

Accelerating image reconstruction using variable splitting methods

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Statistical image reconstruction: a CT revolution

- A picture is worth 1000 words
- (and perhaps several 1000 seconds of computation?)



Thin-slice FBP

Seconds



ASIR

A bit longer



Statistical

Much longer

(Same sinogram, so all at same **dose**)

Outline

- **Image denoising** (review)
- **Image restoration**
Antonios Matakos, Sathish Ramani, JF, IEEE T-IP, May 2013
Accelerated edge-preserving image restoration without boundary artifacts
- **Low-dose X-ray CT image reconstruction**
Sathish Ramani & JF, IEEE T-MI, Mar. 2012
A splitting-based iterative algorithm for accelerated statistical X-ray CT reconstruction
- **Model-based MR image reconstruction**
Sathish Ramani & JF, IEEE T-MI, Mar. 2011
Parallel MR image reconstruction using augmented Lagrangian methods
- **Image in-painting (e.g., from cutset sampling) using sparsity**
(premier! inspired by work of D. Neuhoff and M. Prelee)

Image denoising

Denoising using sparsity

Measurement model:

$$\underbrace{\mathbf{y}}_{\text{observed}} = \underbrace{\mathbf{x}}_{\text{unknown}} + \underbrace{\boldsymbol{\varepsilon}}_{\text{noise}}$$

Object model: assume $\mathbf{Q}\mathbf{x}$ is sparse (compressible) for some **orthogonal** sparsifying transform \mathbf{Q} , such as an orthogonal wavelet transform (OWT).

Sparsity regularized estimator:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \underbrace{\frac{1}{2} \|\mathbf{y} - \mathbf{x}\|_2^2}_{\text{data fit}} + \beta \underbrace{\|\mathbf{Q}\mathbf{x}\|_p}_{\text{sparsity}} .$$

Regularization parameter β determines trade-off.

Equivalently (because $\mathbf{Q}^{-1} = \mathbf{Q}'$ is an orthonormal matrix):

$$\hat{\mathbf{x}} = \mathbf{Q}'\hat{\boldsymbol{\theta}}, \quad \hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} \frac{1}{2} \|\mathbf{Q}\mathbf{y} - \boldsymbol{\theta}\|_2^2 + \beta \|\boldsymbol{\theta}\|_p = \text{shrink}(\mathbf{Q}\mathbf{y} : \beta, p)$$

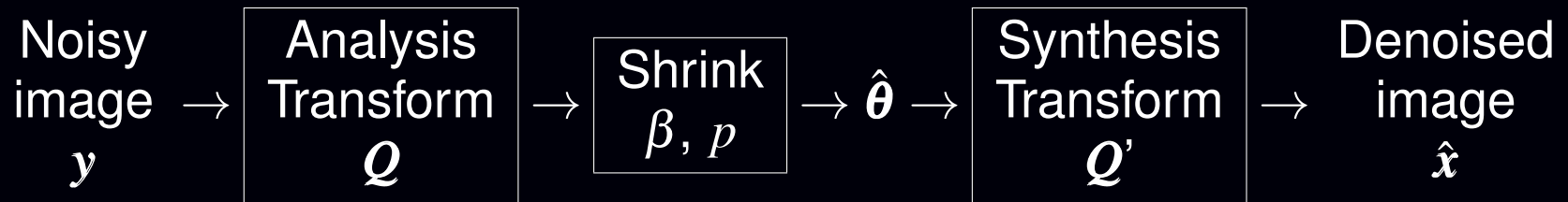
Non-iterative solution!

Orthogonal transform thresholding

Equation:

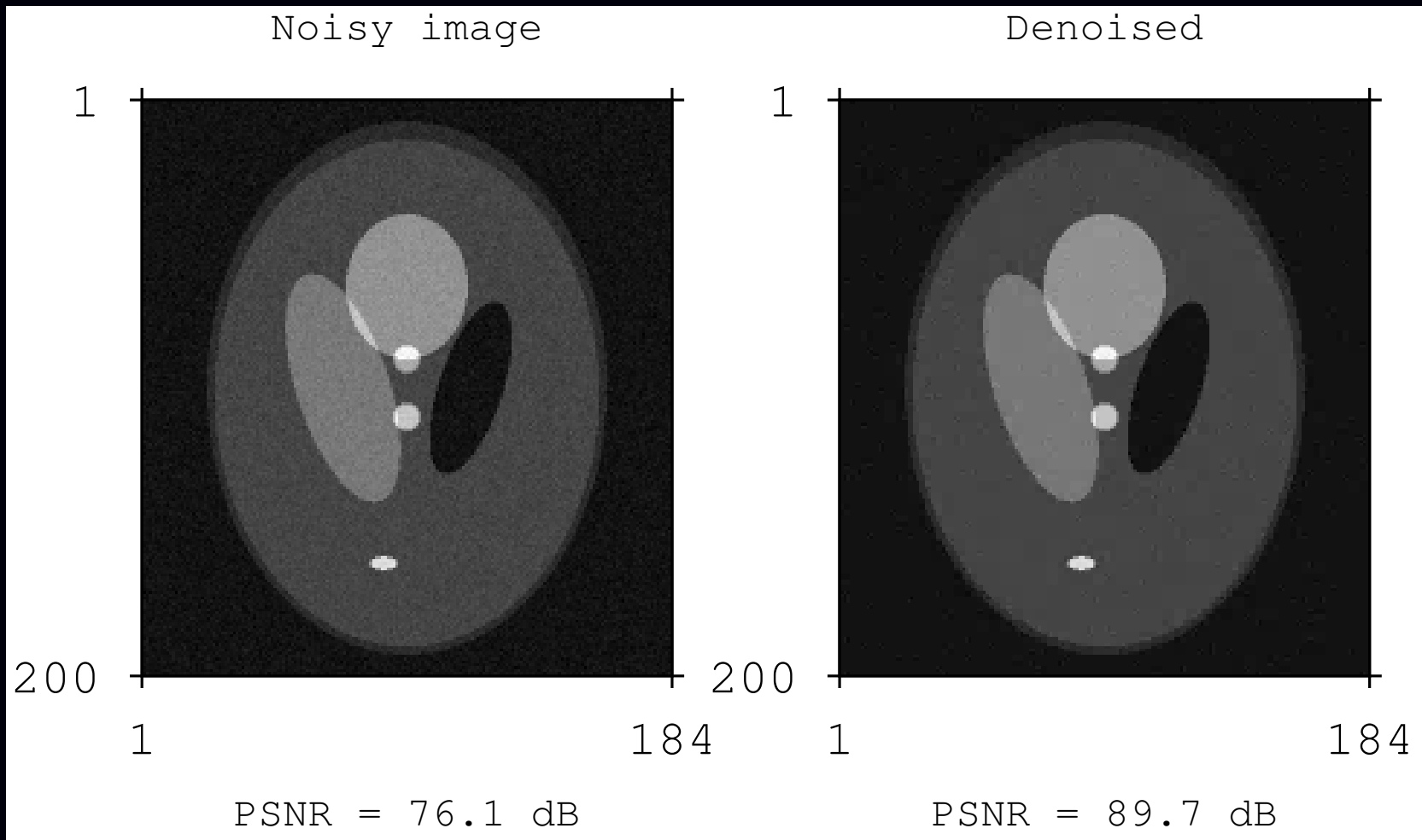
$$\hat{x} = Q' \text{shrink}(Qy : \beta, p)$$

Block diagram:



todo: show shrink function for $p = 1$ and $p = 0$

Hard thresholding example



$p = 0$, orthonormal Haar wavelets

Sparsity using shift-invariant models

Analysis form:

Assume $\mathbf{R}\mathbf{x}$ is sparse for some sparsifying transform \mathbf{R} .

Often \mathbf{R} is a “tall” matrix, *e.g.*, finite differences along horizontal and vertical directions, *i.e.*, anisotropic total variation (TV).

Often \mathbf{R} is shift invariant: $\|\mathbf{R}\mathbf{x}\|_p = \|\mathbf{R} \text{circshift}(\mathbf{x})\|_p$ and $\mathbf{R}'\mathbf{R}$ is circulant.

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 + \beta \underbrace{\|\mathbf{R}\mathbf{x}\|_p}_{\text{transform sparsity}}.$$

Synthesis form

Assume $\mathbf{x} = \mathbf{S}\boldsymbol{\theta}$ where coefficient vector $\boldsymbol{\theta}$ is sparse.

Often \mathbf{S} is a “fat” matrix (over-complete dictionary) and $\mathbf{S}'\mathbf{S}$ is circulant.

$$\hat{\mathbf{x}} = \mathbf{S}\hat{\boldsymbol{\theta}}, \quad \hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} \frac{1}{2} \|\mathbf{y} - \mathbf{S}\boldsymbol{\theta}\|_2^2 + \beta \underbrace{\|\boldsymbol{\theta}\|_p}_{\text{sparse coefficients}}$$

Analysis form preferable to synthesis form?

(Elad *et al.*, *Inv. Prob.*, June 2007)

Constrained optimization

Unconstrained estimator (analysis form for illustration):

$$\hat{\mathbf{x}} = \arg \min_x \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 + \beta \|\mathbf{R}\mathbf{x}\|_p.$$

(Nonnegativity constraint or box constraints easily added.)

Equivalent **constrained** optimization problem:

$$\min_{\mathbf{x}, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{R}\mathbf{x}.$$

(Y. Wang *et al.*, SIAM J. Im. Sci., 2008)

(M Afonso, J Bioucas-Dias, M Figueiredo, IEEE T-IP, Sep. 2010)

(The auxiliary variable \mathbf{v} is discarded after optimization; keep only $\hat{\mathbf{x}}$.)

Penalty approach:

$$\hat{\mathbf{x}} = \arg \min_x \min_v \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\mu}{2} \|\mathbf{v} - \mathbf{R}\mathbf{x}\|_2^2.$$

Large μ better enforces the constraint $\mathbf{v} = \mathbf{R}\mathbf{x}$, but can worsen conditioning.

Preferable (?) approach: augmented Lagrangian.

Augmented Lagrangian method: V1

General linearly constrained optimization problem:

$$\min_{\mathbf{u}} \Psi(\mathbf{u}) \text{ sub. to } \mathbf{C}\mathbf{u} = \mathbf{b}.$$

Form *augmented Lagrangian*:

$$L(\mathbf{u}, \boldsymbol{\gamma}) \triangleq \Psi(\mathbf{u}) + \boldsymbol{\gamma}'(\mathbf{C}\mathbf{u} - \mathbf{b}) + \frac{\rho}{2} \|\mathbf{C}\mathbf{u} - \mathbf{b}\|_2^2$$

where $\boldsymbol{\gamma}$ is the *dual variable* or *Lagrange multiplier vector*.

