

# Approximate Variance Images for Penalized-Likelihood Image Reconstruction

Jeffrey A. Fessler  
University of Michigan

NSS/MIC Nov. 1997

## Introduction

- Statistical image reconstruction methods are nonlinear estimators
  - ⇒ space-variant pixel variances
- Potential applications of variance maps:
  - reconstruction algorithm evaluation
  - imaging system design
  - medical diagnosis (confidence)
  - choosing simulation parameters
  - ???
- Fast approximate variance maps may be useful (cf simulations)
- Variance maps for FBP images: well-known but little used...

## Poisson Statistical Model

$$Y_i \sim \text{Poisson}\{\bar{Y}_i(\underline{\Delta}^{\text{true}})\}$$

$$\bar{Y}_i(\underline{\Delta}) = \sum_j c_i g_{ij} \lambda_j + r_i$$

- $Y_i$  measured emission counts
- $\bar{Y}_i$  modeled mean of  $Y_i$
- $\lambda_j$  unknown activity in the  $j$ th pixel
- $\mathbf{G}$  geometric system response  $\mathbf{G} = \{g_{ij}\}$
- $c_i$  ray factors e.g. attenuation and detector efficiency
- $r_i$  random coincidences and scatter.

Image reconstruction: estimate image  $\underline{\Delta}$  from sinogram  $\underline{Y}$

## Penalized-Likelihood Estimators

Log-likelihood:

$$L(\underline{\lambda}, \underline{Y}) = \sum_{i=1}^n y_i \log \bar{Y}_i(\underline{\lambda}) - \bar{Y}_i(\underline{\lambda}) + \text{constant}$$

Estimator:

$$\hat{\underline{\lambda}} = \arg \max_{\underline{\lambda} \geq 0} \Phi(\underline{\lambda}, \underline{Y})$$

Penalized-Likelihood Objective Function:

$$\Phi(\underline{\lambda}, \underline{Y}) = L(\underline{\lambda}, \underline{Y}) - \beta R(\underline{\lambda}),$$

where  $R(\underline{\lambda})$  is a roughness penalty function.

Fast converging algorithms available for finding minimizer  $\hat{\underline{\lambda}}$  of  $\Phi$ .

## Covariance Approximation

Estimator defined implicitly  $\Rightarrow$  no explicit expression for covariance.

Approximation from (Fessler, IEEE Tr. Image Proc., Mar. 1996):

$$\text{Cov}\{\hat{\lambda}\} \approx [\mathbf{F} + \beta \mathbf{R}]^{-1} \mathbf{F} [\mathbf{F} + \beta \mathbf{R}]^{-1}$$

- $\mathbf{F} = \mathbf{G}' \mathbf{D}(u_i) \mathbf{G}$  Fisher-information matrix
- $\mathbf{D}(u_i)$  Diagonal matrix with  $D_{ii} = u_i$
- $u_i = c_i^2 / \bar{Y}_i$  Inverse of measurement variance
- $\mathbf{R} = \nabla^2 R$  Hessian of the penalty.

Covariance approximation improves with increasing scan time.

## Variance

Variance map: image of the diagonal elements of  $\text{Cov}\{\hat{\underline{\lambda}}\}$ .

$$\begin{aligned}\text{Var}\{\hat{\lambda}_j\} &= [\text{Cov}\{\hat{\underline{\lambda}}\}]_{jj} = \underline{e}_j' \text{Cov}\{\hat{\underline{\lambda}}\} \underline{e}_j \\ &\approx \underline{e}_j' [\mathbf{F} + \beta \mathbf{R}]^{-1} \mathbf{F} [\mathbf{F} + \beta \mathbf{R}]^{-1} \underline{e}_j \\ &= \underline{x}' \mathbf{G}' \mathbf{D}(u_i) \mathbf{G} \underline{x} \\ &= \sum_{i=1}^n u_i [\mathbf{G} \underline{x}]_i^2\end{aligned}$$

where  $\underline{e}_j$  is the  $j$ th standard unit vector and

$$[\mathbf{G}' \mathbf{D}(u_i) \mathbf{G} + \beta \mathbf{R}] \underline{x} = \underline{e}_j.$$

One would have to solve this system of equations once for each pixel.

