Accelerated Non-Cartesian SENSE Reconstruction Using a Majorize-Minimize Algorithm Combining Variable-Splitting

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Overview

- Non-Cartesian k-space trajectories
  - Efficient k-space coverage, robustness to motion & off-resonance effects
  - Reconstruction is however more involved than for Cartesian trajectories
  - More computation: NUFFTs perform interpolation in addition to FFTs
- This work: An algorithm for non-Cartesian SENSE reconstruction
  - Combines majorize-minimize strategy with variable-splitting
  - Has reduced need of NUFFTs
- Organization of the talk:
  - Quick overview of regularized SENSE
  - Existing variable-splitting methods for regularized SENSE
  - Proposed majorize-minimize + variable-splitting scheme
  - Experimental results

Regularized SENSE Reconstruction

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_L \end{bmatrix}; \qquad \mathbf{y}_l \in \mathbb{C}^M\text{: data from } l\text{th coil}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 \\ \vdots \\ \mathbf{S}_L \end{bmatrix}; \qquad \mathbf{S}_l \in \mathbb{C}^{N \times N}\text{: diagonal sensitivity matrix for } l\text{th coil}$$

$$\mathbf{F} = \mathbf{I}_L \otimes \mathbf{F}_{\mathrm{u}}; \qquad \mathbf{F}_{\mathrm{u}} \in \mathbb{C}^{M \times N}\text{: non-Cartesian Fourier encoding matrix}$$

- $\Psi(\mathbf{R}\mathbf{x})$ is a suitable regularizer: imposes prior information,
  reduces noise & artifacts,
  e.g., TV, ℓ_1-regularizers, etc.

  $\mathbf{R} \in \mathbb{R}^{P \times N}$: finite differences, wavelet frames, etc.

Regularized SENSE Reconstruction

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- Challenges:
  - Gram matrix $\mathbf{S}'\mathbf{F}'\mathbf{F}\mathbf{S}$ of data-fit term is highly shift-variant
  - $\mathbf{F}'\mathbf{F}$ not circulant, unlike Cartesian case
  - Regularizer Ψ is non-quadratic and often non-differentiable
  - Problem size can be large

- Inherent mathematical structures in the problem
  - $\mathbf{F}'\mathbf{F}$ is Toeplitz for non-Cartesian trajectories (no field inhomogeneity)
  - $\mathbf{S}'\mathbf{S}$ is diagonal
  - $\mathbf{R}'\mathbf{R}$ is circulant (e.g., finite differences with periodic boundary conditions)

- Exploit inherent structures: separate $\mathbf{F}$, $\mathbf{S}$ and $\mathbf{R}$ via variable splitting

Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

- Split-Bregman (SB) type algorithms — Goldstein *et al. SIAM J. Img. Sci.*, 2009

$\mathbf{u}_1 = \mathbf{R}\mathbf{x}$ only decouples $\mathbf{R}$ from regularizer Ψ

Variable Splitting & Augmented Lagrangian

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- Augmented Lagrangian (AL) algorithm — Ramani *et al. IEEE TMI* 2011

  $\mathbf{u}_0 = \mathbf{S}\mathbf{x}$ separates $\mathbf{F}$ and $\mathbf{S}$

  $\mathbf{u}_1 = \mathbf{R}\mathbf{u}_2$ separates $\mathbf{R}$ from regularizer Ψ

  $\mathbf{u}_2 = \mathbf{x}$ separates $\mathbf{R}$ from $\mathbf{S}$

- Equivalent constrained problem: $\min_{\mathbf{z}} J(\mathbf{z})$ s.t. $\mathfrak{B}\mathbf{z} = \mathbf{0}$
  - SB: $\mathfrak{B} \triangleq [\mathbf{I} \;\; -\mathbf{R}]$ and $\mathbf{z} \triangleq \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{x} \end{bmatrix}$
  - AL: $\mathfrak{B} \triangleq \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -\mathbf{S} \\ \mathbf{0} & \mathbf{I} & -\mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} \end{bmatrix}$ and $\mathbf{z} \triangleq \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{x} \end{bmatrix}$

Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

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  $\mathbf{u}_2 = \mathbf{x}$ separates $\mathbf{R}$ from $\mathbf{S}$

- Equivalent constrained problem: $\min_{\mathbf{z}} J(\mathbf{z})$ s.t. $\mathcal{B}\mathbf{z} = \mathbf{0}$
- Augmented Lagrangian function: $\mathcal{L}(\mathbf{z}, \boldsymbol{\eta}) = J(\mathbf{z}) + \frac{\mu}{2}\|\mathcal{B}\mathbf{z} - \boldsymbol{\eta}\|_{\boldsymbol{\Lambda}}^2 + c(\boldsymbol{\eta})$
  - μ is a penalty parameter; $\boldsymbol{\Lambda} \equiv$ relative weighting of constraints
  - $\boldsymbol{\eta}$ is a Lagrange-multiplier-type vector for the constraint $\mathcal{B}\mathbf{z} = \mathbf{0}$

Variable Splitting & Augmented Lagrangian

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- Equivalent constrained problem: $\min_{\mathbf{z}} J(\mathbf{z})$ s.t. $\mathcal{B}\mathbf{z} = \mathbf{0}$

- Augmented Lagrangian function: $\mathcal{L}(\mathbf{z}, \boldsymbol{\eta}) = J(\mathbf{z}) + \frac{\mu}{2}\|\mathcal{B}\mathbf{z} - \boldsymbol{\eta}\|_{\boldsymbol{\Lambda}}^2 + c(\boldsymbol{\eta})$

- Algorithm: Alternating minimization of $\mathcal{L}(\mathbf{z}, \boldsymbol{\eta}^{(k)})$ w.r.t. components of $\mathbf{z}$

  Update $\boldsymbol{\eta}^{(k+1)} = \boldsymbol{\eta}^{(k)} - \mathcal{B}\mathbf{z}^{(k+1)}$

Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

- Split-Bregman (SB) type algorithms — Goldstein *et al. SIAM J. Img. Sci.*, 2009
  - $\mathbf{u}_1$-update corresponds to a denoising problem
  - $\mathbf{x}$-update requires "inverting" the shift-variant matrix $(\mathbf{S}'\mathbf{F}'\mathbf{F}\mathbf{S} + \mathbf{R}'\mathbf{R})$

- Augmented Lagrangian (AL) algorithm — Ramani *et al. IEEE TMI* 2011
  - $\mathbf{u}_0$-update requires "inverting" the Toeplitz matrix $(\mathbf{F}'\mathbf{F} + \mu\mathbf{I})$
  - $\mathbf{u}_1$-update corresponds to a denoising problem
  - $\mathbf{u}_2$-update requires "inverting" the circulant matrix $(\mathbf{R}'\mathbf{R} + \beta\mathbf{I})$
  - $\mathbf{x}$-update requires "inverting" the diagonal matrix $(\mathbf{S}'\mathbf{S} + \gamma\mathbf{I})$

Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

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  - $\mathbf{u}_0$-update requires “inverting” the Toeplitz matrix $(\mathbf{F}'\mathbf{F} + \mu\mathbf{I})$

Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

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- Augmented Lagrangian (AL) algorithm — Ramani *et al. IEEE TMI* 2011
  - $\mathbf{u}_0$-update requires "inverting" the Toeplitz matrix $(\mathbf{F}'\mathbf{F} + \mu\mathbf{I})$
- Non-Cartesian MRI: Need iterative solvers for $\mathbf{x}$ in SB and $\mathbf{u}_0$ in ALA
  - $\mathbf{F}'\mathbf{F} = \mathbf{I}_L \otimes \mathbf{F}_{\mathrm{u}}'\mathbf{F}_{\mathrm{u}} \implies L$ solvers needed for L coils
  - Repeated products with $\mathbf{F}_{\mathrm{u}}'\mathbf{F}_{\mathrm{u}} \equiv$ more computation

    e.g., embedding $\mathbf{F}_{\mathrm{u}}'\mathbf{F}_{\mathrm{u}}$ in a larger circulant matrix $\equiv$ FFTs of larger size
- Proposed approach: Majorize-minimize strategy
  - Replaces $\mathbf{F}'\mathbf{F}$ with a circulant matrix

