Approximate Variance Images for Penalized/Likelihood Image Reconstruction

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Introduction

Statistical image reconstruction methods are nonlinear estimators of space-variant pixel variances. Potential applications of variance maps:

- Fast approximate variance maps may be useful (cf. simulations)
- Choosing simulation parameters
- Medical diagnosis (confidence)
- Imaging system design
- Reconstruction algorithm evaluation

Statistical image reconstruction methods are nonlinear estimators of variance maps for FBP images: well-known but little used.

Fast approximate variance maps may be useful (cf. simulations)

Variance maps for FBP images: well-known but little used.
Image reconstruction: estimate image from sinogram

random coincidences and scatter:

\[ \{ \ell \} = \mathcal{G} \]

geometric system response

unknown activity in the \( j \)th pixel

\[ Y_i \]

modelled mean of \( Y_i \)

measured emission counts

\[ 0 \cdot Y_i + \bigoplus_j c_i g_{ij} = (\overline{Y})_{i,\Lambda} \]

\[ \{ (\text{Poisson}) (\overline{Y})_{i,\Lambda} \} \sim \Lambda \]
Penalized/Likelihood Estimators

Log/Likelihood:

$$L = \sum_{i=1}^{n} \log Y_i$$, \(Y_i = X_i \theta + \epsilon_i\),

where \(\epsilon_i\) is a roughness penalty function.

Penalized-Likelihood Objective Function:

$$0 < \lambda \max \Phi = \lambda$$

Estimator:

$$\hat{T} = \arg \max_{\theta} \left\{ (\lambda, \nabla) \Phi \right\}$$

Fast converging algorithms available for finding a minimizer of \(\Phi\).

Log-Likelihood:

$$\log Y_i \prod_{i=1}^{n} = (\lambda, \nabla) T$$
Covariance approximation improves with increasing scan time.

\( \Delta = R \bullet \nabla \delta \)

\( \Lambda / \psi n = n \bullet \)

\( n = n \bullet D \bullet \)

\( D = \) Diagonal matrix with \( D_{ii} = u_i \)

\( \nabla^2 \) Hessian of the penalty.

\( \nabla^2 \) Inverse of measurement variance

Fisher-information matrix

\( \mathcal{C}(n)D\mathcal{C} = \mathcal{F} \bullet \)

\( \text{Cov} \{ \nabla \} \approx \{ \nabla \} \text{Cov} \)


Estimator defined implicitly – no explicit expression for covariance.
Variance

\[ \hat{\varphi} = \overline{x[\mathcal{H} + \mathcal{G}(^n\!x)D,\mathcal{G}]} \]

where \( \hat{\varphi} \) is the \( j \)-th standard unit vector and

\[ \overline{\left[ \sum_{u=1}^{\infty} x^n \right]} = \overline{\mathcal{G}(^n\!x)D,\mathcal{G}} \]

\[ \hat{\varphi} \cdot [\mathcal{H} + \mathcal{A}] R^{-1} [\mathcal{H} + \mathcal{A}] \approx \]

\[ \{ \overline{\gamma} \} \hat{\varphi} \hat{\varphi}^T = \{ \gamma \} \hat{\varphi} \hat{\varphi}^T = \{ \varphi \} \hat{\varphi} \hat{\varphi}^T \]

\{ \overline{\gamma} \} \text{ Variance map: Image of the diagonal elements of Covariances}
\[
\begin{align*}
\text{where } R & = \sum_j D_j \approx \{\nabla\} \text{Cov} \\
\end{align*}
\]

**New Covariance Approximation**

For homoscedastic Gaussian noise, the \(R_j\)'s would all be equal.

**Normalized Backprojection of Inverse Ray Variances**

\[
\text{is the effective certainty of the } j\text{th pixel.} \\
\]

\[
\left( \frac{\ell}{\sigma^2} \right) \text{ where } \ell = \frac{1}{\sum \frac{\ell}{\sigma^2}} \\
\]

\[
\left( \frac{\ell}{\sigma^2} \right) \text{D}(\ell, \sigma^2) \approx G \left( \frac{1}{\sigma^2} \right) \text{D}(\ell, \sigma^2) = F \\
\]

Proposed Variance Approximation

\[ \text{Var} \left( \frac{\partial}{\partial \theta} f \right) \approx \text{certain expression} \]

For shift-invariant systems:

- The functions are all identical.
- Variables are computed using FFTs.
- All object-dependent factors are contained in the system geometry.

In PET the function(s) depend only on the system geometry and reconstruction with regularization parameter \( \lambda \).

All object-dependent factors are contained in the system geometry.

\[ \text{Var} \left( \frac{\partial}{\partial \theta} f \right) \approx \text{certain expression} \]

where

\[ \text{certain expression} \approx \{ f(y) \} \text{Var} \]
Table of $\sigma^2(\eta)$ for Simulated PET System
Simulation

- 2000 realizations
- PET digital emission phantom / nonuniform attenuation
- Modified quadratic penalty
- 10 iterations of PML-SAGE-3
- Nonnegativity enforced
Standard Deviation Maps

Empirical

Predicted
Center Horizontal Profile

Std. Dev. of Reconstructions

- ▲ ▲ FBP Empirical
- — PL Predicted
- ○ ○ PL Empirical

Pixel Std. Dev.

Horizontal Pixel
Mismatch in cold spot where nonnegativity constraint is very active.
Autocorrelation Functions

Autocorrelation Profiles for Center Pixel

FBP

Horizontal

Vertical

PL–SAGE

Horizontal

Vertical

Pixel Offset from Center

Correlation

Correlation
Autocorrelation Function: Radial Average

- PL Predicted
- PL–SAGE Empirical
- FBP

(For Center Pixel)
Summary and Future Work

Fast approximation for pixel variances in penalized/likelihood or penalized weighted least-squares image reconstruction methods

Very fast for shift-invariant systems

Over-estimates variance in low-count regions

Refinement needed for asymmetric autocorrelation functions

Extend to 3D and shift-variant systems

Preprints: http://www.eecs.umich.edu/~fessler/

When is it useful?

- Fast approximation for pixel variances in penalized-likelihood or penalized weighted least-squares image reconstruction methods
- Very fast for shift-invariant systems
- Over-estimates variance in low-count regions
- Refinement needed for asymmetric autocorrelation functions
- Extend to 3D and shift-variant systems
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