Accelerated Non-Cartesian SENSE Reconstruction Using a Majorize-Minimize Algorithm Combining Variable-Splitting

Sathish Ramani and Jeffrey A. Fessler

Electrical Engineering and Computer Science Department
University of Michigan, Ann Arbor, MI, USA

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Overview

- Non-Cartesian k-space trajectories
  - Efficient k-space coverage, robustness to motion & off-resonance effects
  - Reconstruction is however more involved than for Cartesian trajectories
  - More computation: NUFFTs perform interpolation in addition to FFTs

- This work: An algorithm for non-Cartesian SENSE reconstruction
  - Combines majorize-minimize strategy with variable-splitting
  - Has reduced need of NUFFTs

- Organization of the talk:
  - Quick overview of regularized SENSE
  - Existing variable-splitting methods for regularized SENSE
  - Proposed majorize-minimize + variable-splitting scheme
  - Experimental results
Regularized SENSE Reconstruction

- Regularized SENSE reconstruction: model-based optimization problem

\[ \hat{x} = \arg \min_x \left\{ J(x) \triangleq \|y - FSx\|_2^2 + \Psi(Rx) \right\} \]

\[ y = \begin{bmatrix} y_1 \\ \vdots \\ y_L \end{bmatrix}; \quad y_l \in \mathbb{C}^M: \text{data from } l\text{th coil} \]

\[ S = \begin{bmatrix} S_1 \\ \vdots \\ S_L \end{bmatrix}; \quad S_l \in \mathbb{C}^{N \times N}: \text{diagonal sensitivity matrix for } l\text{th coil} \]

\[ F = I_L \otimes F_u; \quad F_u \in \mathbb{C}^{M \times N}: \text{non-Cartesian Fourier encoding matrix} \]

- \( \Psi(Rx) \) is a suitable regularizer: imposes prior information,
  reduces noise & artifacts,
  e.g., TV, \( \ell_1 \)-regularizers, etc.

\[ R \in \mathbb{R}^{P \times N}: \text{finite differences, wavelet frames, etc.} \]
Regularized SENSE Reconstruction

- Regularized SENSE reconstruction: model-based optimization problem

\[
\hat{x} = \arg\min_x \left\{ J(x) \triangleq \|y - FSx\|_2^2 + \Psi(Rx) \right\}
\]

- Challenges:
  - Gram matrix \( S'F'FS \) of data-fit term is highly shift-variant
  - \( F'F \) not circulant, unlike Cartesian case
  - Regularizer \( \Psi \) is non-quadratic and often non-differentiable
  - Problem size can be large

- Inherent mathematical structures in the problem
  - \( F'F \) is Toeplitz for non-Cartesian trajectories (no field inhomogeneity)
  - \( S'S \) is diagonal
  - \( R'R \) is circulant (e.g., finite differences with periodic boundary conditions)

- Exploit inherent structures: separate \( F, S \) and \( R \) via variable splitting
Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

\[
\hat{x} = \arg \min_x \left\{ J(x) \overset{\triangle}{=} \| y - FSx \|_2^2 + \Psi(Rx) \right\}
\]

- Split-Bregman (SB) type algorithms

  $u_1 = Rx$ only decouples $R$ from regularizer $\Psi$

Goldstein et al. SIAM J. Img. Sci., 2009
Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

\[
\hat{x} = \arg\min_x \left\{ J(x) \triangleq \|y - FSx\|_2^2 + \Psi(Rx) \right\}
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- Split-Bregman (SB) type algorithms

\[
u_1 = Rx \text{ only decouples } R \text{ from regularizer } \Psi
\]

- Augmented Lagrangian (AL) algorithm

\[
u_0 = Sx \text{ separates } F \text{ and } S \\
u_1 = Ru_2 \text{ separates } R \text{ from regularizer } \Psi \\
u_2 = x \text{ separates } R \text{ from } S
\]

- Equivalent constrained problem:

\[
\min J(z) \text{ s.t. } \mathcal{B}z = 0
\]

- SB: \(\mathcal{B} \triangleq [I \ -R]\) and \(z \triangleq \begin{bmatrix} u_1 \\ x \end{bmatrix}\)