Too expensive (simulations would be cheaper!):  $\therefore$  approximate further.

## Fisher Information Approximation

From (Fessler and Rogers, IEEE Tr. Image Proc., Sep. 1996):

$$F = G' D(u_i) G \approx D(\kappa_j) G' G D(\kappa_j)$$

where  $\kappa_j = \sqrt{\frac{\sum_{i=1}^n g_{ij}^2 u_i}{\sum_{i=1}^n g_{ij}^2}}$  is the “effective certainty” of the  $j$ th pixel.

(Normalized backprojection of inverse ray variances.)

For homoscedastic Gaussian noise, the  $\kappa_j$ 's would all be equal.

## New Covariance Approximation

$$\text{Cov}\{\hat{\Delta}\} \approx D(\kappa_j^{-1}) [G' G + \beta R_2]^{-1} G' G [G' G + \beta R_2]^{-1} D(\kappa_j^{-1})$$

where  $R_2 = D(\kappa_j^{-1}) R D(\kappa_j^{-1})$

## Proposed Variance Approximation

$$\text{Var}\{\hat{\lambda}_j\} \approx \frac{\sigma_j^2(\beta/\kappa_j^2)}{\kappa_j^2}$$

where  $\sigma_j^2(\eta) \triangleq \underline{e}_j' [\mathbf{G}'\mathbf{G} + \eta\mathbf{R}]^{-1} \mathbf{G}'\mathbf{G} [\mathbf{G}'\mathbf{G} + \eta\mathbf{R}]^{-1} \underline{e}_j$

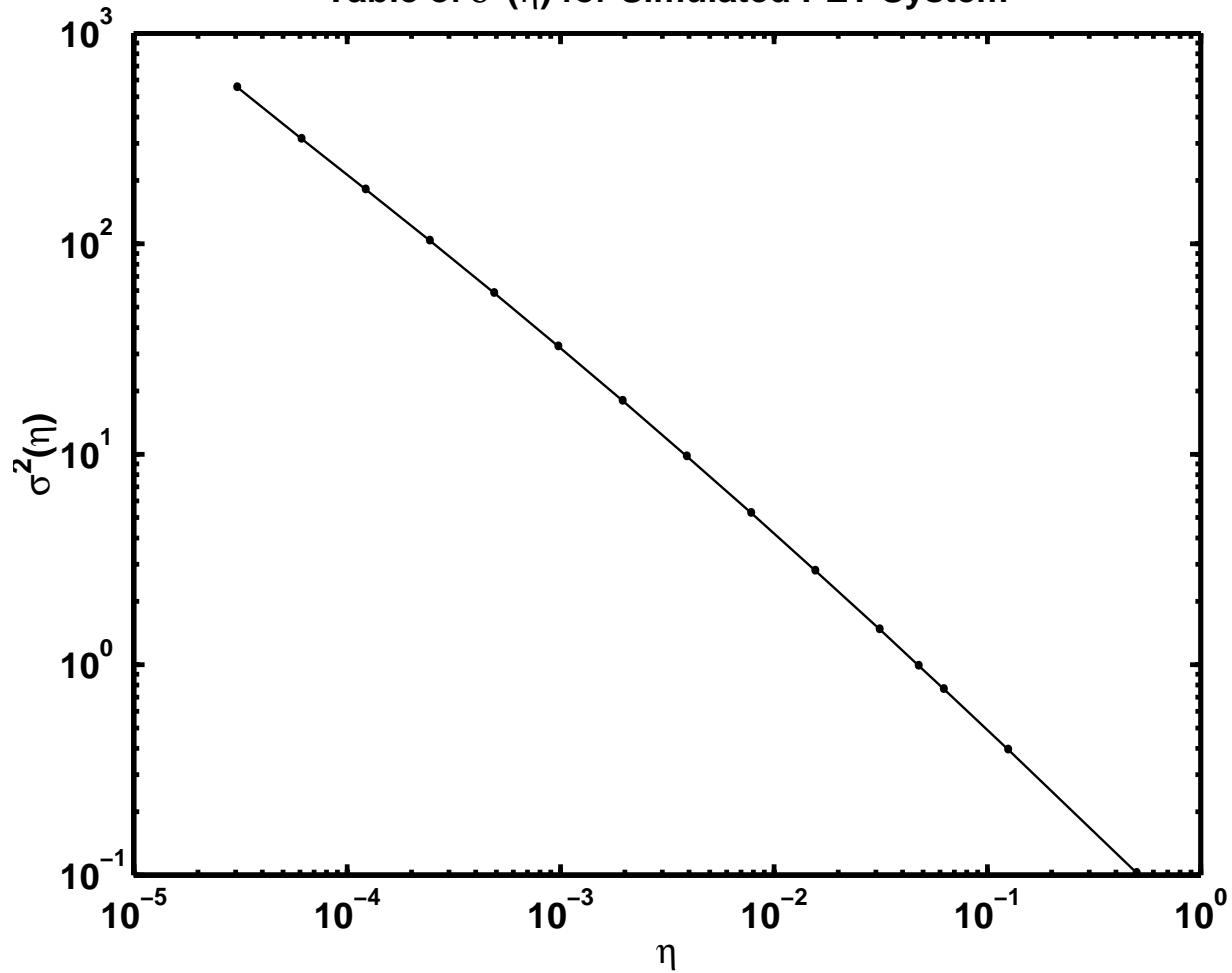
- In PET the  $\sigma_j^2$  function(s) depend only on the system geometry and the penalty function,  $\Rightarrow$  precompute / tabulate once.
- All object-dependent factors are contained in the  $\kappa_j$ 's.
- $\sigma_j^2(\eta)$  is the variance of  $\hat{\lambda}_j$  under homoscedastic Gaussian noise and reconstruction with regularization parameter  $\eta$ .

