\tau)|^2 \, d\tau = \int \left(\sum_{i=1}^{n} x_i b_i^{(m)}(\tau) \right)^2 \, d\tau = \sum_i \sum_j x_i x_j R_{ij},$$

where

$$R_{ij} = \int (b_i^{(m)}(\tau)) \, (b_j^{(m)}(\tau)) \, d\tau.$$

Then the objective (5) can be rewritten

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \, (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}) + \beta \boldsymbol{x}' \boldsymbol{R} \boldsymbol{x},$$

where the definitions of the various vectors and matrices should be obvious. Thus, the estimated B-spline coefficients are given by:

$$\hat{\boldsymbol{x}} = (\boldsymbol{A} \boldsymbol{\Sigma}^{-1} \boldsymbol{A} + \beta \boldsymbol{R})^{-1} \boldsymbol{A}' \boldsymbol{\Sigma}^{-1} \boldsymbol{y}.$$

This latter system of equations is solved using a Cholesky decomposition for banded matrices [8], which requires only $O(n)$ operations. Having solved for the coefficients $\hat{\boldsymbol{x}}$, one can then evaluate $\hat{f}$ at any point τ using (7).

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