Majorize-Minimize Approach

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

- Majorization: $D(\mathbf{y}, \mathbf{S}\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2$

$$\begin{aligned} &\leq D(\mathbf{y}, \mathbf{S}\mathbf{x}_{(j)}) + 2\mathcal{R}\{(\mathbf{x} - \mathbf{x}_{(j)})' \, \nabla D(\mathbf{y}, \mathbf{S}\mathbf{x})\} \\ &\quad + (\mathbf{x} - \mathbf{x}_{(j)})' \mathbf{S}' \mathbf{C}' \mathbf{C} \mathbf{S} (\mathbf{x} - \mathbf{x}_{(j)}) \end{aligned}$$

$$= 2\mathcal{R}\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + c$$

$$\triangleq D_{\mathrm{maj}}(\mathbf{y}, \mathbf{S}\mathbf{x}, \mathbf{x}_{(j)})$$

- Circulant matrix $\mathbf{C} = \mathbf{I}_L \otimes \mathbf{C}_{\mathrm{u}}$ such that $\mathbf{C}_{\mathrm{u}}' \mathbf{C}_{\mathrm{u}} \succeq \mathbf{F}_{\mathrm{u}}' \mathbf{F}_{\mathrm{u}}$

$$\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F} = \mathbf{I}_L \otimes [\mathbf{C}_{\mathrm{u}}'\mathbf{C}_{\mathrm{u}} - \mathbf{F}_{\mathrm{u}}'\mathbf{F}_{\mathrm{u}}] \succeq \mathbf{0}$$

Majorize-Minimize Approach

- Regularized SENSE reconstruction: model-based optimization problem

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ J(\mathbf{x}) \triangleq \|\mathbf{y} - \mathbf{F}\mathbf{S}\mathbf{x}\|_2^2 + \Psi(\mathbf{R}\mathbf{x}) \right\}$$

- $D_{\mathrm{maj}}(\mathbf{y}, \mathbf{S}\mathbf{x}, \mathbf{x}_{(j)}) = 2\Re\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + c$

$$\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F}$$

- Majorizer: $J_{\mathrm{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) \triangleq D_{\mathrm{maj}}(\mathbf{y}, \mathbf{S}\mathbf{x}, \mathbf{x}_{(j)}) + \Psi(\mathbf{R}\mathbf{x})$

$J_{\mathrm{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) \le J(\mathbf{x}) \;\; \forall \;\; \mathbf{x}$ with equality at $\mathbf{x}_{(j)}$

- Majorize-minimize scheme

$$\mathbf{x}_{(j+1)} = \arg\min_{\mathbf{x}} \; J_{\mathrm{maj}}(\mathbf{x}, \mathbf{x}_{(j)})$$

Variable Splitting for Minimization Step

- $J_{\text{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) = 2\mathcal{R}\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x})$

$$\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F}$$

- Minimization of J_{maj} using variable-splitting and augmented Lagrangian

  $\mathbf{u}_0 = \mathbf{S}\mathbf{x}$ separates $\mathbf{C}$ and $\mathbf{S}$

  $\mathbf{u}_1 = \mathbf{R}\mathbf{u}_2$ separates $\mathbf{R}$ from regularizer Ψ

  $\mathbf{u}_2 = \mathbf{x}$ separates $\mathbf{R}$ from $\mathbf{S}$

- Equivalent constrained problem: $\min_{\mathbf{z}} J_{\text{maj}}(\mathbf{z})$ s.t. $\mathbf{A}\mathbf{z} = \mathbf{0}$

$$\mathbf{A} \triangleq \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -\mathbf{S} \\ \mathbf{0} & \mathbf{I} & -\mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{z} \triangleq \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{x} \end{bmatrix}$$

Augmented Lagrangian for Minimization Step

- $J_{\mathrm{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) = 2\mathcal{R}\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x})$

$$\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F}$$

- Minimization of J_{maj} using variable-splitting and augmented Lagrangian

  $\mathbf{u}_0 = \mathbf{S}\mathbf{x}$ separates $\mathbf{C}$ and $\mathbf{S}$

  $\mathbf{u}_1 = \mathbf{R}\mathbf{u}_2$ separates $\mathbf{R}$ from regularizer Ψ

  $\mathbf{u}_2 = \mathbf{x}$ separates $\mathbf{R}$ from $\mathbf{S}$

- Augmented Lagrangian function:

$$\begin{aligned}\mathcal{L}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{x}) \triangleq\ & 2\mathcal{R}\{-\mathbf{u}_0'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{u}_0'\mathbf{C}'\mathbf{C}\mathbf{u}_0 + \Psi(\mathbf{u}_1) \\ & + \mu\|\mathbf{u}_0 - \mathbf{S}\mathbf{x} - \boldsymbol{\eta}_0\|_2^2 + \mu\nu_1\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_2 - \boldsymbol{\eta}_1\|_2^2 \\ & + \mu\nu_2\|\mathbf{u}_2 - \mathbf{x} - \boldsymbol{\eta}_2\|_2^2\end{aligned}$$

$\mu, \nu_1, \nu_2 > 0$ are penalty parameters

Algorithm Summary

- $J_{\mathrm{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) = 2\mathcal{R}\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x})$
- Every jth majorize-minimize iteration involves
  - One product with $\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F} \equiv$ one product with $\mathbf{F}'\mathbf{F}$
  - $k = 1 \cdots K$ iterations of alternating minimization of $\mathcal{L}$ and $\boldsymbol{\eta}_{0,1,2}$-updates
  - *Warm-starting* of constraint variables $\mathbf{u}_{0,1,2}$ and $\boldsymbol{\eta}_{0,1,2}$
- Every kth iteration of alternating minimization involves
  - Inversion of circulant matrices $(\mathbf{C}'\mathbf{C} + \mu\mathbf{I})$ & $(\mathbf{R}'\mathbf{R} + \nu_2/\nu_1\mathbf{I})$
  - Inversion of diagonal matrix $(\mathbf{S}'\mathbf{S} + \nu_2\mathbf{I})$
  - Denoising problem that admits closed-form solutions for many Ψ
  - Trivial Lagrange-multiplier $\boldsymbol{\eta}_{0,1,2}$-updates

Algorithm Summary

- $J_{\text{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) = 2\mathcal{R}\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x})$
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  - Inversion of diagonal matrix $(\mathbf{S}'\mathbf{S} + \nu_2\mathbf{I})$
  - Denoising problem that admits closed-form solutions for many Ψ
  - Trivial Lagrange-multiplier $\boldsymbol{\eta}_{0,1,2}$-updates
- Acceleration using two-step strategy — Beck *et al.* SIAM J. Img. Sci., 2009

$$\mathbf{x}_{(j+1)} = \arg\min_{\mathbf{x}} J_{\text{maj}}(\mathbf{x}, \mathbf{w}_{(j)});$$

Form J_{maj} at $\mathbf{w}_{(j)}$ instead of $\mathbf{x}_{(j)}$

$$\mathbf{w}_{(j)} = \mathbf{x}_{(j)} + (a_j - 1)/a_{j+1}(\mathbf{x}_{(j)} - \mathbf{x}_{(j-1)}); \qquad a_{j+1} = [1 + \sqrt{1 + 4a_j^2}]/2$$