- AL: \(\mathcal{B} \triangleq \begin{bmatrix} I & 0 & 0 & -S \\ 0 & I & -R & 0 \\ 0 & 0 & I & -I \end{bmatrix}\) and \(z \triangleq \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ x \end{bmatrix}\)
Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

\[ \hat{x} = \arg \min_x \left\{ J(x) \triangleq \|y - FSx\|^2_2 + \Psi(Rx) \right\} \]

- Split-Bregman (SB) type algorithms  
  Goldstein \textit{et al.} \textit{SIAM J. Img. Sci.}, 2009

\[ u_1 = Rx \text{ only decouples } R \text{ from regularizer } \Psi \]

- Augmented Lagrangian (AL) algorithm  
  Ramani \textit{et al.} \textit{IEEE TMI} 2011

\[ u_0 = Sx \text{ separates } F \text{ and } S \]
\[ u_1 = Ru_2 \text{ separates } R \text{ from regularizer } \Psi \]
\[ u_2 = x \text{ separates } R \text{ from } S \]

- Equivalent constrained problem:

\[ \min_z J(z) \text{ s.t. } \mathcal{B}z = 0 \]

- Augmented Lagrangian function:

\[ \mathcal{L}(z, \eta) = J(z) + \frac{\mu}{2} \| \mathcal{B}z - \eta \|_\Lambda^2 + c(\eta) \]

- \( \mu \) is a penalty parameter;  \( \Lambda \equiv \text{relative weighting of constraints} \)

- \( \eta \) is a Lagrange-multiplier-type vector for the constraint \( \mathcal{B}z = 0 \)
Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

\[ \hat{x} = \arg \min_x \left\{ J(x) \triangleq \| y - FSx \|_2^2 + \Psi(Rx) \right\} \]

- Split-Bregman (SB) type algorithms

  \[ u_1 = Rx \] only decouples \( R \) from regularizer \( \Psi \)

- Augmented Lagrangian (AL) algorithm

  \[ u_0 = Sx \] separates \( F \) and \( S \)
  \[ u_1 = Ru_2 \] separates \( R \) from regularizer \( \Psi \)
  \[ u_2 = x \] separates \( R \) from \( S \)

- Equivalent constrained problem:

  \[ \min_z J(z) \text{ s.t. } Bz = 0 \]

- Augmented Lagrangian function:

  \[ \mathcal{L}(z, \eta) = J(z) + \frac{\mu}{2} \| Bz - \eta \|_2^2 + c(\eta) \]

- Algorithm: Alternating minimization of \( \mathcal{L}(z, \eta^{(k)}) \) w.r.t. components of \( z \)

  Update \( \eta^{(k+1)} = \eta^{(k)} - Bz^{(k+1)} \)
Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

\[
\hat{x} = \arg\min_x \left\{ J(x) \triangleq \|y - FSx\|^2_2 + \Psi(Rx) \right\}
\]

- Split-Bregman (SB) type algorithms  
  - \( u_1 \)-update corresponds to a denoising problem
  - \( x \)-update requires “inverting” the shift-variant matrix \( (S'F'FS + R'R) \)

- Augmented Lagrangian (AL) algorithm  
  - \( u_0 \)-update requires “inverting” the Toeplitz matrix \( (F'F + \mu I) \)
  - \( u_1 \)-update corresponds to a denoising problem
  - \( u_2 \)-update requires “inverting” the circulant matrix \( (R'R + \beta I) \)
  - \( x \)-update requires “inverting” the diagonal matrix \( (S'S + \gamma I) \)
Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

\[ \hat{x} = \arg \min_x \left\{ J(x) \triangleq \| y - FSx \|_2^2 + \Psi(Rx) \right\} \]

- Split-Bregman (SB) type algorithms

  - x-update requires “inverting” the shift-variant matrix \( (S'F'FS + R'R) \)

- Augmented Lagrangian (AL) algorithm

  - \( u_0 \)-update requires “inverting” the Toeplitz matrix \( (F'F + \mu I) \)