AL method alternates between minimizing over \mathbf{u} and gradient ascent on $\boldsymbol{\gamma}$:

$$\begin{aligned} \mathbf{u}^{(n+1)} &= \arg \min_{\mathbf{u}} L(\mathbf{u}, \boldsymbol{\gamma}^{(n)}) \\ \boldsymbol{\gamma}^{(n+1)} &= \boldsymbol{\gamma}^{(n)} + \rho (\mathbf{C}\mathbf{u}^{(n+1)} - \mathbf{b}). \end{aligned}$$

Desirable convergence properties.

AL penalty parameter ρ affects convergence *rate*, not solution!

Unfortunately, minimizing over \mathbf{u} is impractical here:

$$\mathbf{v} = \mathbf{R}\mathbf{x} \text{ equivalent to } \mathbf{C}\mathbf{u} = \mathbf{b}, \quad \mathbf{C} = [\mathbf{R} \quad -\mathbf{I}], \quad \mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}, \quad \mathbf{b} = \mathbf{0}.$$

Augmented Lagrangian method: V2

General linearly constrained optimization problem:

$$\min_{\mathbf{u}} \Psi(\mathbf{u}) \text{ sub. to } \mathbf{C}\mathbf{u} = \mathbf{b}.$$

Form (modified) *augmented Lagrangian* by completing the square:

$$L(\mathbf{u}, \boldsymbol{\eta}) \triangleq \Psi(\mathbf{u}) + \frac{\rho}{2} \|\mathbf{C}\mathbf{u} - \boldsymbol{\eta}\|_2^2 + \mathbf{C}\boldsymbol{\eta},$$

where $\boldsymbol{\eta} \triangleq \mathbf{b} - \frac{1}{\rho}\boldsymbol{\gamma}$ is a modified *dual variable* or *Lagrange multiplier vector*.

AL method alternates between minimizing over \mathbf{u} and gradient ascent on $\boldsymbol{\eta}$:

$$\begin{aligned} \mathbf{u}^{(n+1)} &= \arg \min_{\mathbf{u}} L(\mathbf{u}, \boldsymbol{\gamma}^{(n)}) \\ \boldsymbol{\eta}^{(n+1)} &= \boldsymbol{\eta}^{(n)} - (\mathbf{C}\mathbf{u}^{(n+1)} - \mathbf{b}). \end{aligned}$$

Desirable convergence properties.

AL penalty parameter ρ affects convergence *rate*, not solution!

Unfortunately, minimizing over \mathbf{u} is impractical here:

$$\mathbf{v} = \mathbf{R}\mathbf{x} \text{ equivalent to } \mathbf{C}\mathbf{u} = \mathbf{b}, \quad \mathbf{C} = [\mathbf{R} \quad -\mathbf{I}], \quad \mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}, \quad \mathbf{b} = \mathbf{0}.$$

Alternating direction method of multipliers (ADMM)

When \mathbf{u} has multiple component vectors, e.g., $\mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}$,
rewrite (modified) augmented Lagrangian in terms of all component vectors:

$$\begin{aligned} L(\mathbf{x}, \mathbf{v}; \boldsymbol{\eta}) &= \Psi(\mathbf{x}, \mathbf{v}) + \frac{\rho}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}\|_2^2 \\ &= \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho}{2} \underbrace{\|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}\|_2^2}_{\text{cf. penalty!}} \end{aligned}$$

because here $\mathbf{C}\mathbf{u} = \mathbf{R}\mathbf{x} - \mathbf{v}$.

Alternate between minimizing over each *component* vector:

$$\mathbf{x}^{(n+1)} = \arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{v}^{(n)}, \boldsymbol{\eta}^{(n)})$$

$$\mathbf{v}^{(n+1)} = \arg \min_{\mathbf{v}} L(\mathbf{x}^{(n+1)}, \mathbf{v}, \boldsymbol{\eta}^{(n)})$$

$$\boldsymbol{\eta}^{(n+1)} = \boldsymbol{\eta}^{(n)} + (\mathbf{R}\mathbf{x}^{(n+1)} - \mathbf{v}^{(n+1)}).$$

Reasonably desirable convergence properties. (Inexact inner minimizations!)

Sufficient conditions on matrix \mathbf{C} .

(Eckstein & Bertsekas, *Math. Prog.*, Apr. 1992)

(Douglas and Rachford, *Tr. Am. Math. Soc.*, 1956, heat conduction problems)

ADMM for image denoising

Augmented Lagrangian:

$$L(\mathbf{x}, \mathbf{v}; \boldsymbol{\eta}) = \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}\|_2^2$$

Update of primal variable (unknown image):

$$\mathbf{x}^{(n+1)} = \arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{v}^{(n)}, \boldsymbol{\eta}^{(n)}) = \underbrace{[\mathbf{I} + \rho \mathbf{R}' \mathbf{R}]^{-1}}_{\text{Wiener filter}} (\mathbf{y} + \rho \mathbf{R}' (\mathbf{v}^{(n)} + \boldsymbol{\eta}^{(n)}))$$

Update of auxiliary variable:

(No “corner rounding” needed for ℓ_1 .)

$$\mathbf{v}^{(n+1)} = \arg \min_{\mathbf{v}} L(\mathbf{x}^{(n+1)}, \mathbf{v}, \boldsymbol{\eta}^{(n)}) = \text{shrink}(\mathbf{R}\mathbf{x}^{(n+1)} - \boldsymbol{\eta}^{(n)}; \beta/\rho, p)$$

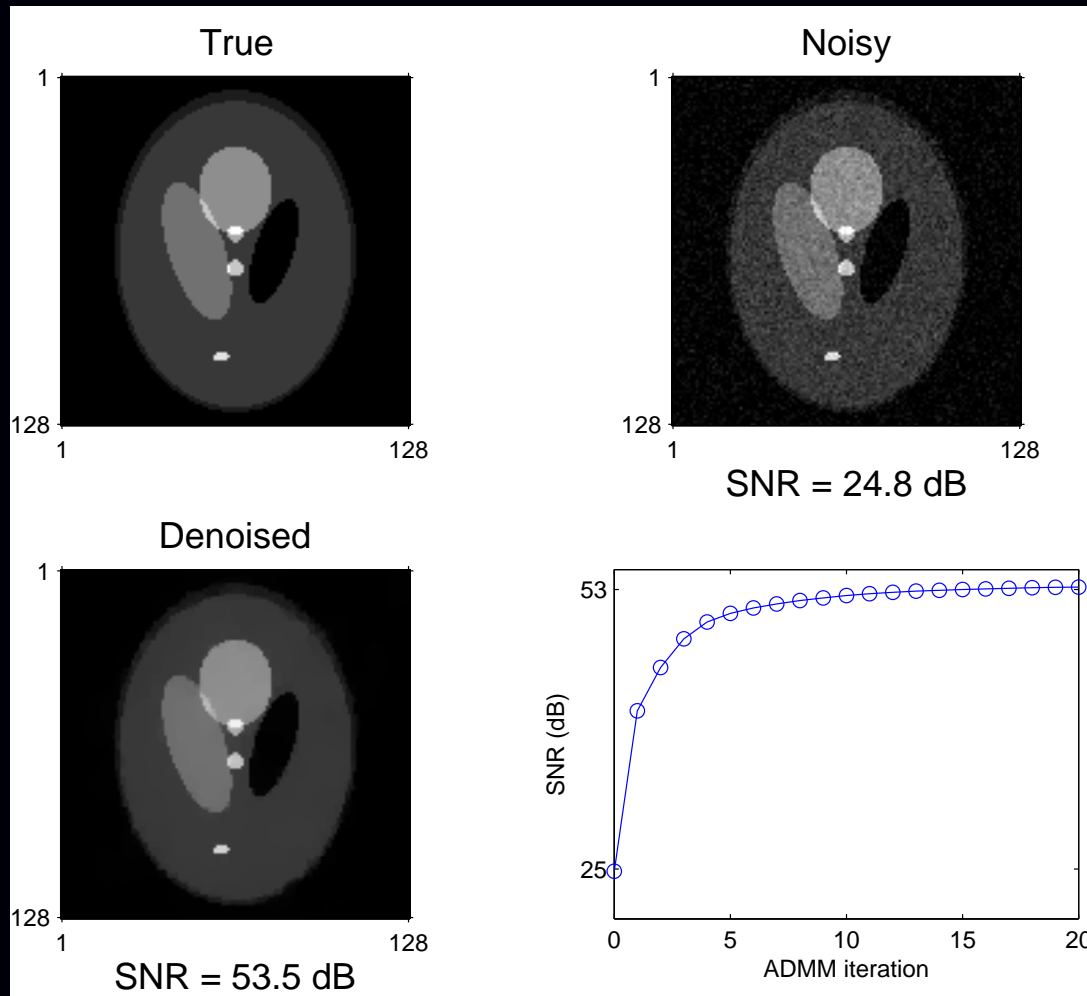
Update of multiplier: $\boldsymbol{\eta}^{(n+1)} = \boldsymbol{\eta}^{(n)} + (\mathbf{R}\mathbf{x}^{(n+1)} - \mathbf{v}^{(n+1)})$

Equivalent to “*split Bregman*” approach.

(Goldstein & Osher, SIAM J. Im. Sci. 2009)

Each update is simple and exact (non-iterative) if $[\mathbf{I} + \rho \mathbf{R}' \mathbf{R}]^{-1}$ is easy.

ADMM image denoising example



\mathbf{R} : horizontal and vertical finite differences (anisotropic TV),
 $p = 1$ (i.e., ℓ_1), $\beta = 1/2$, $\rho = 1$ (condition number of $(\mathbf{I} + \rho\mathbf{R}'\mathbf{R})$ is 9)

ADMM image denoising iterates



Image restoration

Image restoration models

Unrealistic model:

$$\underbrace{\mathbf{y}}_{\text{observed}} = \underbrace{\mathbf{A}}_{\text{blur}} \underbrace{\mathbf{x}}_{\text{unknown}} + \underbrace{\boldsymbol{\varepsilon}}_{\text{noise}}$$

Measured blurry image \mathbf{y} and unknown image \mathbf{x} have the same size. \mathbf{A} is a circulant matrix corresponding to a shift-invariant blur model.

Somewhat more realistic measurement model:

$$\mathbf{y} = \mathbf{T}\mathbf{A}\mathbf{x} + \boldsymbol{\varepsilon}$$

Measured blurry image \mathbf{y} is smaller than unknown image \mathbf{x} . \mathbf{T} is a (fat) “truncation” matrix, akin to $[\mathbf{0} \ \mathbf{I} \ \mathbf{0}]$.