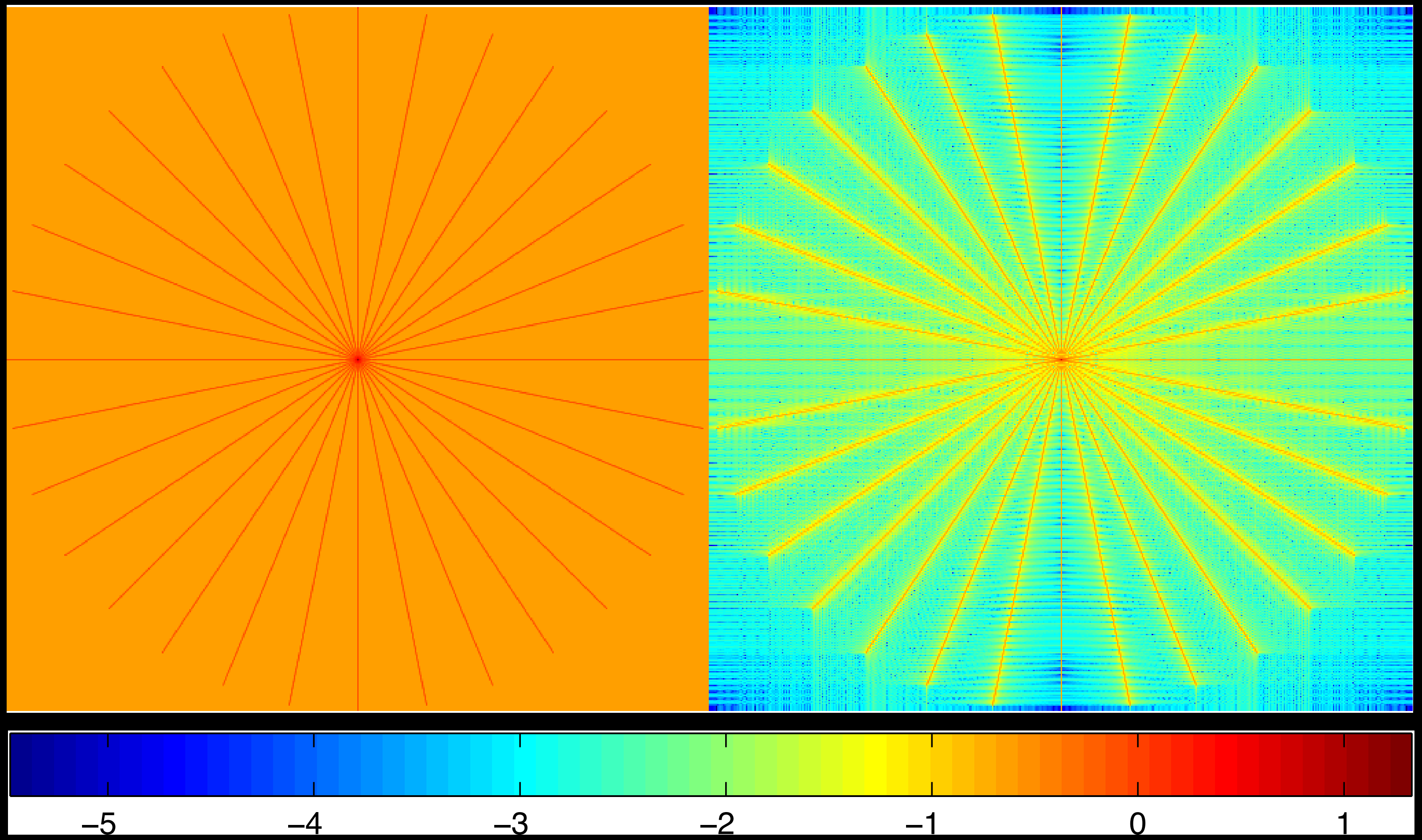
Construction of the Majorizer

- Obtain circulant $\widetilde{\mathbf{C}}$ such that $\widetilde{\mathbf{C}}'\widetilde{\mathbf{C}} \approx \mathbf{F}'\mathbf{F}$ in Frobenius-norm

  Chan *et al.* SIAM J. Sci. Stat. Comp. 1988

- Find $\alpha > 0$ such that $\alpha\widetilde{\mathbf{C}}'\widetilde{\mathbf{C}} \succeq \mathbf{F}'\mathbf{F}$ using Power method
  - Requires matrix-vector products with $\mathbf{F}'\mathbf{F}$
  - Depends only on the trajectory: can be precomputed for various trajectories
- Desired circulant matrix in the majorizer: $\mathbf{C} \triangleq \sqrt{\alpha}\widetilde{\mathbf{C}}$
- Illustration for a radial trajectory with 16 spokes containing 512 samples each

Frequency response associated with $\mathbf{C}_\mathrm{u}'\mathbf{C}_\mathrm{u}$ ($\log_{10}$-scale)



Magnitude of FFT of response of $\mathbf{F}_\mathrm{u}'\mathbf{F}_\mathrm{u}$ to a unit impulse at the image center ($\log_{10}$-scale)

Algorithms Compared in This Work

- Compared proposed methods with recent splitting-based algorithms
  - Split-Bregman (SB-n) — Goldstein *et al.* SIAM J. Img. Sci. 2009
  - Bregman Operator Splitting (BOS) — Zhang *et al.* SIAM J. Img. Sci. 2010
  - Augmented Lagrangian (AL-n) — Ramani *et al.* IEEE TMI 2011
  - Proposed: MAjorize-Minimize AL (MAMAL-K)
  - Proposed: Majorize-minimize AL with Two-Step (MALTS-K)

  n inner PCG iterations for SB and AL
  K inner AL iterations for MAMAL & MALTS

- Circulant preconditioner using $\widetilde{\mathbf{C}}'\widetilde{\mathbf{C}}$ for inner-linear-systems in SB & AL

- Denoising-like step involves shrinkage: common to all algorithms
  - Automatically set penalty parameters of all algorithms to obtain same shrinkage-threshold

Simulation with Analytical Shepp-Logan Phantom

- Simulated noisy data using analytical Shepp-Logan phantom

  Guerquin-Kern *et al.* IEEE TMI 2012

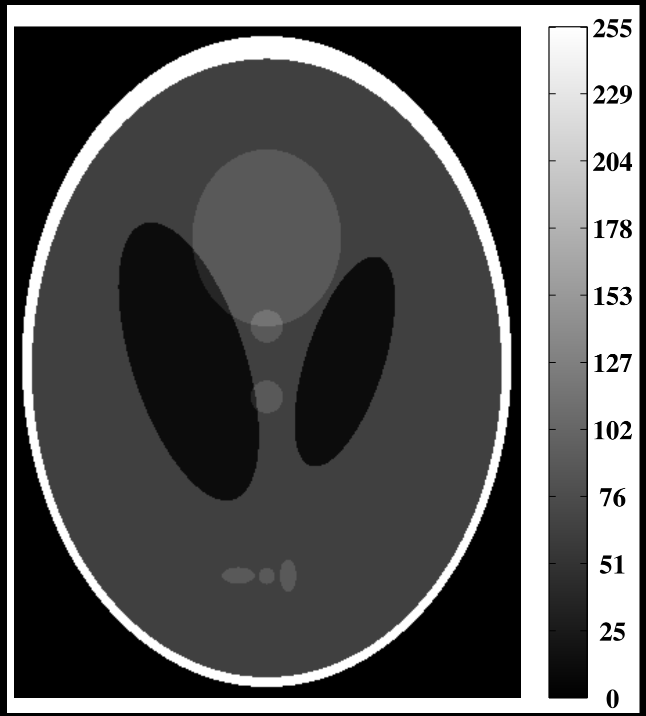
  - Radial trajectory: 16 spokes each with 512 samples $\approx 32\times$ acceleration
  - $L = 8$ coils with simulated sensitivity maps
  - SNR of data = 40 dB; SNR = $10\log_{10}(\|\mathbf{y}_{\text{true}}\|_2^2/N\sigma^2)$

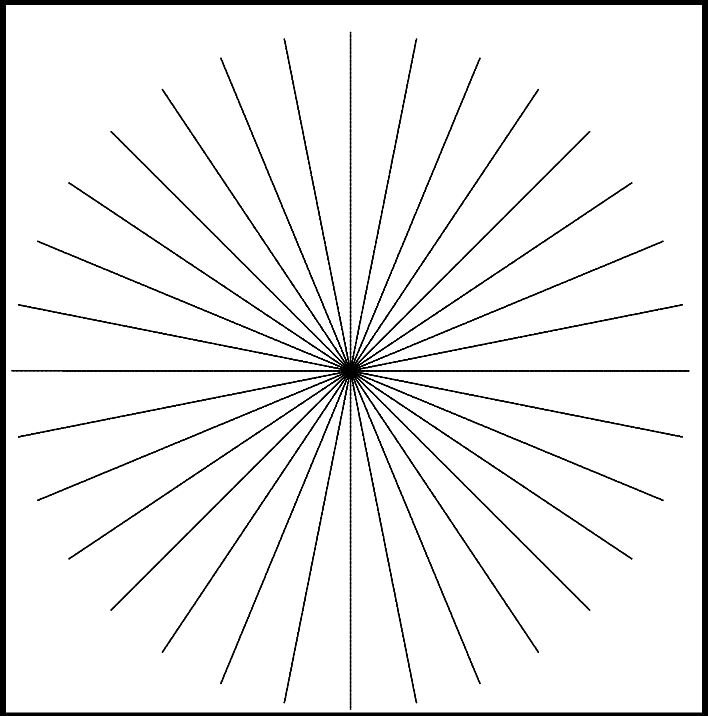
- Simulated 32×32 Cartesian low-resolution data for body and surface coils
  - Estimated smoothed sensitivity maps Allison *et al.* IEEE TMI 2013