Goldstein et al. SIAM J. Img. Sci., 2009

Ramani et al. IEEE TMI 2011
Variable Splitting & Augmented Lagrangian

- Regularized SENSE reconstruction: model-based optimization problem

\[ \hat{x} = \arg \min_x \left\{ J(x) \triangleq \|y - FSx\|^2_2 + \Psi(Rx) \right\} \]

- Split-Bregman (SB) type algorithms \cite{Goldstein2009}
  - \(x\)-update requires “inverting” the shift-variant matrix \((S'F'FS + R'R)\)

- Augmented Lagrangian (AL) algorithm \cite{Ramani2011}
  - \(u_0\)-update requires “inverting” the Toeplitz matrix \((F'F + \mu I)\)

- Non-Cartesian MRI: Need iterative solvers for \(x\) in SB and \(u_0\) in ALA

  - \(F'F = I_L \otimes F'_u F_u \implies L\) solvers needed for \(L\) coils
  - Repeated products with \(F'_u F_u \equiv\) more computation
    - e.g., embedding \(F'_u F_u\) in a larger circulant matrix \(\equiv\) FFTs of larger size

- Proposed approach: Majorize-minimize strategy
  - Replaces \(F'F\) with a circulant matrix
Majorize-Minimize Approach

- Regularized SENSE reconstruction: model-based optimization problem

\[
\hat{x} = \arg\min_x \left\{ J(x) \triangleq \|y - FSx\|_2^2 + \Psi(Rx) \right\}
\]

- Majorization: \( D(y, Sx) \triangleq \|y - FSx\|_2^2 \)

\[
\leq D(y, Sx(j)) + 2\Re\{(x - x(j))' \nabla D(y, Sx)\}
+ (x - x(j))'S'C'C_S(x - x(j)) \\
= 2\Re\{-x'S'[F'y + MSx(j)]\} + x'S'C'C_Sx + c \\
\triangleq D_{maj}(y, Sx, x(j))
\]

- Circulant matrix \( C = I_L \otimes C_u \) such that \( C_u'C_u \succeq F_u'F_u \)

\[
M \triangleq C'C - F'F = I_L \otimes [C_u'C_u - F_u'F_u] \succeq 0
\]
Majorize-Minimize Approach

- Regularized SENSE reconstruction: model-based optimization problem

\[
\hat{x} = \arg \min_x \left\{ J(x) \triangleq \|y - FSx\|_2^2 + \Psi(Rx) \right\}
\]

\[
D_{maj}(y, Sx, x_{(j)}) = 2R\{-x'S'[F'y + MSx_{(j)}]\} + x'S'C'CSx + c
\]

\[
M \triangleq C'C - F'F
\]

- Majorizer: \[
J_{maj}(x, x_{(j)}) \triangleq D_{maj}(y, Sx, x_{(j)}) + \Psi(Rx)
\]

\[
J_{maj}(x, x_{(j)}) \leq J(x) \quad \forall \quad x \quad \text{with equality at} \quad x_{(j)}
\]

- Majorize-minimize scheme

\[
x_{(j+1)} = \arg \min_x J_{maj}(x, x_{(j)})
\]
Variable Splitting for Minimization Step

- \( J_{\text{maj}}(x, x_{(j)}) = 2\Re\{-x'S'[F'y + MSx_{(j)}]\} + x'S'C'CSx + \Psi(Rx) \)

\[ M \triangleq C'C - F'F \]

- Minimization of \( J_{\text{maj}} \) using variable-splitting and augmented Lagrangian

\[ u_0 = Sx \text{ separates } C \text{ and } S \]
\[ u_1 = Ru_2 \text{ separates } R \text{ from regularizer } \Psi \]
\[ u_2 = x \text{ separates } R \text{ from } S \]

- Equivalent constrained problem:

\[ \min_z J_{\text{maj}}(z) \text{ s.t. } Az = 0 \]

\[ A \triangleq \begin{bmatrix} I & 0 & 0 & -S \\ 0 & I & -R & 0 \\ 0 & 0 & I & -I \end{bmatrix} \quad \text{and} \quad z \triangleq \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ x \end{bmatrix} \]
Augmented Lagrangian for Minimization Step