(S. Reeves, IEEE T-IP, Oct. 2005)



Image restoration with sparsity regularization

Regularized estimator:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \underbrace{\frac{1}{2} \|\mathbf{y} - \mathbf{TAx}\|_2^2}_{\text{data fit}} + \beta \underbrace{\|\mathbf{Rx}\|_p}_{\text{sparsity}}.$$

Basic equivalent **constrained** optimization problem:

$$\min_{\mathbf{x}, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{TAx}\|_2^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{Rx}.$$

Corresponding (modified) augmented Lagrangian (cf. “split Bregman”):

$$L(\mathbf{x}, \mathbf{v}; \boldsymbol{\eta}) = \frac{1}{2} \|\mathbf{y} - \mathbf{TAx}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho}{2} \|\mathbf{Rx} - \mathbf{v} - \boldsymbol{\eta}\|_2^2$$

ADMM update of primal variable (unknown image):

$$\mathbf{x}^{(n+1)} = \arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{v}^{(n)}, \boldsymbol{\eta}^{(n)}) = [\mathbf{A}'\mathbf{T}'\mathbf{TA} + \rho\mathbf{R}'\mathbf{R}]^{-1} (\mathbf{A}'\mathbf{T}'\mathbf{y} + \rho\mathbf{R}'(\mathbf{v}^{(n)} + \boldsymbol{\eta}^{(n)}))$$

Simple if $\mathbf{A}'\mathbf{A}$ and $\mathbf{R}'\mathbf{R}$ are circulant and $\mathbf{T} = \mathbf{I}$ (unrealistic).

Otherwise need iterative inner (quadratic) minimization: PCG.

Improved ADMM for image restoration

New equivalent **constrained** optimization problem:

$$\min_{\mathbf{x}, \mathbf{u}, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{T} \mathbf{u}\|_2^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{R}\mathbf{x}, \quad \mathbf{u} = \mathbf{A}\mathbf{x}.$$

(Antonios Matakos, Sathish Ramani, JF, IEEE T-IP, 2013, to appear)

Corresponding (modified) augmented Lagrangian:

$$L(\mathbf{x}, \mathbf{u}, \mathbf{v}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \frac{1}{2} \|\mathbf{y} - \mathbf{T} \mathbf{u}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho_1}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}_1\|_2^2 + \frac{\rho_2}{2} \|\mathbf{A}\mathbf{x} - \mathbf{u} - \boldsymbol{\eta}_2\|_2^2$$

ADMM update of primal variable (unknown image):

$$\arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = [\rho_2 \mathbf{A}'\mathbf{A} + \rho_1 \mathbf{R}'\mathbf{R}]^{-1} (\rho_1 \mathbf{R}'(\mathbf{v} + \boldsymbol{\eta}_1) + \rho_2 \mathbf{A}'(\mathbf{u} + \boldsymbol{\eta}_2))$$

Simple if $\mathbf{A}'\mathbf{A}$ and $\mathbf{R}'\mathbf{R}$ are circulant. No inner iterations needed!

ADMM update of new auxiliary variable \mathbf{u} :

$$\arg \min_{\mathbf{u}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \underbrace{[\mathbf{T}'\mathbf{T} + \rho_2 \mathbf{I}]^{-1}}_{\text{diagonal}} (\mathbf{T}'\mathbf{y} + \rho_2(\mathbf{A}\mathbf{x} - \boldsymbol{\eta}_2))$$

\mathbf{v} update is shrinkage again. Very easy to code!

Image restoration results: quality



Measurement y



Using circulant model
with boundary preprocessing



ADMM with Reeves model

15×15 pixel uniform blur, $50\text{dB BSNR} = 10\log(\text{Var}\{\mathbf{TAx}\} / \sigma^2)$,
isotropic TV regularization, $\beta = 2^{-17}$

Qualitatively confirms Reeves model is preferable.

Image restoration results: iterations

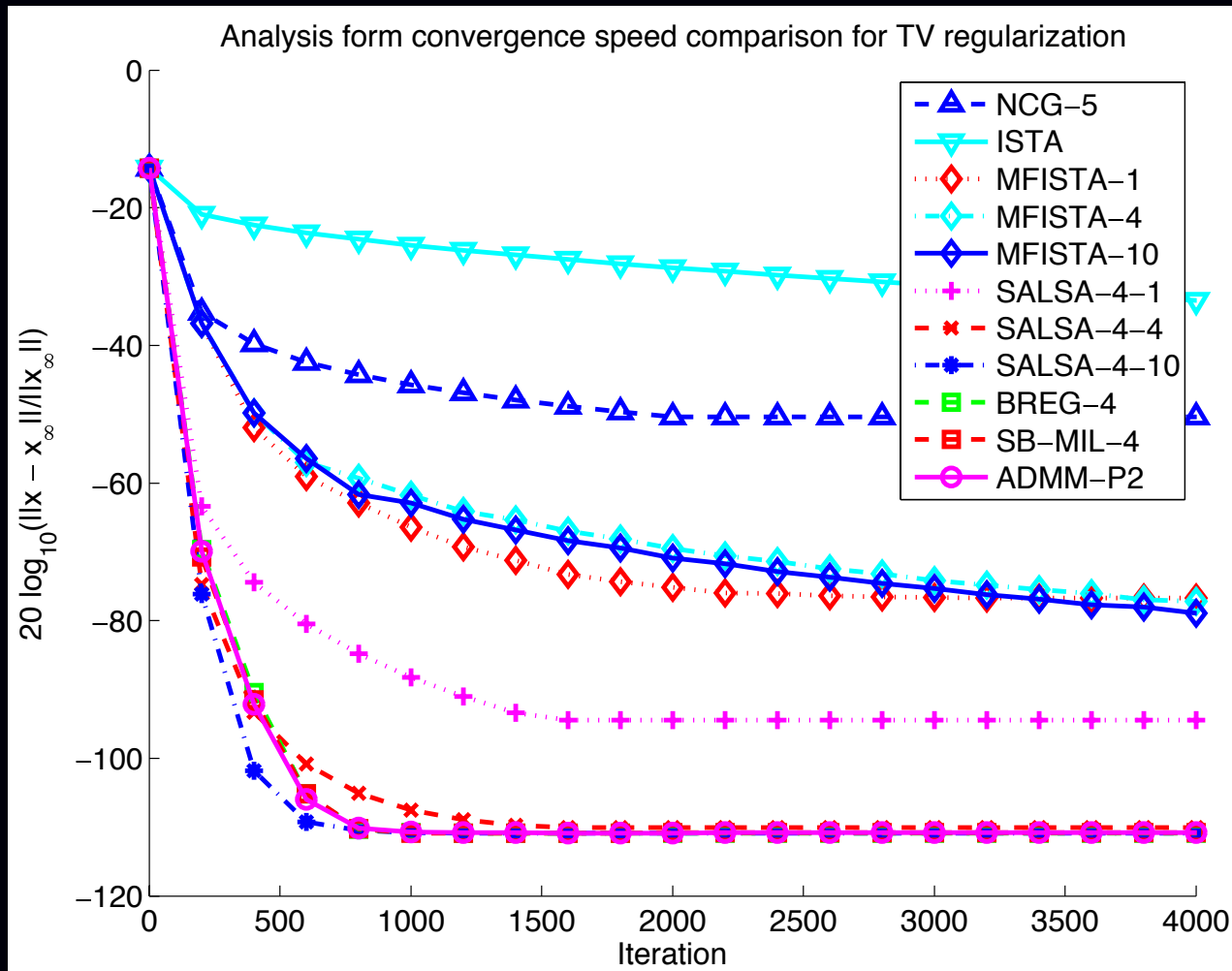
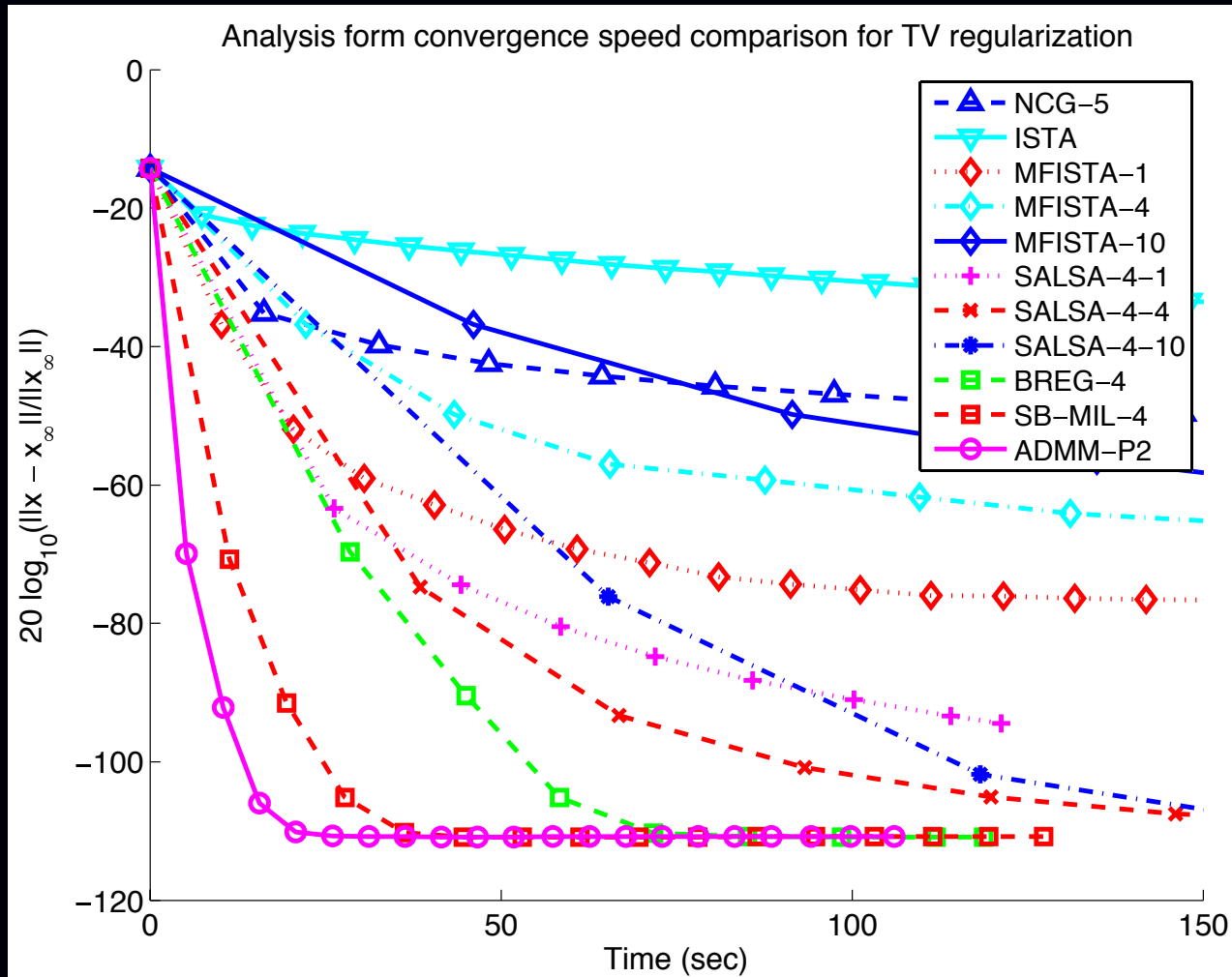


Image restoration results: speed



Proposed ADMM is fast due to non-iterative inner updates.