For shift-invariant systems:

- The  $\sigma_j^2$  functions are all identical
- $\sigma^2(\eta)$  easily computed using FFTs

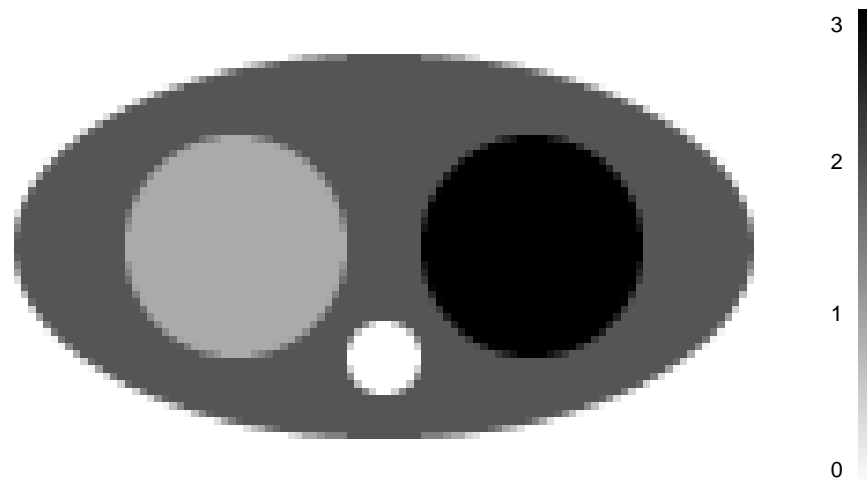
# Example "Table"