- Initialization: sum-of-squares (SoS) of conjugate phase (CP) reconstruction
- Regularization parameter adjusted manually
- Reconstructed 512×512 images using TV regularizer
- Ran 1000 iterations of SB-10 to obtain a solution $\mathbf{x}^\star$
- Computed NRMSD w.r.t. $\mathbf{x}^\star$ as $\dfrac{\|\mathbf{x}_{(j)} - \mathbf{x}^\star\|_2}{\|\mathbf{x}^\star\|_2}$

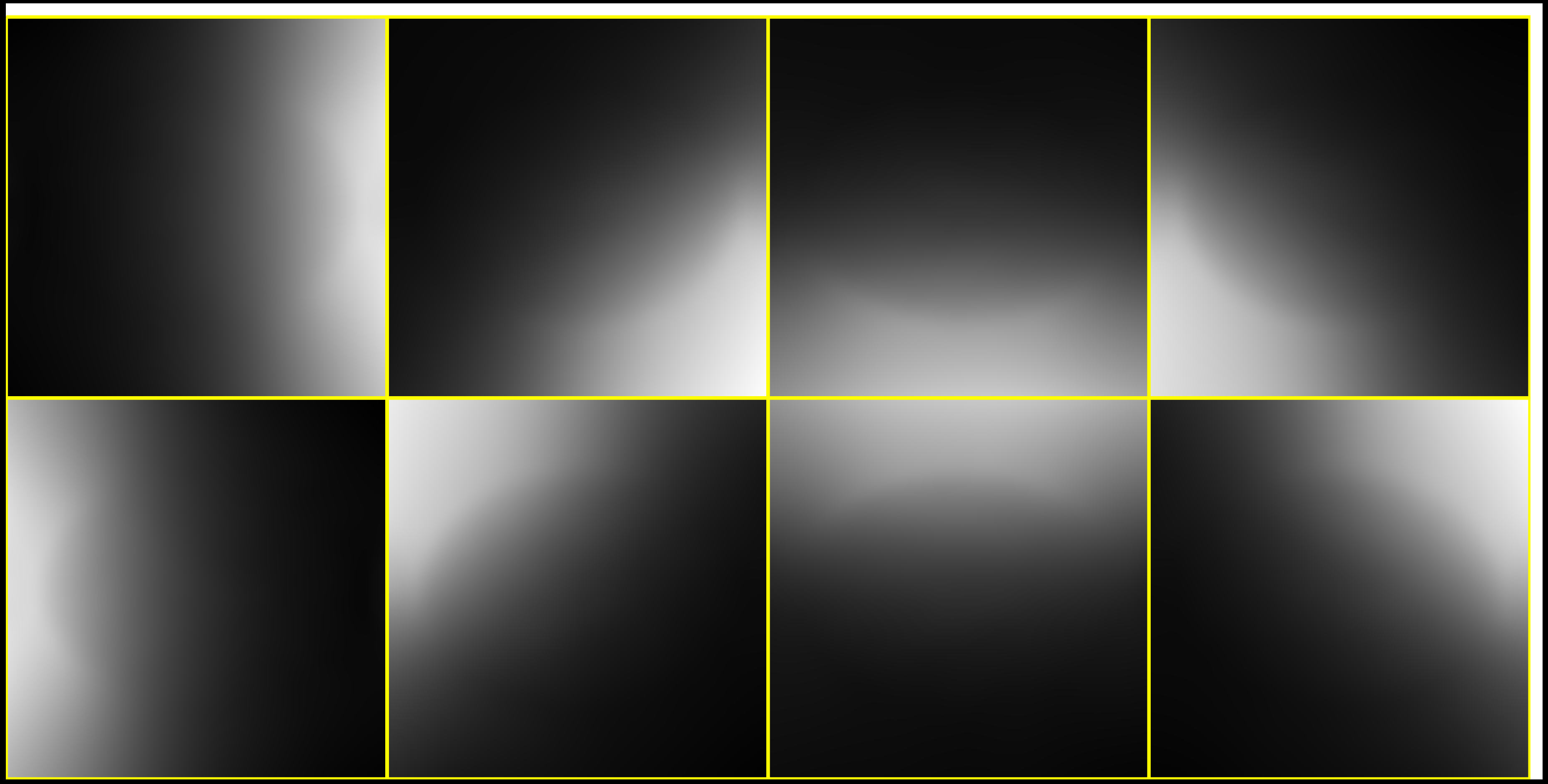
Simulation with Analytical Shepp-Logan Phantom