X-ray CT image reconstruction

Low-dose X-ray CT image reconstruction

Regularized estimator:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \succeq 0} \underbrace{\frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2}_{\text{data fit}} + \beta \underbrace{\|\mathbf{R}\mathbf{x}\|_p}_{\text{sparsity}}.$$

Complications:

- $\mathbf{A}'\mathbf{A}$ is not circulant (but “approximately Toeplitz” in 2D)
- $\mathbf{A}'\mathbf{W}\mathbf{A}$ is highly shift variant due to huge dynamic range of weighting \mathbf{W}
- Non-quadratic (edge-preserving) regularization $\|\cdot\|_p$
- Nonnegativity constraint
- Large problem size

Basic ADMM for X-ray CT

Basic equivalent **constrained** optimization problem (*cf.* split Bregman):

$$\min_{\mathbf{x} \geq 0, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{Ax}\|_{\mathbf{W}}^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{Rx}.$$

Corresponding (modified) augmented Lagrangian (*cf.* “split Bregman”):

$$L(\mathbf{x}, \mathbf{v}; \boldsymbol{\eta}) = \frac{1}{2} \|\mathbf{y} - \mathbf{Ax}\|_{\mathbf{W}}^2 + \beta \|\mathbf{v}\|_p + \frac{\rho}{2} \|\mathbf{Rx} - \mathbf{v} - \boldsymbol{\eta}\|_2^2$$

ADMM update of primal variable (unknown image):

$$\mathbf{x}^{(n+1)} = \arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{v}^{(n)}, \boldsymbol{\eta}^{(n)}) = [\mathbf{A}'\mathbf{WA} + \rho\mathbf{R}'\mathbf{R}]^{-1} (\mathbf{A}'\mathbf{W}'\mathbf{y} + \rho\mathbf{R}'(\mathbf{v}^{(n)} + \boldsymbol{\eta}^{(n)}))$$

- Ignoring nonnegativity constraint
- $[\mathbf{A}'\mathbf{WA} + \rho\mathbf{R}'\mathbf{R}]^{-1}$ requires iteration (*e.g.*, PCG) but hard to precondition
- Auxiliary variable $\mathbf{v} = \mathbf{Rx}$ is enormous in 3D CT

Improved ADMM for X-ray CT

$$\min_{\mathbf{x} \geq \mathbf{0}, \mathbf{u}, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{R}\mathbf{x}, \quad \mathbf{u} = \mathbf{A}\mathbf{x}.$$

Corresponding (modified) augmented Lagrangian:

$$L(\mathbf{x}, \mathbf{u}, \mathbf{v}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \beta \|\mathbf{v}\|_p + \frac{\rho_1}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}_1\|_2^2 + \frac{\rho_2}{2} \|\mathbf{A}\mathbf{x} - \mathbf{u} - \boldsymbol{\eta}_2\|_2^2$$

ADMM update of primal variable (ignoring nonnegativity):

$$\arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = [\rho_2 \mathbf{A}'\mathbf{A} + \rho_1 \mathbf{R}'\mathbf{R}]^{-1} (\rho_1 \mathbf{R}'(\mathbf{v} + \boldsymbol{\eta}_1) + \rho_2 \mathbf{A}'(\mathbf{u} + \boldsymbol{\eta}_2))$$

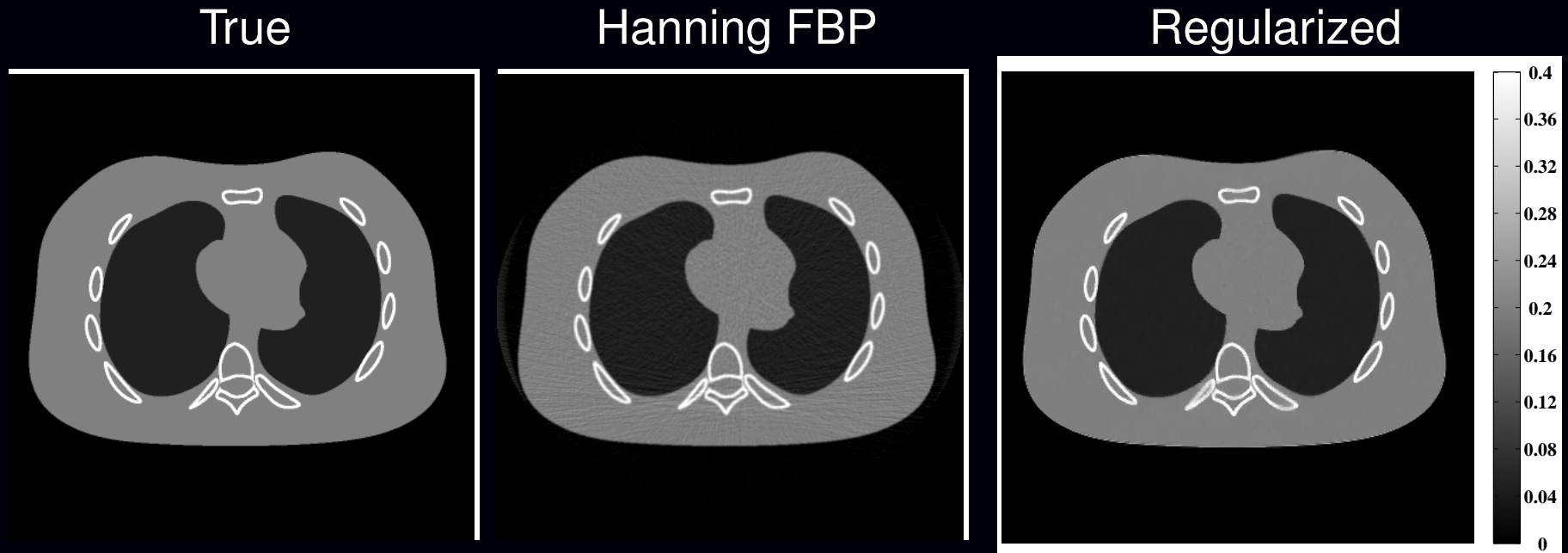
For 2D CT, $[\rho_2 \mathbf{A}'\mathbf{A} + \rho_1 \mathbf{R}'\mathbf{R}]^{-1}$ is approximately Toeplitz so a circulant preconditioner is very effective.

ADMM update of auxiliary variable \mathbf{u} :

$$\arg \min_{\mathbf{u}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \underbrace{[\mathbf{W} + \rho_2 \mathbf{I}]^{-1}}_{\text{diagonal}} (\mathbf{W}\mathbf{y} + \rho_2(\mathbf{A}\mathbf{x} - \boldsymbol{\eta}_2))$$

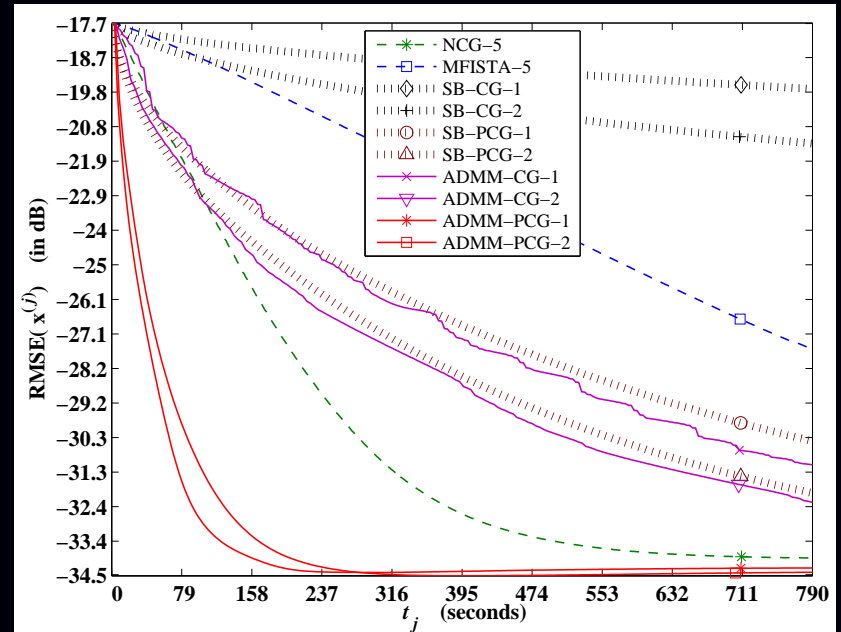
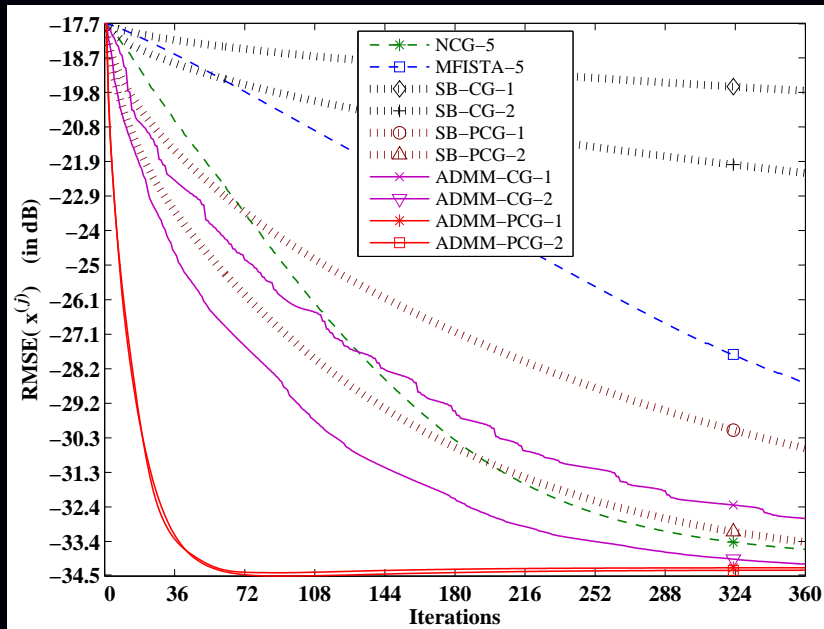
\mathbf{v} update is shrinkage again. Reasonably simple to code.

2D X-ray CT image reconstruction results: quality



PWLS with ℓ_1 regularization of shift-invariant Haar wavelet transform.
No nonnegativity constraint, but probably unimportant if well-regularized.

2D X-ray CT image reconstruction results: speed



Circulant preconditioner for $[\rho_2 \mathbf{A}'\mathbf{A} + \rho_1 \mathbf{R}'\mathbf{R}]^{-1}$ is crucial to acceleration.

Similar results for real head CT scan in paper.