Table of  $\sigma^2(\eta)$  for Simulated PET System

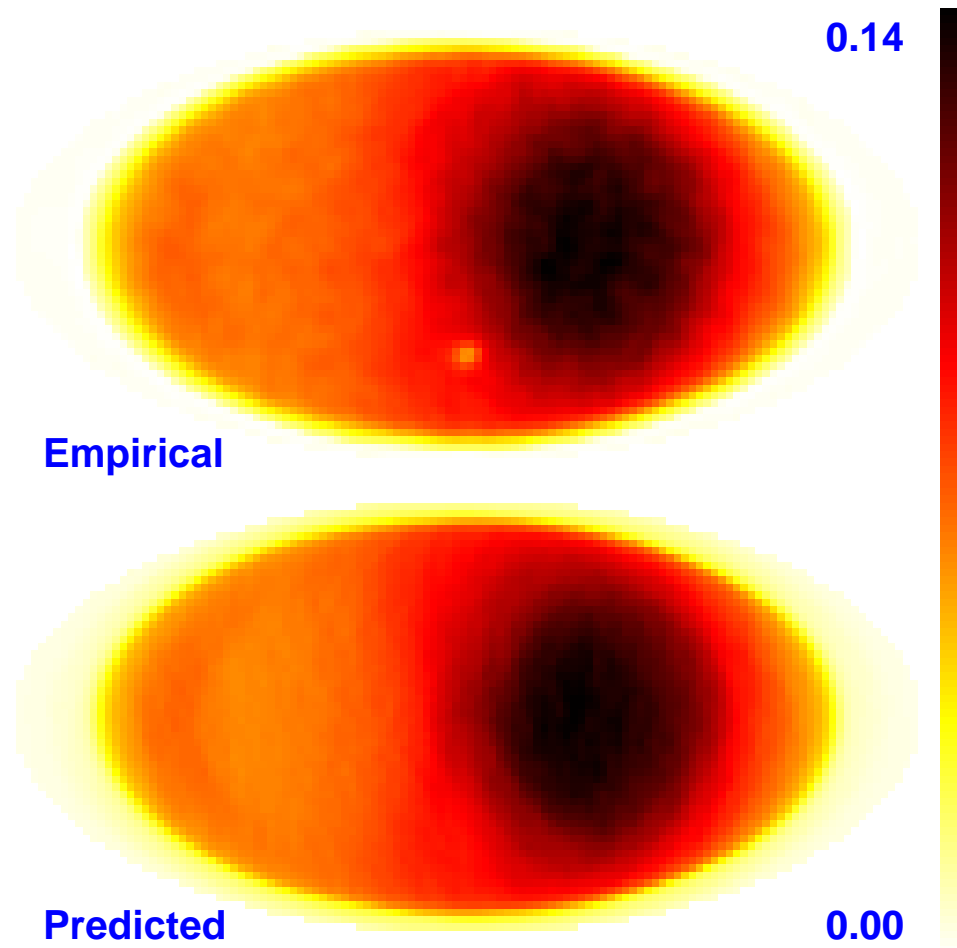


## Simulation

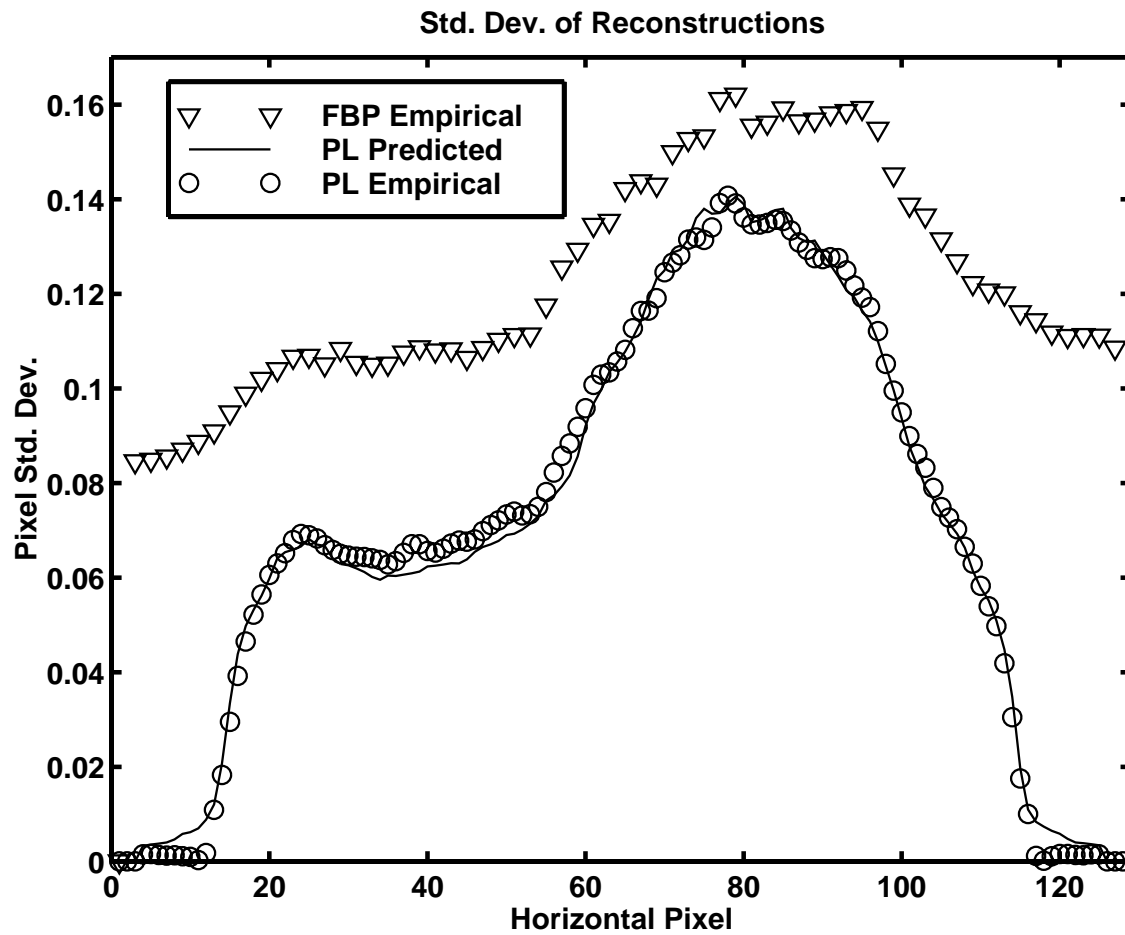
- 2000 realizations
- PET digital emission phantom / nonuniform attenuation
- Modified quadratic penalty
- 10 iterations of PML-SAGE-3
- Nonnegativity enforced