- $J_{maj}(x, x(j)) = 2\Re\{-x'S'[F'y + MSx(j)]\} + x'S'C'CSx + \Psi(Rx)$

  \[ M \triangleq C'C - F'F \]

- Minimization of $J_{maj}$ using variable-splitting and augmented Lagrangian
  
  \[ u_0 = Sx \text{ separates } C \text{ and } S \]
  \[ u_1 = Ru_2 \text{ separates } R \text{ from regularizer } \Psi \]
  \[ u_2 = x \text{ separates } R \text{ from } S \]

- Augmented Lagrangian function:

  \[ \mathcal{L}(u_0, u_1, u_2, x) \triangleq 2\Re\{-u'_0[F'y + MSx(j)]\} + u'_0C'Cu_0 + \Psi(u_1) \]
  
  \[ + \mu \| u_0 - Sx - \eta_0 \|_2^2 + \mu \nu_1 \| u_1 - Ru_2 - \eta_1 \|_2^2 \]
  
  \[ + \mu \nu_2 \| u_2 - x - \eta_2 \|_2^2 \]

  \[ \mu, \nu_1, \nu_2 > 0 \text{ are penalty parameters} \]
Algorithm Summary

- $J_{\text{maj}}(\mathbf{x}, \mathbf{x}_{(j)}) = 2\Re\{-\mathbf{x}'\mathbf{S}'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}_{(j)}]\} + \mathbf{x}'\mathbf{S}'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x})$

- Every $j$th majorize-minimize iteration involves
  - One product with $\mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F} \equiv$ one product with $\mathbf{F}'\mathbf{F}$
  - $k = 1 \cdots K$ iterations of alternating minimization of $\mathcal{L}$ and $\eta_{0,1,2}$-updates
  - Warm-starting of constraint variables $\mathbf{u}_{0,1,2}$ and $\eta_{0,1,2}$

- Every $k$th iteration of alternating minimization involves
  - Inversion of circulant matrices $(\mathbf{C}'\mathbf{C} + \mu\mathbf{I})$ & $(\mathbf{R}'\mathbf{R} + \nu_2/\nu_1\mathbf{I})$
  - Inversion of diagonal matrix $(\mathbf{S}'\mathbf{S} + \nu_2\mathbf{I})$
  - Denoising problem that admits closed-form solutions for many $\Psi$
  - Trivial Lagrange-multiplier $\eta_{0,1,2}$-updates
Algorithm Summary

- \( J_{\text{maj}}(\mathbf{x}, \mathbf{x}(j)) = 2\Re\{-\mathbf{x}'S'[\mathbf{F}'\mathbf{y} + \mathbf{M}\mathbf{S}\mathbf{x}(j)]\} + \mathbf{x}'S'\mathbf{C}'\mathbf{C}\mathbf{S}\mathbf{x} + \Psi(\mathbf{R}\mathbf{x}) \)

- Every \( j \)th majorize-minimize iteration involves
  - One product with \( \mathbf{M} \triangleq \mathbf{C}'\mathbf{C} - \mathbf{F}'\mathbf{F} \equiv \) one product with \( \mathbf{F}'\mathbf{F} \)
  - \( k = 1 \cdots K \) iterations of alternating minimization of \( \mathcal{L} \) and \( \eta_{0,1,2} \)-updates
  - *Warm-starting* of constraint variables \( \mathbf{u}_{0,1,2} \) and \( \eta_{0,1,2} \)

- Every \( k \)th iteration of alternating minimization involves
  - Inversion of circulant matrices \( (\mathbf{C}'\mathbf{C} + \mu\mathbf{I}) \) & \( (\mathbf{R}'\mathbf{R} + \nu_2/\nu_1\mathbf{I}) \)
  - Inversion of diagonal matrix \( (\mathbf{S}'\mathbf{S} + \nu_2\mathbf{I}) \)
  - Denoising problem that admits closed-form solutions for many \( \Psi \)
  - Trivial Lagrange-multiplier \( \eta_{0,1,2} \)-updates