Lower-memory ADMM for X-ray CT

$$\min_{\mathbf{x}, \mathbf{u}, \mathbf{z} \succeq \mathbf{0}} \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \beta \|\mathbf{Rz}\|_p \quad \text{sub. to } \mathbf{z} = \mathbf{x}, \quad \mathbf{u} = \mathbf{Ax}.$$

(M McGaffin, S Ramani, JF, SPIE 2012)

Corresponding (modified) augmented Lagrangian:

$$L(\mathbf{x}, \mathbf{u}, \mathbf{z}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \beta \|\mathbf{Rz}\|_p + \frac{\rho_1}{2} \|\mathbf{x} - \mathbf{z} - \boldsymbol{\eta}_1\|_2^2 + \frac{\rho_2}{2} \|\mathbf{Ax} - \mathbf{u} - \boldsymbol{\eta}_2\|_2^2$$

ADMM update of primal variable (nonnegativity not required, use PCG):

$$\arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{z}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = [\rho_2 \mathbf{A}'\mathbf{A} + \rho_1 \mathbf{I}]^{-1} (\rho_1 (\mathbf{z} + \boldsymbol{\eta}_1) + \rho_2 \mathbf{A}'(\mathbf{u} + \boldsymbol{\eta}_2)).$$

ADMM update of auxiliary variable \mathbf{z} :

$$\arg \min_{\mathbf{z} \succeq \mathbf{0}} L(\mathbf{x}, \mathbf{u}, \mathbf{z}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \arg \min_{\mathbf{z} \succeq \mathbf{0}} \frac{\rho_1}{2} \|\mathbf{x} - \mathbf{z} - \boldsymbol{\eta}_1\|_2^2 + \beta \|\mathbf{Rz}\|_p.$$

Use nonnegatively constrained, edge-preserving image denoising.

ADMM updates of auxiliary variables \mathbf{u} and \mathbf{v} same as before.

Variations...

3D X-ray CT image reconstruction results

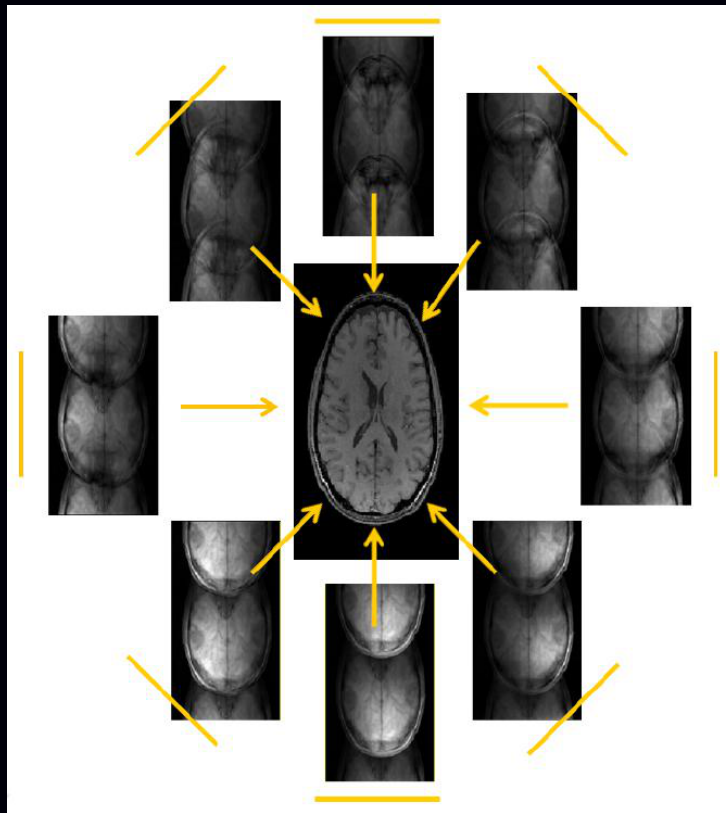
Awaiting better preconditioner for $[\rho_2 \mathbf{A}'\mathbf{A} + \rho_1 \mathbf{I}]^{-1}$

Image reconstruction for parallel MRI

Parallel MRI

Undersampled Cartesian k-space, multiple receive coils, ...

(Pruessmann *et al.*, MRM, Nov. 1999)



Compressed sensing parallel MRI \equiv further (random) under-sampling

Lustig *et al.*, IEEE Sig. Proc. Mag., Mar. 2008

Model-based image reconstruction in parallel MRI

Regularized estimator:

$$\hat{\mathbf{x}} = \arg \min_x \underbrace{\frac{1}{2} \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2}_{\text{data fit}} + \beta \underbrace{\|\mathbf{R}\mathbf{x}\|_p}_{\text{sparsity}}.$$

\mathbf{F} is under-sampled DFT matrix (fat)

Features:

- coil sensitivity matrix \mathbf{S} is block diagonal (Pruessmann *et al.*, MRM, Nov. 1999)
- $\mathbf{F}'\mathbf{F}$ is circulant

Complications:

- Data-fit Hessian $\mathbf{S}'\mathbf{F}'\mathbf{F}\mathbf{S}$ is highly shift variant due to coil sensitivity maps
- Non-quadratic (edge-preserving) regularization $\|\cdot\|_p$
- Complex quantities
- Large problem size (if 3D)

Basic ADMM for parallel MRI

Basic equivalent **constrained** optimization problem (*cf.* split Bregman):

$$\min_{\mathbf{x}, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{R}\mathbf{x}.$$

Corresponding (modified) augmented Lagrangian (*cf.* “split Bregman”):

$$L(\mathbf{x}, \mathbf{v}; \boldsymbol{\eta}) = \frac{1}{2} \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}\|_2^2$$

(Skipping technical details about complex vectors.)

ADMM update of primal variable (unknown image):

$$\mathbf{x}^{(n+1)} = \arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{v}^{(n)}, \boldsymbol{\eta}^{(n)}) = [\mathbf{S}'\mathbf{F}'\mathbf{F}\mathbf{S} + \rho\mathbf{R}'\mathbf{R}]^{-1} (\mathbf{S}'\mathbf{F}'\mathbf{y} + \rho\mathbf{R}'(\mathbf{v}^{(n)} + \boldsymbol{\eta}^{(n)}))$$

- $[\mathbf{S}'\mathbf{F}'\mathbf{F}\mathbf{S} + \rho\mathbf{R}'\mathbf{R}]^{-1}$ requires iteration (*e.g.*, PCG) but hard to precondition
- (Trivial for single coil case with $\mathbf{S} = \mathbf{I}$.)
- The “problem” matrix is on opposite side:
 - MRI: **FS**
 - Restoration: **TA**

Improved ADMM for parallel MRI

$$\min_{\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{z}} \frac{1}{2} \|\mathbf{y} - \mathbf{F}\mathbf{u}\|_2^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{R}\mathbf{z}, \quad \mathbf{u} = \mathbf{S}\mathbf{x}, \quad \mathbf{z} = \mathbf{x}.$$

Corresponding (modified) augmented Lagrangian:

$$\frac{1}{2} \|\mathbf{y} - \mathbf{F}\mathbf{u}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho_1}{2} \|\mathbf{R}\mathbf{z} - \mathbf{v} - \boldsymbol{\eta}_1\|_2^2 + \frac{\rho_2}{2} \|\mathbf{S}\mathbf{x} - \mathbf{u} - \boldsymbol{\eta}_2\|_2^2 + \frac{\rho_3}{2} \|\mathbf{x} - \mathbf{z} - \boldsymbol{\eta}_3\|_2^2$$

ADMM update of primal variable

$$\arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{z}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) = \underbrace{[\rho_2 \mathbf{S}'\mathbf{S} + \rho_3 \mathbf{I}]^{-1}}_{\text{diagonal}} (\rho_2 \mathbf{S}'(\mathbf{u} + \boldsymbol{\eta}_2) + \rho_3(\mathbf{z} + \boldsymbol{\eta}_3))$$

ADMM update of auxiliary variables:

$$\arg \min_{\mathbf{u}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{z}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) = \underbrace{[\mathbf{F}'\mathbf{F} + \rho_2 \mathbf{I}]^{-1}}_{\text{circulant}} (\mathbf{F}'\mathbf{y} + \rho_2(\mathbf{S}\mathbf{x} - \boldsymbol{\eta}_2))$$

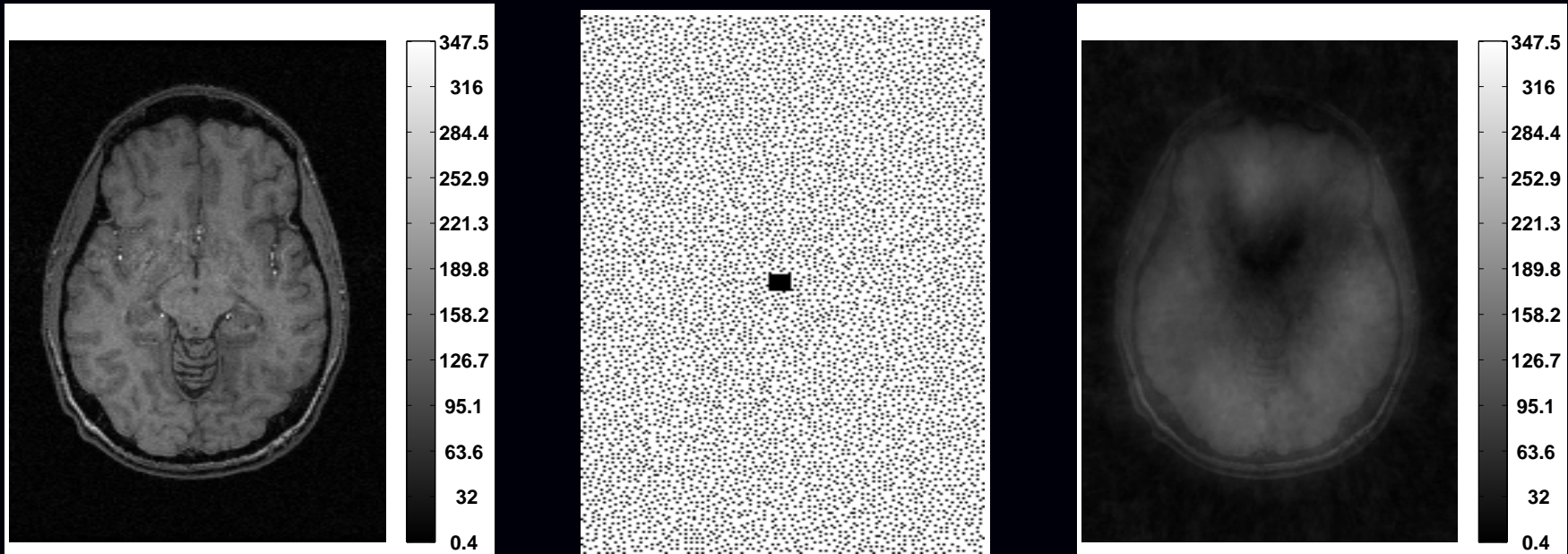
$$\arg \min_{\mathbf{z}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{z}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) = \underbrace{[\rho_1 \mathbf{R}'\mathbf{R} + \rho_3 \mathbf{I}]^{-1}}_{\text{circulant}} (\rho_1 \mathbf{R}'(\mathbf{v} + \boldsymbol{\eta}_1) + \rho_3(\mathbf{x} - \boldsymbol{\eta}_3))$$

\mathbf{v} update is shrinkage again.