Noisefree discretized phantom





Radial trajectory; 16 spokes with 512 samples each

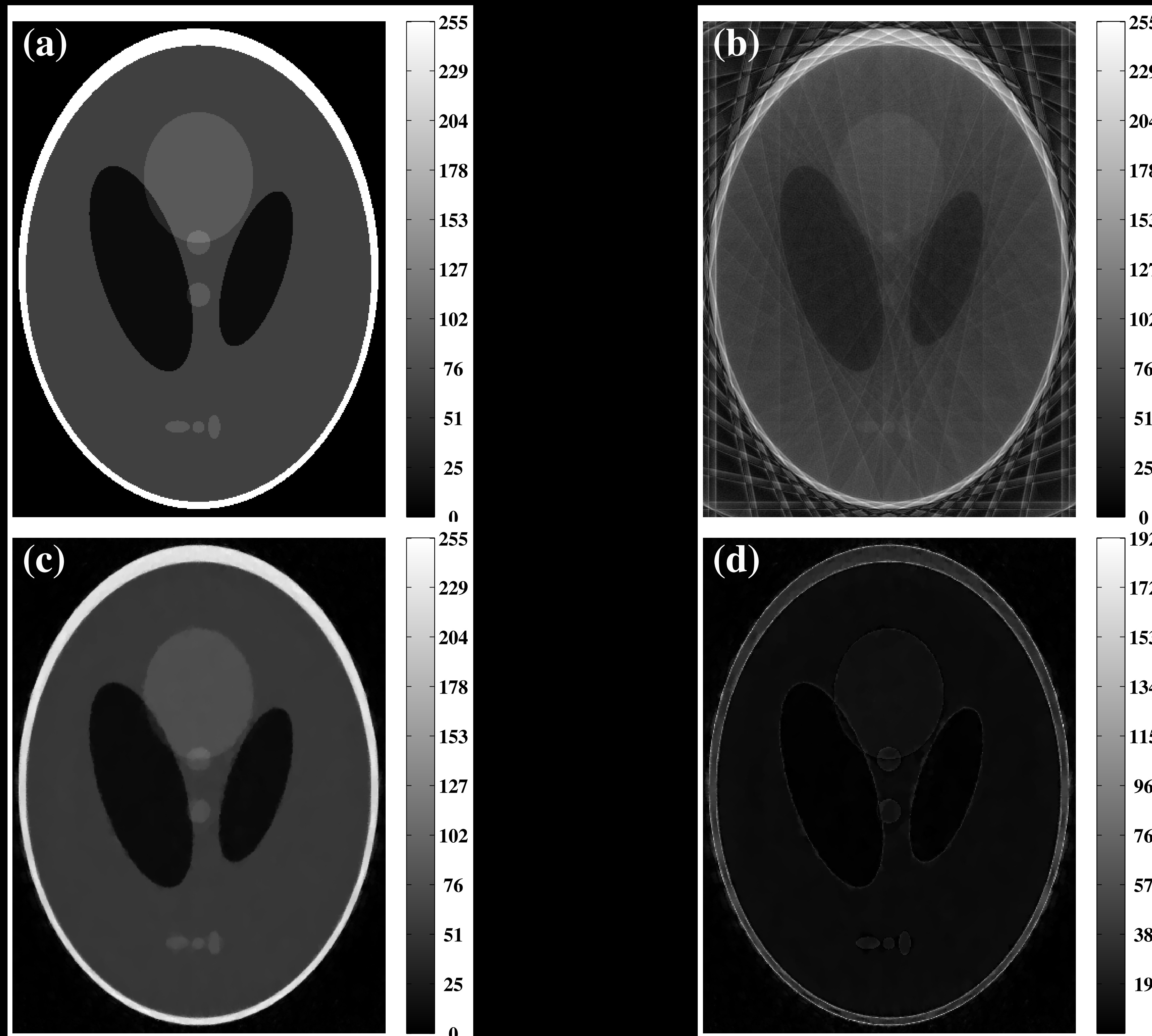


Estimated sensitivity maps

Simulation with Analytical Shepp-Logan Phantom

Noisefree phantom

SoS of CP Reconstruction

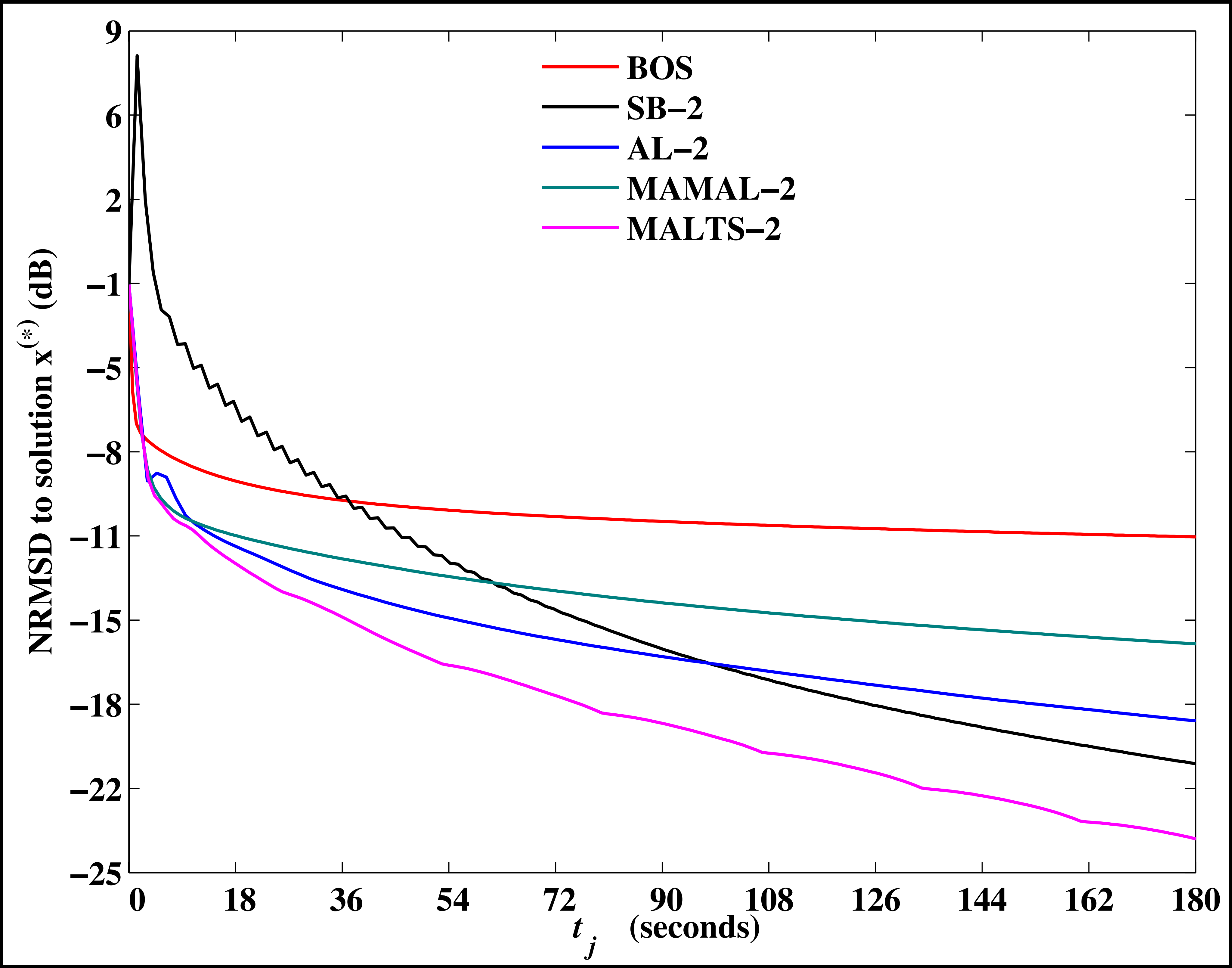


Regularized reconstruction $\mathbf{x}^\star$

Absolute difference

Simulation with Analytical Shepp-Logan Phantom

Plot of NRMSD as function of runtime

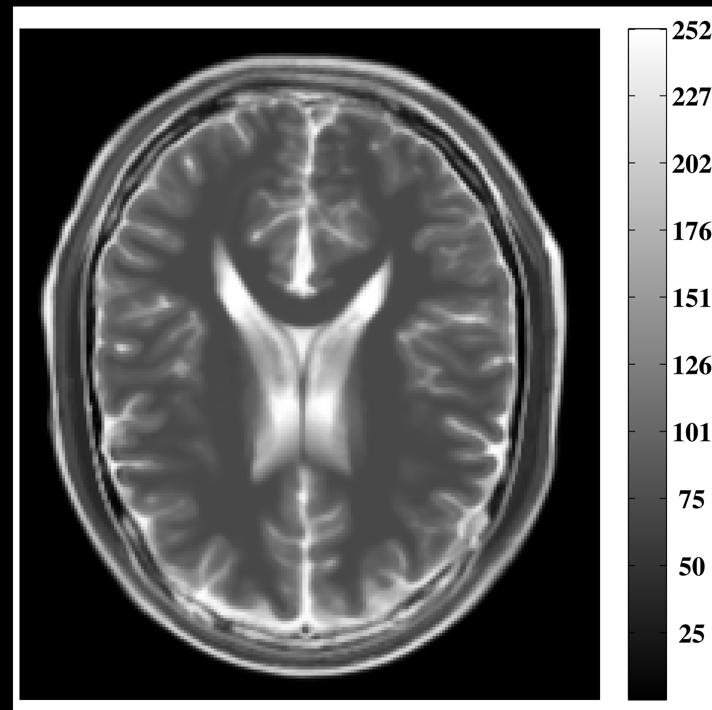


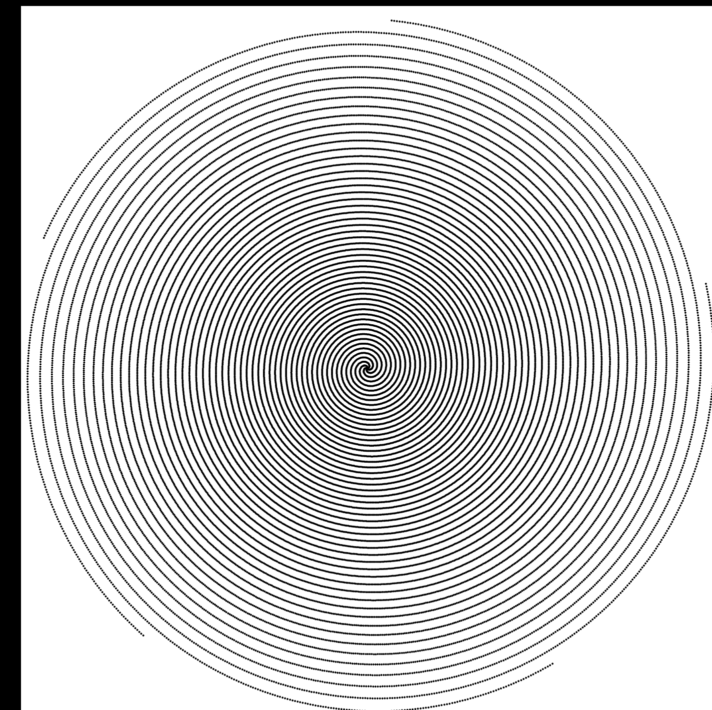
Simulation with a T_2-weighted Brainweb Image

- Simulated noisy data from a 2048×2048 *interpolated* T_2-weighted image
  - Variable density spiral; 5 interleaves; reduction factor $\approx$ 5
  - $L = 8$ coils with simulated sensitivity maps
  - SNR of data = 50 dB; SNR = $10 \log_{10}(\|\mathbf{y}_{\text{true}}\|_2^2 / N\sigma^2)$
- Simulated 32×32 Cartesian low-resolution data for body and surface coils
  - Estimated smoothed sensitivity maps Allison *et al.* IEEE TMI 2013
- Initialization: sum-of-squares (SoS) of conjugate phase (CP) reconstruction
- Regularization parameter adjusted manually
- Reconstructed 256×256 images using ℓ_1-regularizer
- Ran 1000 iterations of SB-10 to obtain a solution $\mathbf{x}^\star$
- Computed NRMSD w.r.t. $\mathbf{x}^\star$ as $\dfrac{\|\mathbf{x}_{(j)} - \mathbf{x}^\star\|_2}{\|\mathbf{x}^\star\|_2}$