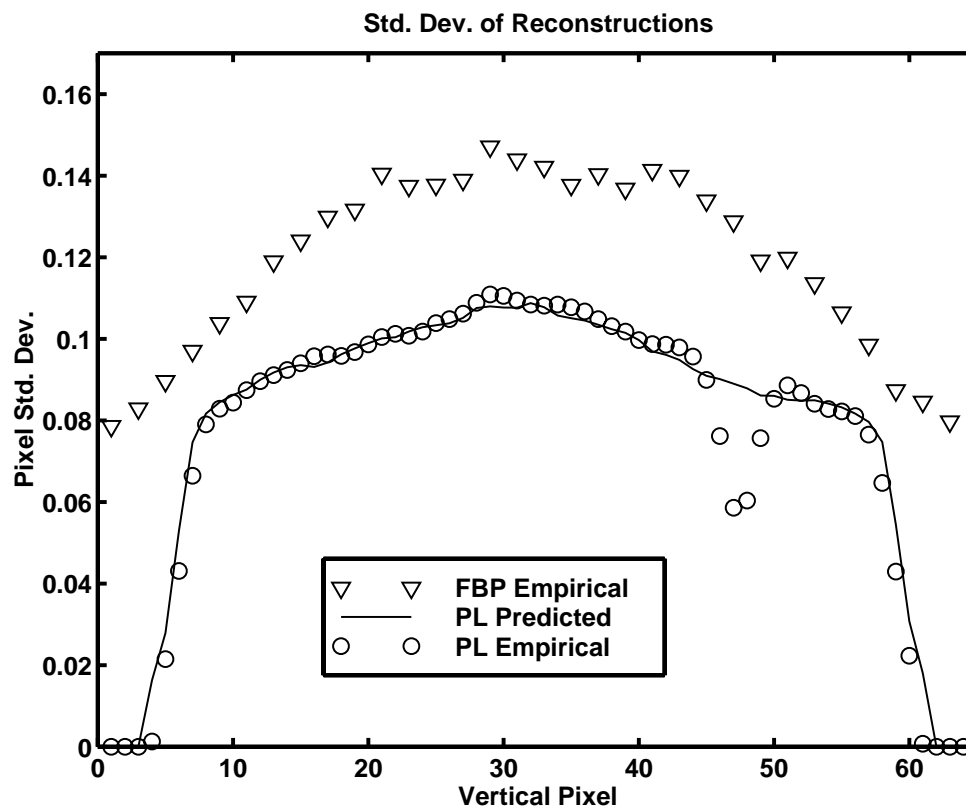
# Standard Deviation Maps



# Center Horizontal Profile

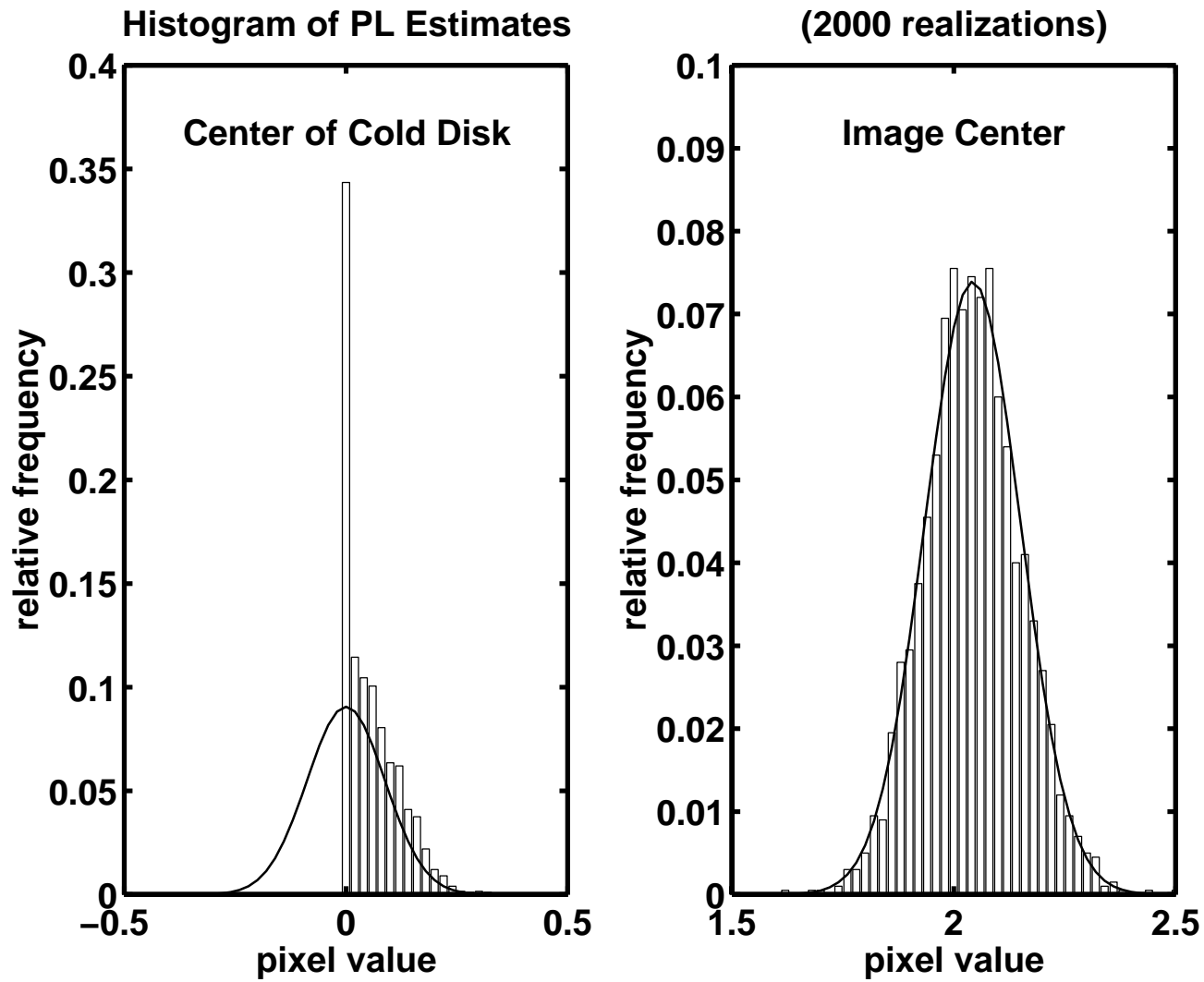