Simple, but does not satisfy sufficient conditions.

(Sathish Ramani & JF, IEEE T-MI, Mar. 2011)

2.5D parallel MR image reconstruction results: data

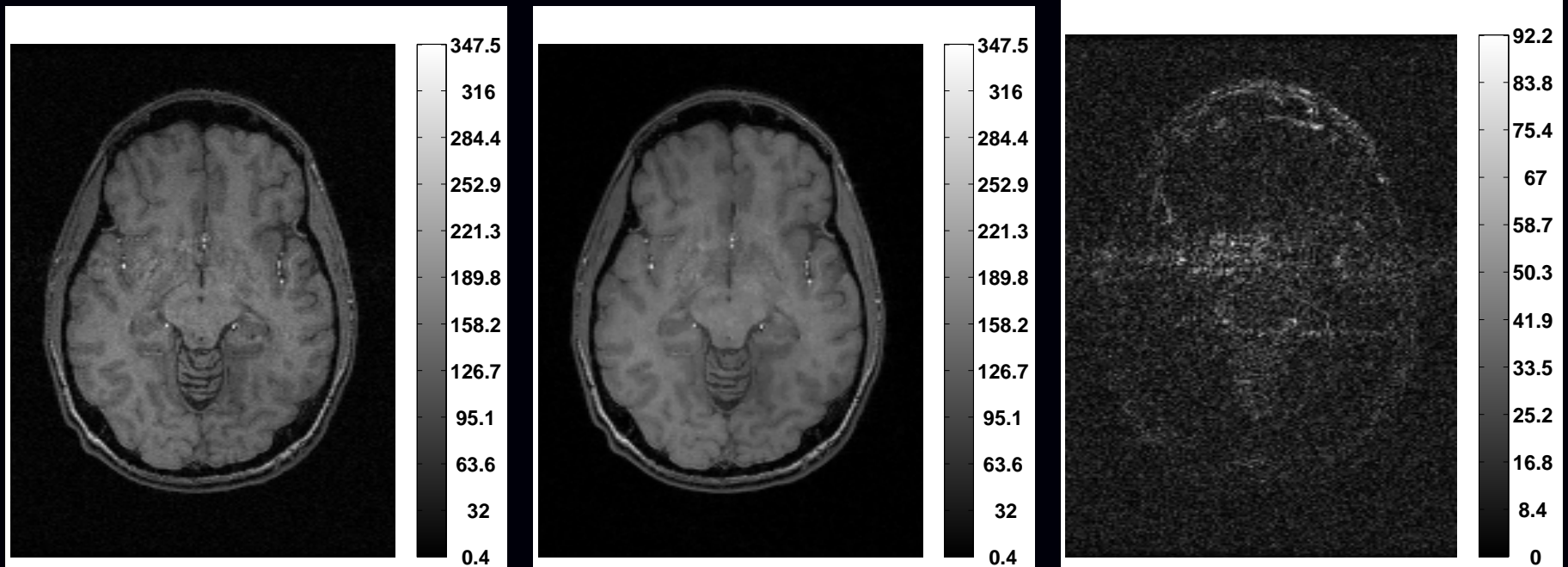


Fully sampled body coil image of human brain

Poisson-disk-based k-space sampling, 16% sampling (acceleration 6.25)

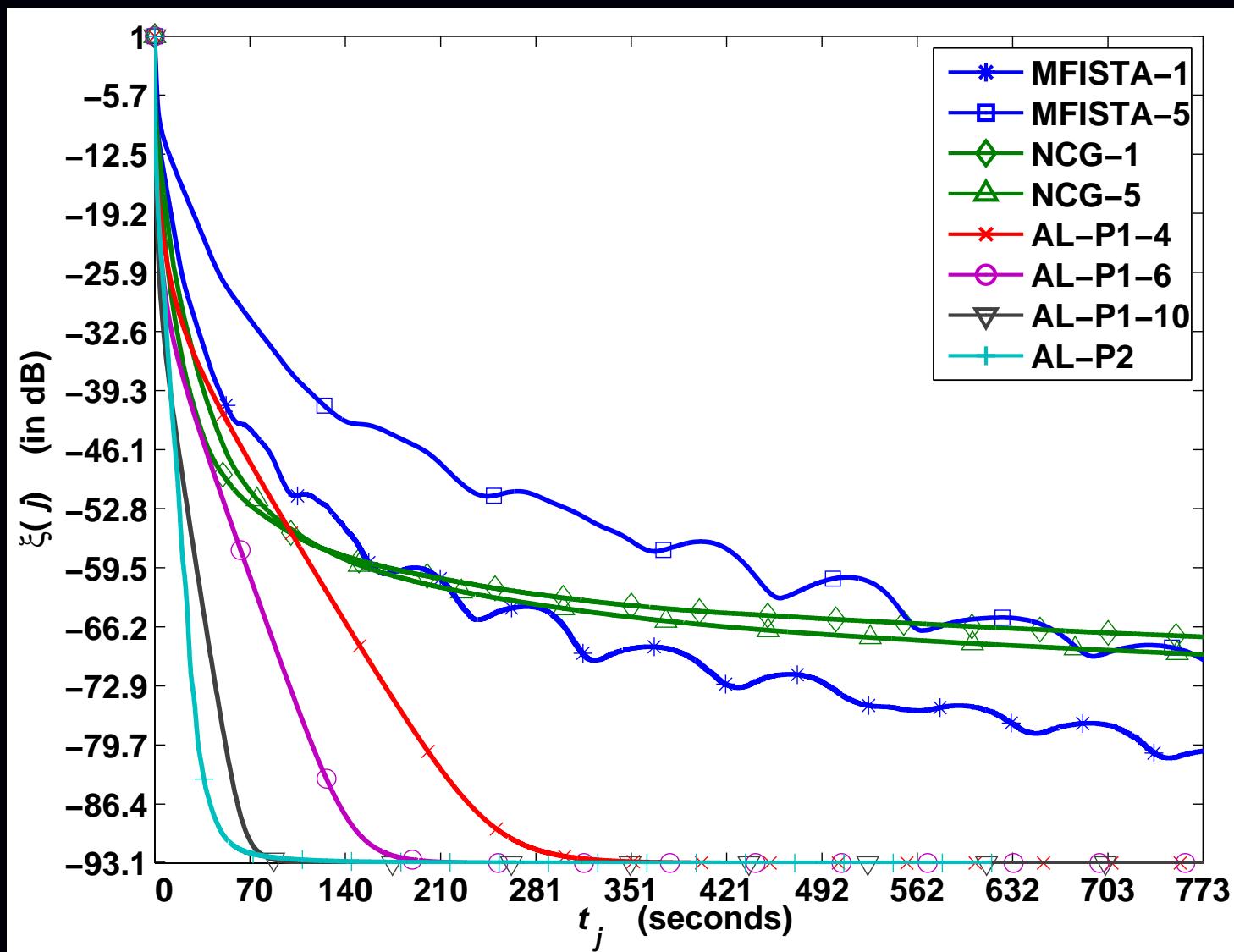
Square-root of sum-of-squares inverse FFT of zero-filled k-space data

2.5D parallel MR image reconstruction results: IQ



- Fully sampled body coil image of human brain
- Regularized reconstruction $x^{(\infty)}$ (1000s of iterations of MFISTA)
(A Beck & M Teboulle, *SIAM J. Im. Sci*, 2009)
Combined TV and ℓ_1 norm of two-level undecimated Haar wavelets
- Difference image magnitude

2.5D parallel MR image reconstruction results: speed



AL approach converges to $x^{(\infty)}$ much faster than MFISTA and CG

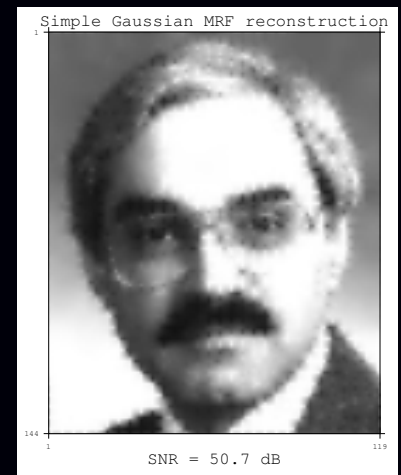
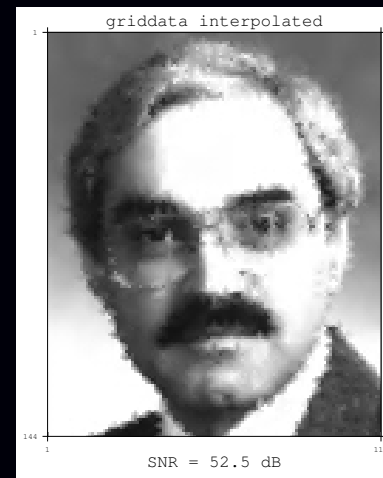
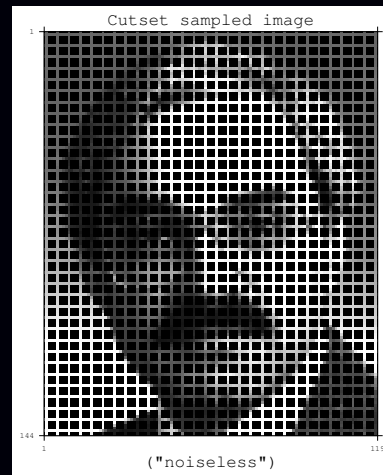
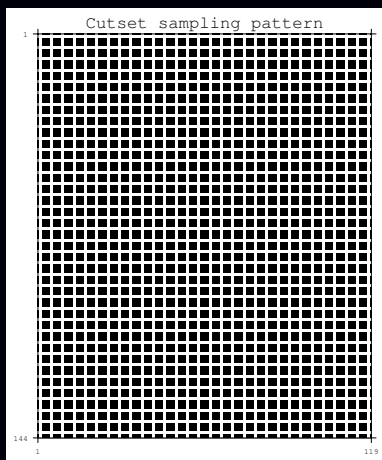
Image in-painting - Premier

Imaging in-painting

Measurement model:

$$\underbrace{y}_{\text{observed}} = \underbrace{S}_{\text{sampling}} \underbrace{x}_{\text{unknown}} + \underbrace{\varepsilon}_{\text{noise (?)}}$$

S is an identity matrix with rows removed, keeping only those row corresponding to image samples



Goal is to recover unknown image x from samples y .