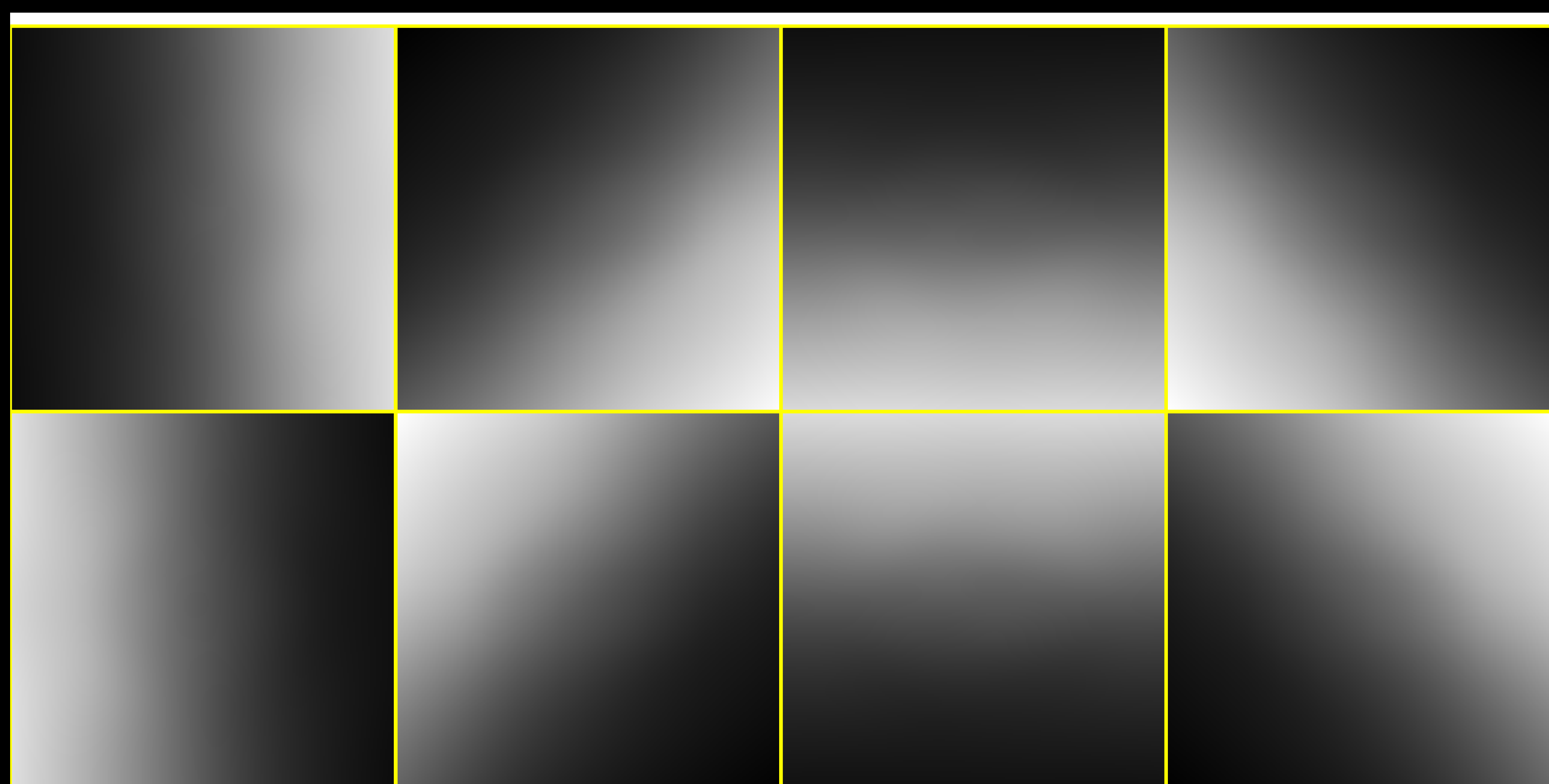
Simulation with a T_2-weighted Brainweb Image