# Center Vertical Profile

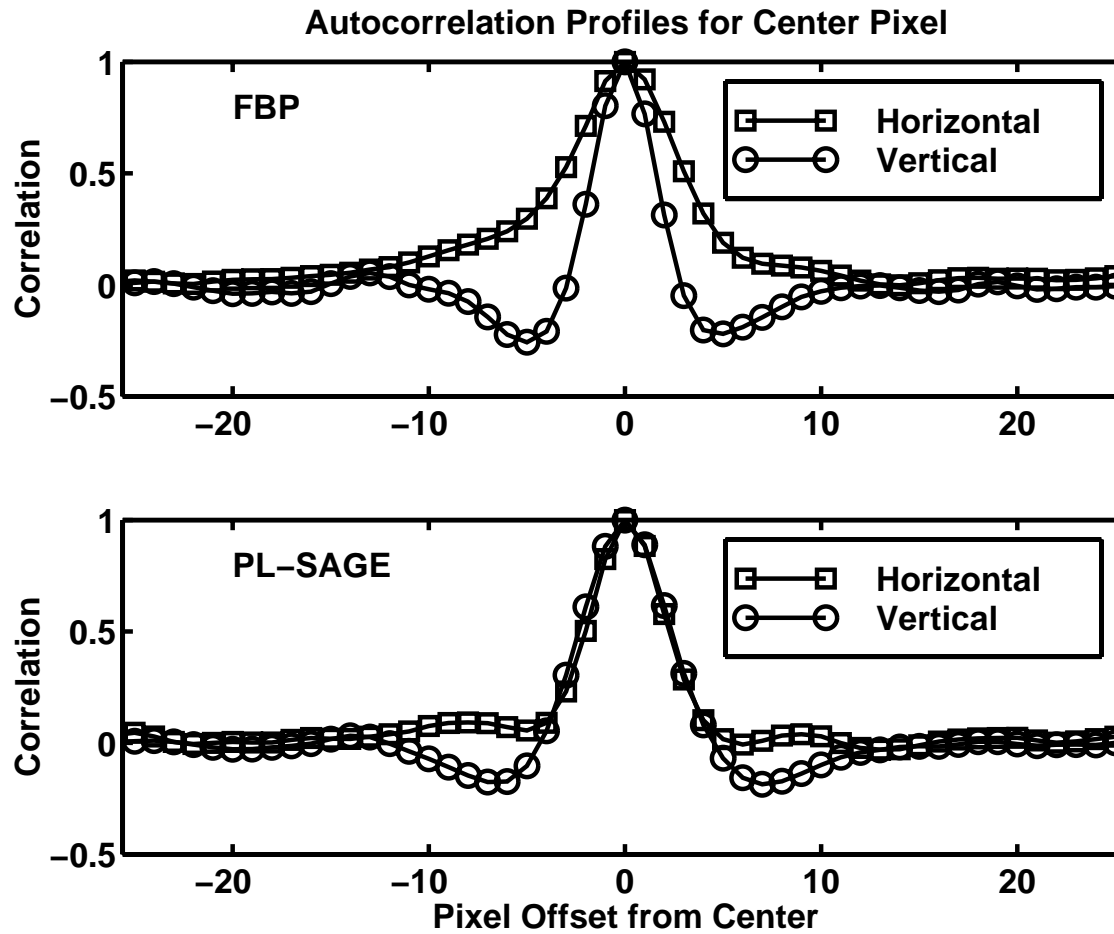


Mismatch in cold spot where nonnegativity constraint is very active.