(“in-painting”)

Image in-painting with sparsity regularization

Regularized estimator:

$$\hat{\mathbf{x}} = \arg \min_x \underbrace{\frac{1}{2} \|\mathbf{y} - \mathbf{S}\mathbf{x}\|_2^2}_{\text{data fit}} + \beta \underbrace{\|\mathbf{R}\mathbf{x}\|_p}_{\text{sparsity}}.$$

Equivalent **constrained** optimization problem:

$$\min_{\mathbf{x}, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{S}\mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{R}\mathbf{x}.$$

Corresponding (modified) augmented Lagrangian:

$$L(\mathbf{x}, \mathbf{v}; \boldsymbol{\eta}) = \frac{1}{2} \|\mathbf{y} - \mathbf{S}\mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}\|_2^2$$

Update of auxiliary variable \mathbf{v} is shrinkage again:

$$\mathbf{v}^{(n+1)} = \arg \min_{\mathbf{v}} L(\mathbf{x}^{(n+1)}, \mathbf{v}, \boldsymbol{\eta}^{(n)}) = \text{shrink}(\mathbf{R}\mathbf{x}^{(n+1)} - \boldsymbol{\eta}^{(n)}; \beta/\rho, p)$$

Update of multiplier $\boldsymbol{\eta}$: $\boldsymbol{\eta}^{(n+1)} = \boldsymbol{\eta}^{(n)} + (\mathbf{R}\mathbf{x}^{(n+1)} - \mathbf{v}^{(n+1)})$

Primal variable \mathbf{x} ?

ADMM for image in-painting

$$L(\mathbf{x}, \mathbf{v}; \boldsymbol{\eta}) = \frac{1}{2} \|\mathbf{y} - \mathbf{S}\mathbf{x}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}\|_2^2$$

ADMM update of primal variable (unknown image):

$$\mathbf{x}^{(n+1)} = \arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{v}^{(n)}, \boldsymbol{\eta}^{(n)}) = [\mathbf{S}'\mathbf{S} + \rho\mathbf{R}'\mathbf{R}]^{-1} (\mathbf{S}'\mathbf{y} + \rho\mathbf{R}'(\mathbf{v}^{(n)} + \boldsymbol{\eta}^{(n)}))$$

Feasible?

$[\mathbf{S}'\mathbf{S} + \rho\mathbf{R}'\mathbf{R}]$ is *sparse*

Tridiagonal in 1D problems $\implies O(N)$ solver.

Block tridiagonal in 2D problems \implies use sparse Cholesky decomposition, *i.e.* “\” in MATLAB, if “not too large.”

(Michael Allison, Sathish Ramani, JF, IEEE T-MI, Mar. 2013)

(Michael Allison, JF, ISMRM 2012)

Alternate ADMM for image in-painting

$$\hat{\mathbf{x}} = \arg \min_x \underbrace{\frac{1}{2} \|\mathbf{y} - \mathbf{S}\mathbf{x}\|_2^2}_{\text{data fit}} + \beta \underbrace{\|\mathbf{R}\mathbf{x}\|_p}_{\text{sparsity}}.$$

Usual split: $\mathbf{v} = \mathbf{R}\mathbf{x}$

Q: What additional auxiliary variable should we introduce to simplify the updates?

Alternate ADMM for image in-painting

Alternate equivalent **constrained** optimization problem:

$$\min_{\mathbf{x}, \mathbf{u}, \mathbf{v}} \frac{1}{2} \|\mathbf{y} - \mathbf{S}\mathbf{u}\|_2^2 + \beta \|\mathbf{v}\|_p \quad \text{sub. to } \mathbf{v} = \mathbf{R}\mathbf{x}, \quad \mathbf{u} = \mathbf{x}.$$

Corresponding (modified) augmented Lagrangian:

$$L(\mathbf{x}, \mathbf{u}, \mathbf{v}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \frac{1}{2} \|\mathbf{y} - \mathbf{S}\mathbf{u}\|_2^2 + \beta \|\mathbf{v}\|_p + \frac{\rho_1}{2} \|\mathbf{R}\mathbf{x} - \mathbf{v} - \boldsymbol{\eta}_1\|_2^2 + \frac{\rho_2}{2} \|\mathbf{x} - \mathbf{u} - \boldsymbol{\eta}_2\|_2^2$$

ADMM update of primal variable (unknown image):

$$\arg \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = [\rho_2 \mathbf{I} + \rho_1 \mathbf{R}'\mathbf{R}]^{-1} (\rho_1 \mathbf{R}'(\mathbf{v} + \boldsymbol{\eta}_1) + \rho_2(\mathbf{u} + \boldsymbol{\eta}_2))$$

Simple if $\mathbf{R}'\mathbf{R}$ is circulant.

ADMM update of new auxiliary variable \mathbf{u} :

$$\arg \min_{\mathbf{u}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \underbrace{[\mathbf{S}'\mathbf{S} + \rho_2 \mathbf{I}]^{-1}}_{\text{diagonal}} (\mathbf{S}'\mathbf{y} + \rho_2(\mathbf{x} - \boldsymbol{\eta}_2))$$

\mathbf{v} update is shrinkage again. Very easy to code!

Image in-painting results

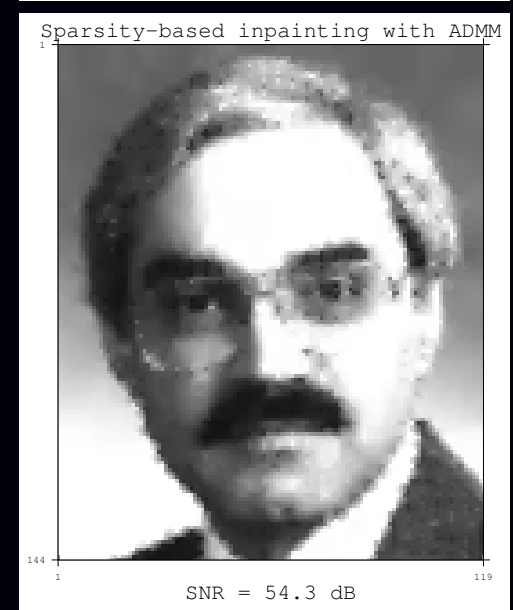
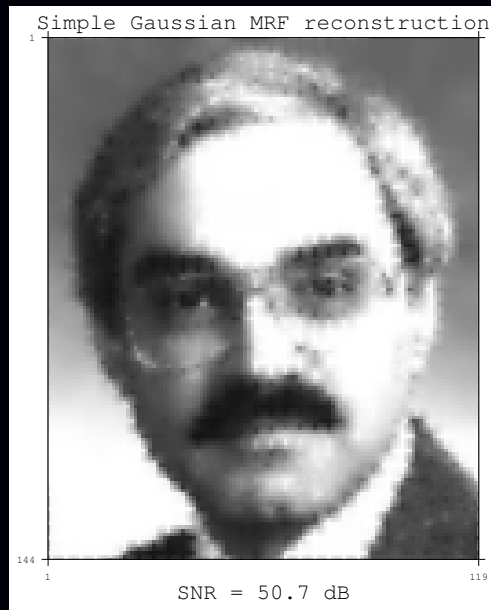
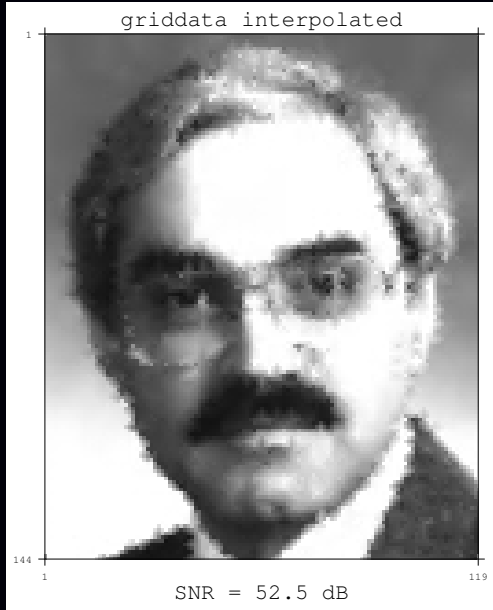
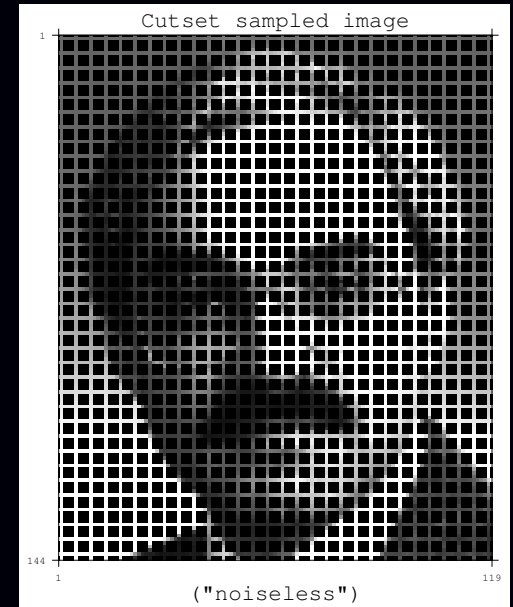
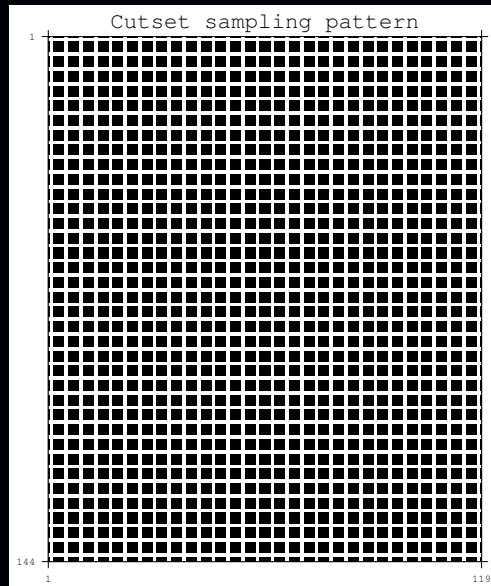
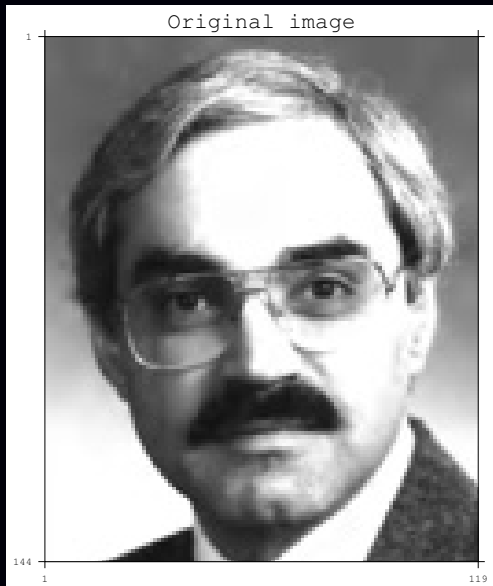
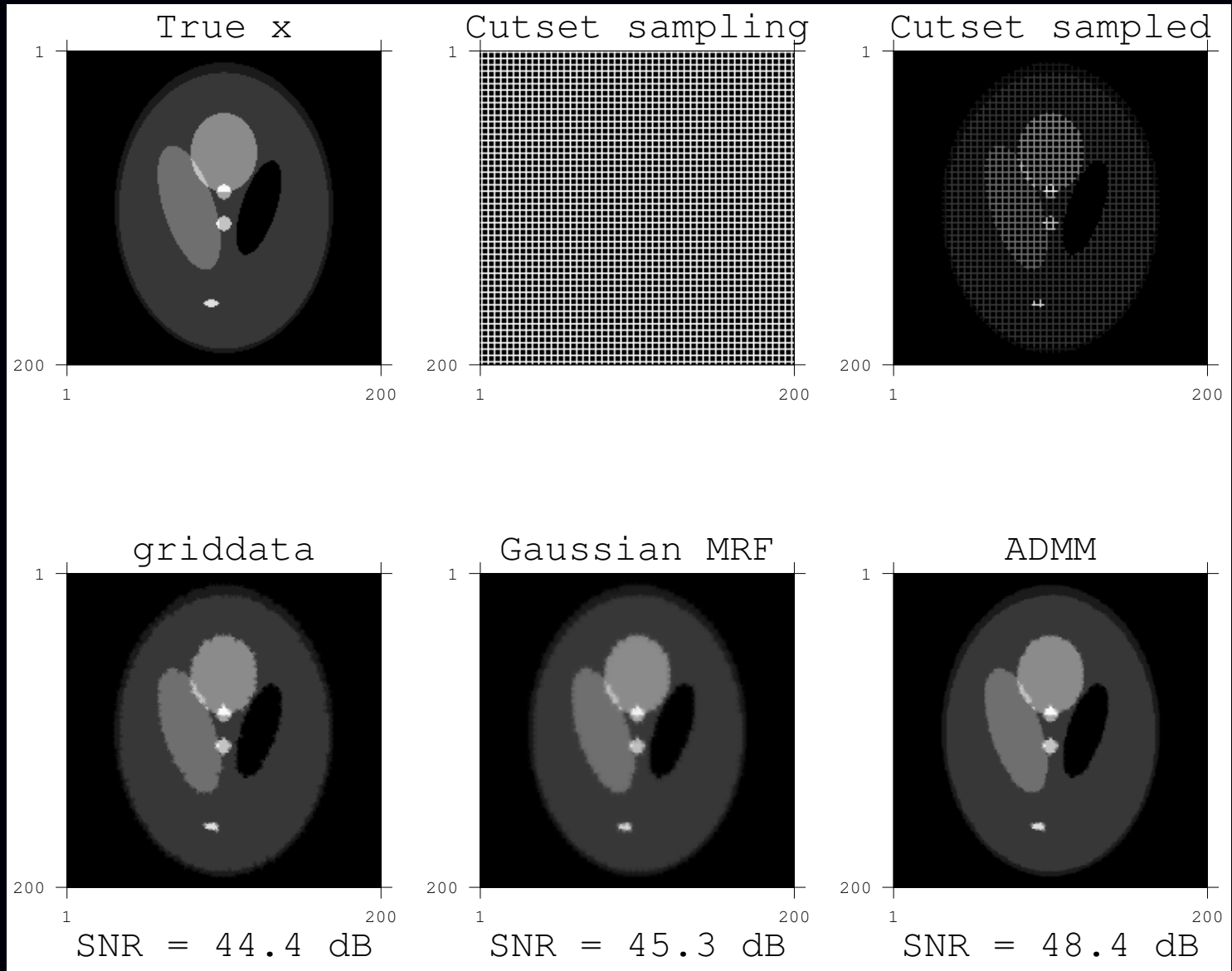