Noisefree
T_2-weighted
test image





Variable density spiral
5 interleaves
reduction factor $\approx$ 5

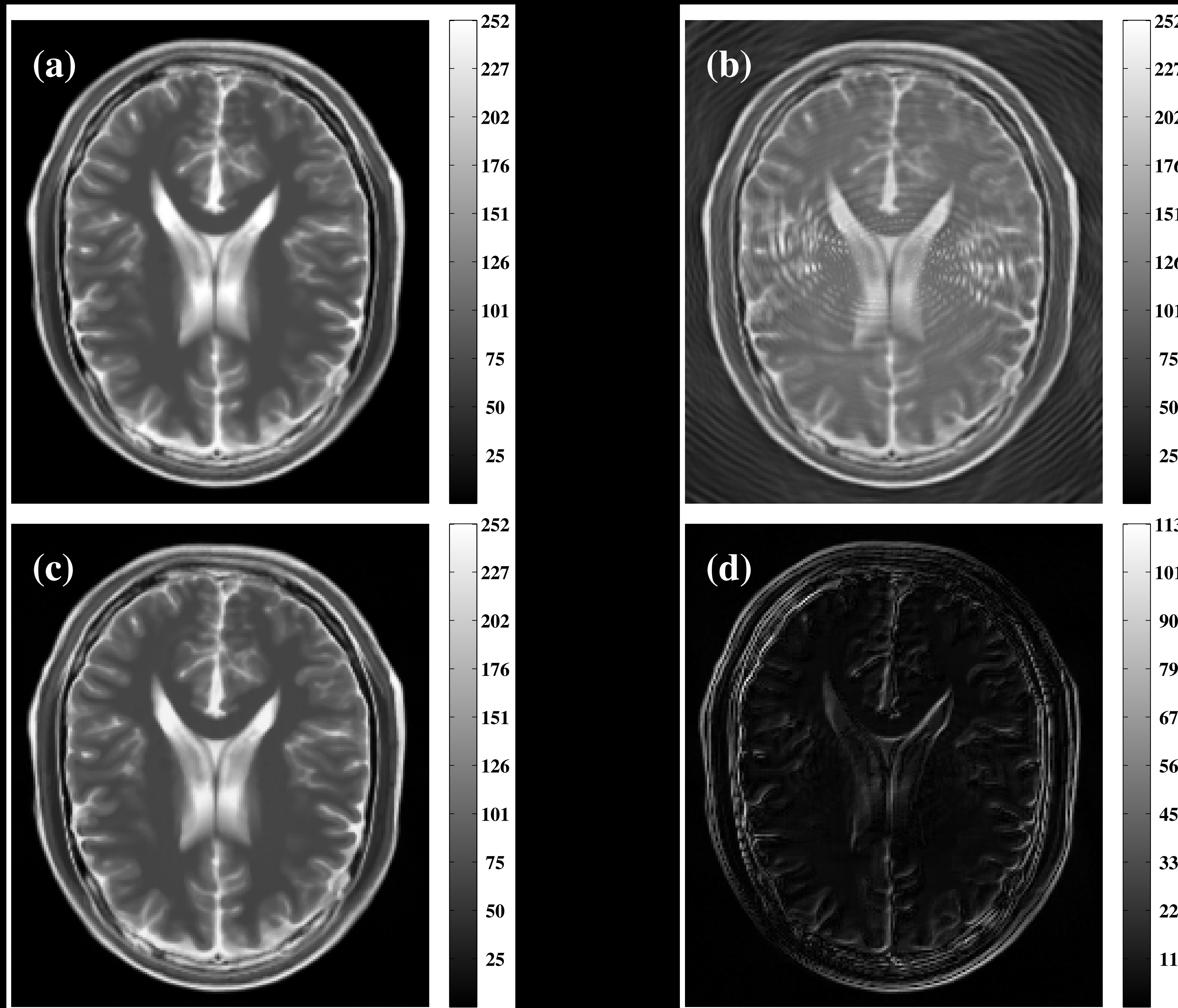


Estimated sensitivity maps

Simulation with a T_2-weighted Brainweb Image

Noisefree T_2-weighted image

SoS of CP Reconstruction

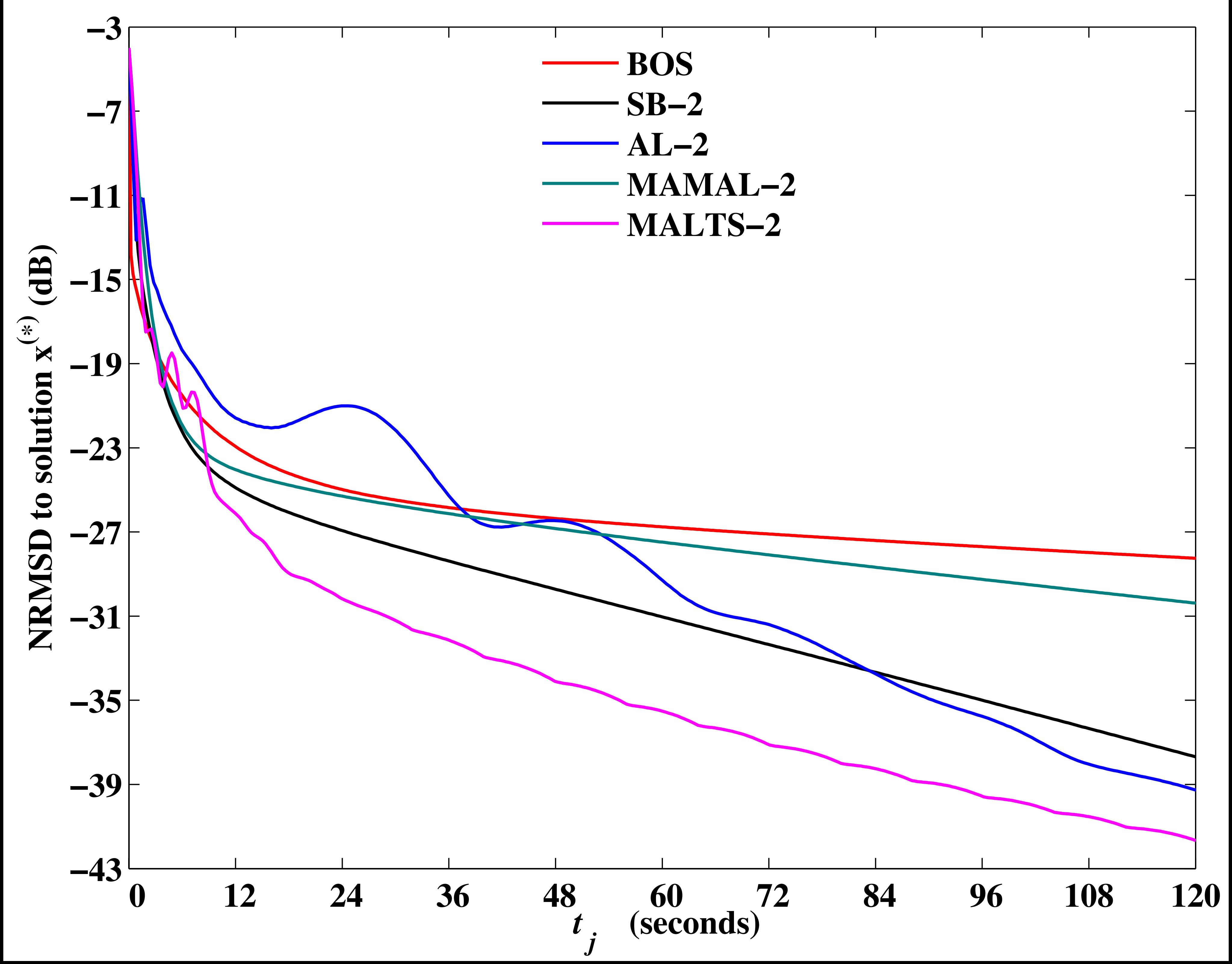


Regularized reconstruction $\mathbf{x}^{\star}$

Absolute difference

Simulation with a T_2-weighted Brainweb Image

Plot of NRMSD as function of runtime

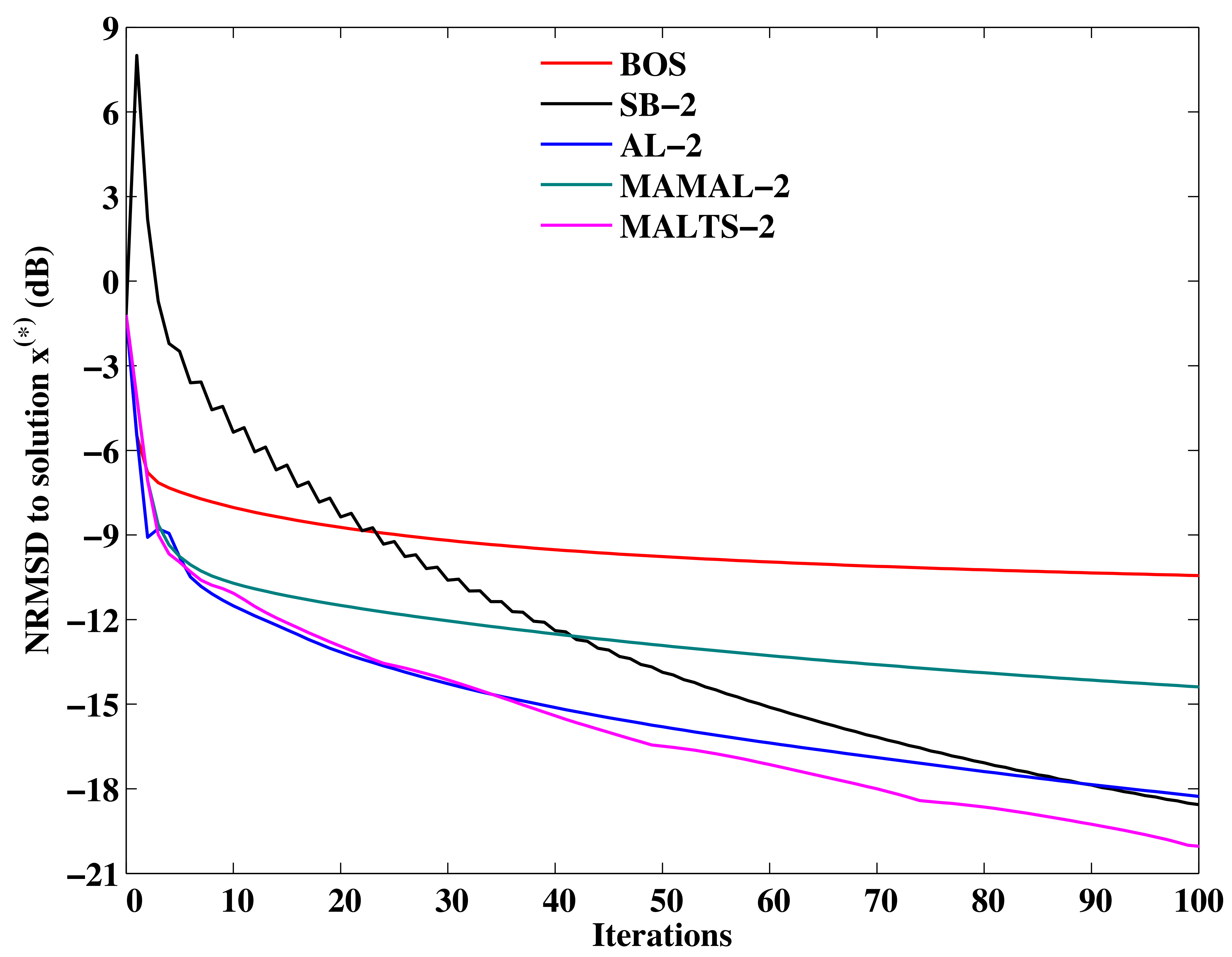


Summary & Conclusion

- Regularized SENSE reconstruction for non-Cartesian trajectories
  - Some existing variable-splitting methods can exploit inherent structures in the problem
  - These require repeated NUFFT-type computations for non-Cartesian MRI
- Our method combines majorize-minimize & variable-splitting concepts
  - Appropriate majorization can lead to an auxiliary cost function that is ”more amenable” to variable splitting
  - Proposed method: *noniterative update steps,*
    also amenable to two-step acceleration
- Preliminary results indicate faster convergence of the proposed method
- Useful for 3D non-Cartesian trajectories
- Future work: find *suitably tight* circulant majorizers