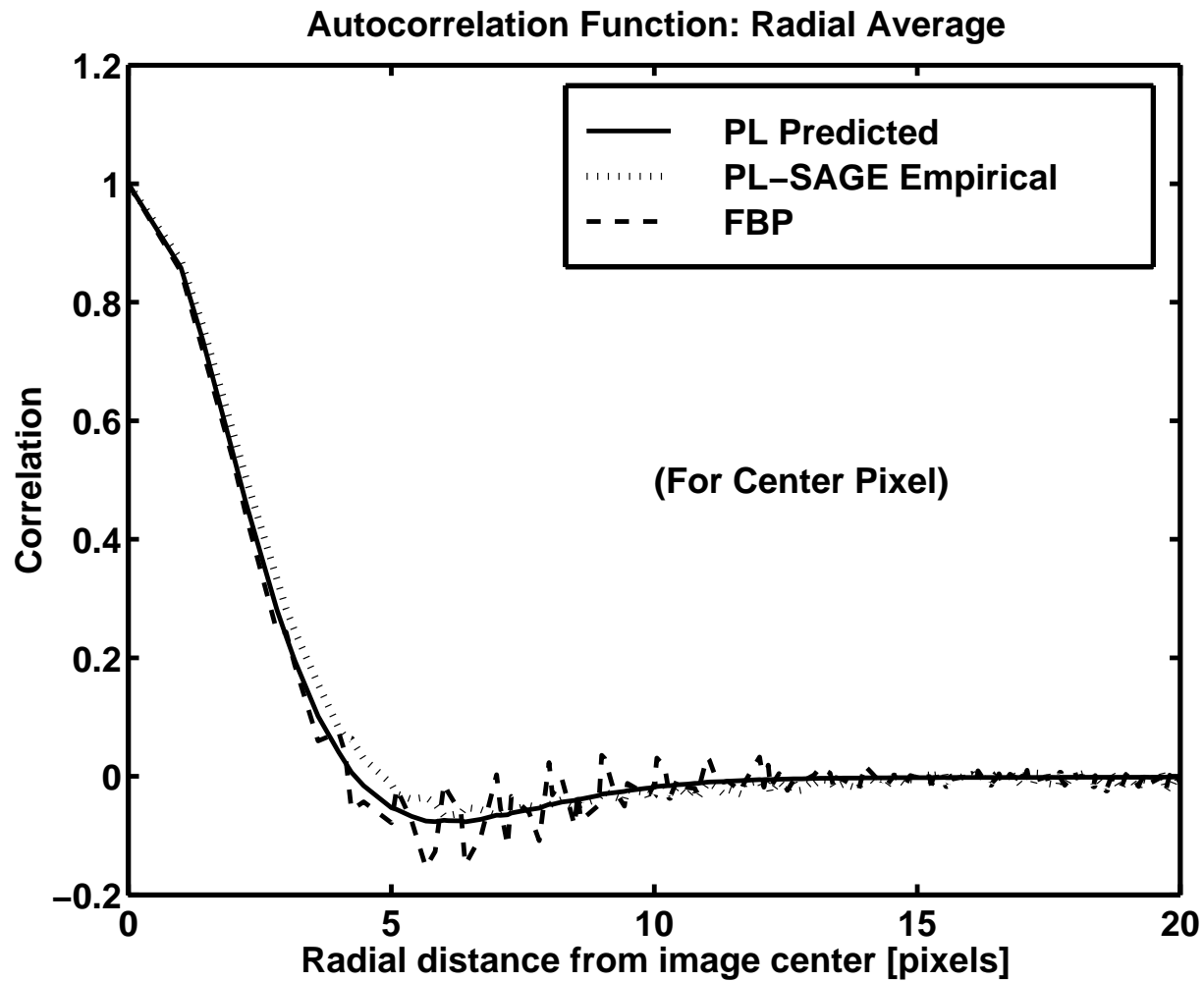
# Histograms



# Autocorrelation Functions



# Autocorrelation Function: Radial Average



## Summary and Future Work

- Fast approximation for pixel variances in penalized-likelihood or penalized weighted least-squares image reconstruction methods
- Very fast for shift-invariant systems
- Over-estimates variance in low-count regions
- Refinement needed for asymmetric autocorrelation functions
- Extend to 3D and shift-variant systems
- When is it useful?

Preprints: <http://www.eecs.umich.edu/~fessler/>