Image in-painting results



anisotropic TV regularization, $\beta = 2^{-4}$

Image in-painting alternatives

Constrained formulation:

$$\arg \min_x \|\mathbf{R}\mathbf{x}\|_p \text{ sub. to } \mathbf{y} = \mathbf{S}\mathbf{x}$$

Also can be solved using ADMM.

Better regularizers (sparsity transforms) than anisotropic TV...

Current and future directions with ADMM

- Motion-compensated image reconstruction: $\mathbf{y} = \mathbf{AT}(\boldsymbol{\alpha})\mathbf{x} + \boldsymbol{\varepsilon}$
(J H Cho, S Ramani, JF, 2nd CT meeting, 2012)
(J H Cho, S Ramani, JF, IEEE Stat. Sig. Proc. W., 2012)
- Dynamic image reconstruction
- Improved preconditioners for ADMM for 3D CT
(M McGaffin and JF, Submitted to Fully 3D 2013)
- Combining ADMM with ordered subsets (OS) methods
(H Nien and JF, Submitted to Fully 3D 2013)
- Generalize parallel MRI algorithm to include spatial support constraint
(M Le, S Ramani, JF, To appear at ISMRM 2013)
- Non-Cartesian MRI (combine optimization transfer and variable splitting)
(S Ramani and JF, ISBI 2013, to appear.)
- SPECT-CT reconstruction with non-local means regularizer
(S Y Chun, Y K Dewaraja, JF, Submitted to Fully 3D 2013)
- Estimation of coil sensitivity maps (quadratic problem!)
(M J Allison, S Ramani, JF, IEEE T-MI, 2013, to appear)
- L1-SPIRiT for non-Cartesian parallel MRI (D S Weller, S Ramani, JF, IEEE T-MI, 2013, submitted)
- Multi-frame super-resolution
- Selection of AL penalty parameter ρ to optimize convergence rate
- Other non-ADMM methods...



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- Mitch Goodsitt
- Ella Kazerooni
- Tom Chenevert
- Charles Meyer

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Bibliography

- [1] A. Matakos, S. Ramani, and J. A. Fessler. Accelerated edge-preserving image restoration without boundary artifacts. *IEEE Trans. Im. Proc.*, 22(5):2019–29, May 2013.
- [2] S. Ramani and J. A. Fessler. A splitting-based iterative algorithm for accelerated statistical X-ray CT reconstruction. *IEEE Trans. Med. Imag.*, 31(3):677–88, March 2012.
- [3] S. Ramani and J. A. Fessler. Parallel MR image reconstruction using augmented Lagrangian methods. *IEEE Trans. Med. Imag.*, 30(3):694–706, March 2011.
- [4] M. Elad, P. Milanfar, and R. Rubinstein. Analysis versus synthesis in signal priors. *Inverse Prob.*, 23(3):947–68, June 2007.
- [5] I. W. Selesnick and Mário A T Figueiredo. Signal restoration with overcomplete wavelet transforms: comparison of analysis and synthesis priors. In *Proc. SPIE 7446 Wavelets XIII*, page 74460D, 2009. Wavelets XIII.
- [6] Y. Wang, J. Yang, W. Yin, and Y. Zhang. A new alternating minimization algorithm for total variation image reconstruction. *SIAM J. Imaging Sci.*, 1(3):248–72, 2008.
- [7] M. V. Afonso, José M Bioucas-Dias, and Mário A T Figueiredo. Fast image recovery using variable splitting and constrained optimization. *IEEE Trans. Im. Proc.*, 19(9):2345–56, September 2010.
- [8] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Found. & Trends in Machine Learning*, 3(1):1–122, 2010.
- [9] J. Eckstein and D. P. Bertsekas. On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Mathematical Programming*, 55(1-3):293–318, April 1992.
- [10] J. Douglas and H. H. Rachford. On the numerical solution of heat conduction problems in two and three space variables. *tams*, 82(2):421–39, July 1956.
- [11] T. Goldstein and S. Osher. The split Bregman method for L1-regularized problems. *SIAM J. Imaging Sci.*, 2(2):323–43, 2009.
- [12] S. J. Reeves. Fast image restoration without boundary artifacts. *IEEE Trans. Im. Proc.*, 14(10):1448–53, October 2005.
- [13] M. G. McGaffin, S. Ramani, and J. A. Fessler. Reduced memory augmented Lagrangian algorithm for 3D iterative X-ray CT image reconstruction. In *Proc. SPIE 8313 Medical Imaging 2012: Phys. Med. Im.*, page 831327, 2012.
- [14] K. P. Pruessmann, M. Weiger, M. B. Scheidegger, and P. Boesiger. SENSE: sensitivity encoding for fast MRI. *Mag. Res. Med.*, 42(5):952–62, November 1999.
- [15] M. Lustig, D. L. Donoho, J. M. Santos, and J. M. Pauly. Compressed sensing MRI. *IEEE Sig. Proc. Mag.*, 25(2):72–82, March 2008.
- [16] A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.*, 2(1):183–202, 2009.
- [17] M. J. Allison, S. Ramani, and J. A. Fessler. Accelerated regularized estimation of MR coil sensitivities using augmented Lagrangian methods. *IEEE Trans. Med. Imag.*, 32(3):556–64, March 2013.
- [18] M. J. Allison and J. A. Fessler. Accelerated computation of regularized field map estimates. In *Proc. Intl. Soc. Mag. Res. Med.*, page 0413, 2012.
- [19] J. H. Cho, S. Ramani, and J. A. Fessler. Motion-compensated image reconstruction with alternating minimization. In *Proc.* 51

2nd Intl. Mtg. on image formation in X-ray CT, pages 330–3, 2012.

- [20] J. H. Cho, S. Ramani, and J. A. Fessler. Alternating minimization approach for multi-frame image reconstruction. In *IEEE Workshop on Statistical Signal Processing*, pages 225–8, 2012.
- [21] M. McGaffin and J. A. Fessler. Sparse shift-varying FIR preconditioners for fast volume denoising. In *Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med*, 2013. To appear.
- [22] H. Nien and J. A. Fessler. Combining augmented Lagrangian method with ordered subsets for X-ray CT image reconstruction. In *Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med*, 2013. To appear.
- [23] M. Le, S. Ramani, and J. A. Fessler. An efficient variable splitting based algorithm for regularized SENSE reconstruction with support constraint. In *Proc. Intl. Soc. Mag. Res. Med.*, page 2654, 2013. To appear.
- [24] S. Ramani and J. A. Fessler. Accelerated non-Cartesian SENSE reconstruction using a majorize-minimize algorithm combining variable-splitting. In *Proc. IEEE Intl. Symp. Biomed. Imag.*, pages 700–3, 2013.
- [25] S. Y. Chun, Y. K. Dewaraja, and J. A. Fessler. Alternating direction method of multiplier for emission tomography with non-local regularizers. In *Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med*, 2013. To appear.
- [26] D. Weller, S. Ramani, and J. A. Fessler. Augmented Lagrangian with variable splitting for faster non-Cartesian L1-SPIRiT MR image reconstruction. *IEEE Trans. Med. Imag.*, 2013. Submitted.

Back-up slide(s)

System matrix / Gram matrix