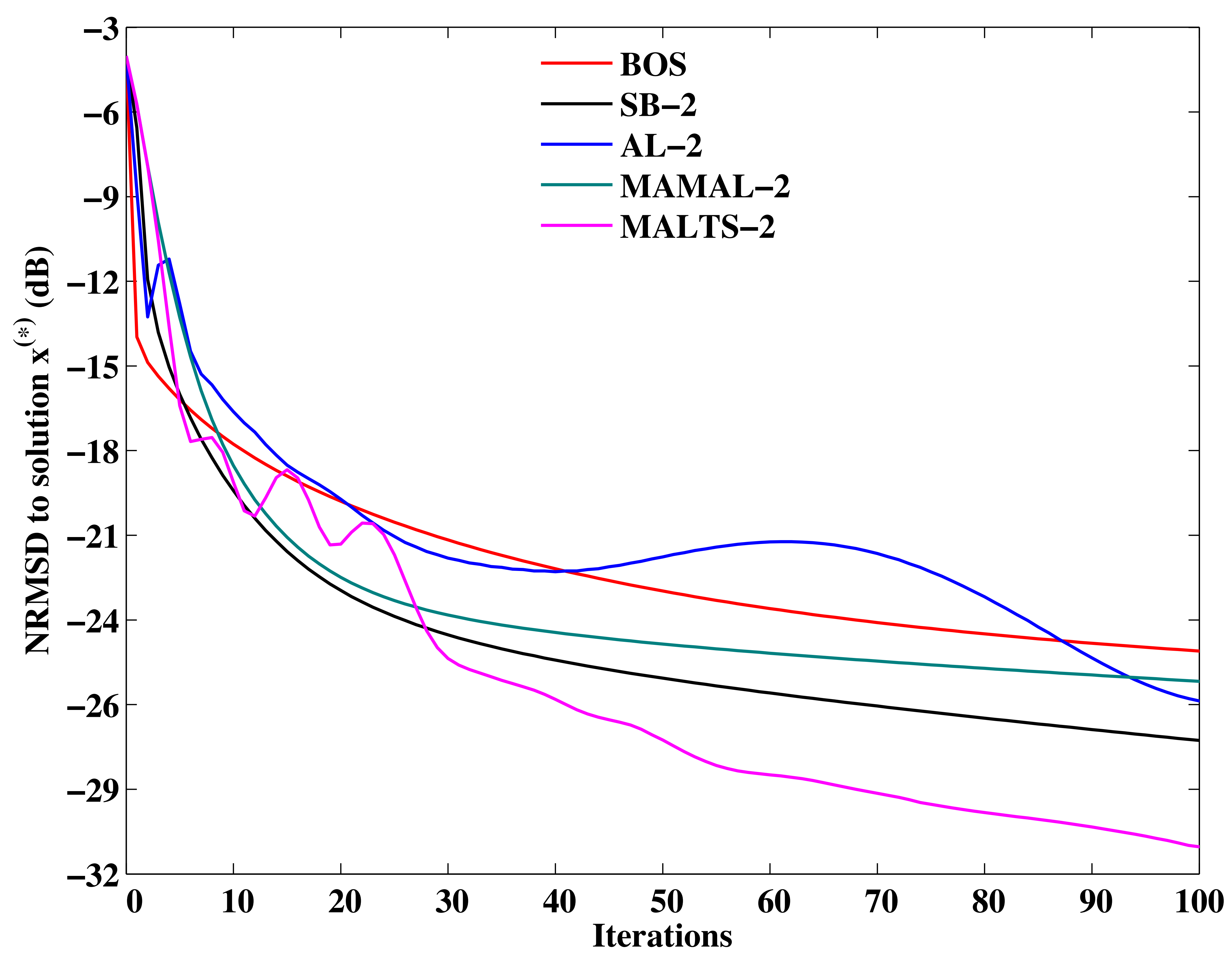
Simulation with Analytical Shepp-Logan Phantom

Plot of NRMSD as function of iterations



Simulation with a T_2-weighted Brainweb Image

Plot of NRMSD as function of iterations



Alternating Minimization of Augmented Lagrangian

- $J_{\text{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) = 2\mathcal{R}\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x})$

$$\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F}$$

- Augmented Lagrangian function:

$$\begin{aligned}\mathcal{L}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{x}) \triangleq\ & 2\mathcal{R}\{-\mathbf{u}_0'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{u}_0'\mathbf{C}'\mathbf{C}\mathbf{u}_0 + \Psi(\mathbf{u}_1) \\ & + \mu\|\mathbf{u}_0 - \mathbf{S}\mathbf{x} - \boldsymbol{\eta}_0\|_2^2 + \mu\nu_1\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_2 - \boldsymbol{\eta}_1\|_2^2 \\ & + \mu\nu_2\|\mathbf{u}_2 - \mathbf{x} - \boldsymbol{\eta}_2\|_2^2\end{aligned}$$

- Alternating minimization of $\mathcal{L}$ at jth iteration of majorize-minimize step

$$\mathbf{u}_{0\,(k+1)} = (\mathbf{C}'\mathbf{C} + \mu\mathbf{I})^{-1}[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)} + \mathbf{S}\mathbf{x}_{(j,k)} + \boldsymbol{\eta}_{0\,(k)}]$$

$$\mathbf{u}_{1\,(k+1)} = \arg\min_{\mathbf{u}_1} \Psi(\mathbf{u}_1) + \mu\nu_1\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_{2\,(k)} - \boldsymbol{\eta}_{1\,(k)}\|_2^2$$ (denoising problem)

$$\mathbf{u}_{2\,(k+1)} = (\mathbf{R}'\mathbf{R} + \nu_2/\nu_1\mathbf{I})^{-1}[\mathbf{R}(\mathbf{u}_{1\,(k+1)} - \boldsymbol{\eta}_{1\,(k)}) + \nu_2/\nu_1(\mathbf{x}_{(j,k)} + \boldsymbol{\eta}_{2\,(k)})]$$

$$\mathbf{x}_{(j,k+1)} = (\mathbf{S}'\mathbf{S} + \nu_2\mathbf{I})^{-1}[\mathbf{S}(\mathbf{u}_{0\,(k+1)} - \boldsymbol{\eta}_{0\,(k)}) + \nu_2(\mathbf{u}_{2\,(k+1)} - \boldsymbol{\eta}_{2\,(k)})]$$

Alternating Minimization of Augmented Lagrangian

- $J_{\text{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) = 2\mathcal{R}\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x})$

$$\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F}$$

- Alternating minimization of $\mathcal{L}$ at jth iteration of majorize-minimize step

$$\mathbf{u}_{0\,(k+1)} = (\mathbf{C}'\mathbf{C} + \mu\mathbf{I})^{-1}[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)} + \mathbf{S}\mathbf{x}_{(j,k)} + \boldsymbol{\eta}_{0\,(k)}]$$

$$\mathbf{u}_{1\,(k+1)} = \arg\min_{\mathbf{u}_1} \Psi(\mathbf{u}_1) + \mu\nu_1\|\mathbf{u}_1 - \mathbf{R}\mathbf{u}_{2\,(k)} - \boldsymbol{\eta}_{1\,(k)}\|_2^2 \quad \text{(denoising problem)}$$

$$\mathbf{u}_{2\,(k+1)} = (\mathbf{R}'\mathbf{R} + \nu_2/\nu_1\mathbf{I})^{-1}[\mathbf{R}(\mathbf{u}_{1\,(k+1)} - \boldsymbol{\eta}_{1\,(k)}) + \nu_2/\nu_1(\mathbf{x}_{(j,k)} + \boldsymbol{\eta}_{2\,(k)})]$$

$$\mathbf{x}_{(j,k+1)} = (\mathbf{S}'\mathbf{S} + \nu_2\mathbf{I})^{-1}[\mathbf{S}(\mathbf{u}_{0\,(k+1)} - \boldsymbol{\eta}_{0\,(k)}) + \nu_2(\mathbf{u}_{2\,(k+1)} - \boldsymbol{\eta}_{2\,(k)})]$$

- Every kth iteration of alternating minimization involves
  - Inversion of circulant matrices $(\mathbf{C}'\mathbf{C} + \mu\mathbf{I})$ & $(\mathbf{R}'\mathbf{R} + \nu_2/\nu_1\mathbf{I})$
  - Inversion of diagonal matrix $(\mathbf{S}'\mathbf{S} + \nu_2\mathbf{I})$
  - Denoising problem that admits closed-form solutions for many Ψ
  - Trivial Lagrange-multiplier $\boldsymbol{\eta}_{0,1,2}$-updates