- **Acceleration using two-step strategy**  
  Beck *et al.* SIAM J. Img. Sci., 2009

\[
\begin{align*}
\mathbf{x}(j+1) &= \arg\min_{\mathbf{x}} J_{\text{maj}}(\mathbf{x}, \mathbf{w}(j)); \\
\mathbf{w}(j) &= \mathbf{x}(j) + (a_j - 1)/a_{j+1}(\mathbf{x}(j) - \mathbf{x}(j-1)); \\
a_{j+1} &= \frac{[1 + \sqrt{1 + 4a_j^2}]/2}{2}
\end{align*}
\]
Construction of the Majorizer

- Obtain circulant $\tilde{C}$ such that $\tilde{C}'\tilde{C} \approx F'F$ in Frobenius-norm

- Find $\alpha > 0$ such that $\alpha\tilde{C}'\tilde{C} \succeq F'F$ using Power method
  - Requires matrix-vector products with $F'F$
  - Depends only on the trajectory: can be precomputed for various trajectories

- Desired circulant matrix in the majorizer: $C \overset{\Delta}{=} \sqrt{\alpha}\tilde{C}$
- Illustration for a radial trajectory with 16 spokes containing 512 samples each

Frequency response associated with $C'_u C_u$ (log$_{10}$-scale)

Magnitude of FFT of response of $F'_u F_u$ to a unit impulse at the image center (log$_{10}$-scale)
Algorithms Compared in This Work

- Compared proposed methods with recent splitting-based algorithms
  - Split-Bregman (SB-\(n\))  
  - Bregman Operator Splitting (BOS)  
    Zhang et al. SIAM J. Img. Sci. 2010
  - Augmented Lagrangian (AL-\(n\))  
    Ramani et al. IEEE TMI 2011
  - Proposed: MAjorize-Minimize AL (MAMAL-\(K\))
  - Proposed: Majorize-minimize AL with Two-Step (MALTS-\(K\))

  \(n\) inner PCG iterations for SB and AL
  \(K\) inner AL iterations for MAMAL & MALTS

- Circulant preconditioner using \(\tilde{C}'\tilde{C}\) for inner-linear-systems in SB & AL

- Denoising-like step involves shrinkage: common to all algorithms
  - Automatically set penalty parameters of all algorithms to obtain same shrinkage-threshold
Simulation with Analytical Shepp-Logan Phantom

- Simulated noisy data using analytical Shepp-Logan phantom

  Guerquin-Kern et al. IEEE TMI 2012

  - Radial trajectory: 16 spokes each with 512 samples \( \approx 32 \times \) acceleration
  - \( L = 8 \) coils with simulated sensitivity maps
  - SNR of data = 40 dB; 
    \[
    \text{SNR} = 10 \log_{10}(\frac{\|y_{\text{true}}\|^2}{N \sigma^2})
    \]

- Simulated \( 32 \times 32 \) Cartesian low-resolution data for body and surface coils

  - Estimated smoothed sensitivity maps
    Allison et al. IEEE TMI 2013

- Initialization: sum-of-squares (SoS) of conjugate phase (CP) reconstruction

- Regularization parameter adjusted manually

- Reconstructed \( 512 \times 512 \) images using TV regularizer

- Ran 1000 iterations of SB-10 to obtain a solution \( x^* \)

- Computed NRMSD w.r.t. \( x^* \) as 
  \[
  \frac{\|x^{(j)} - x^*\|_2}{\|x^*\|_2}
  \]
Simulation with Analytical Shepp-Logan Phantom

Noisefree discretized phantom

Radial trajectory; 16 spokes with 512 samples each

Estimated sensitivity maps
Simulation with Analytical Shepp-Logan Phantom

(a) Noisefree phantom

(b) SoS of CP Reconstruction

(c) Regularized reconstruction $x^*$

(d) Absolute difference
Simulation with Analytical Shepp-Logan Phantom

Plot of NRMSD as function of runtime

NRMSD to solution $x^\ast$ (dB) vs. runtime $t_j$ (seconds)

- BOS
- SB–2
- AL–2
- MAMAL–2
- MALTS–2
Simulation with a $T_2$-weighted Brainweb Image

- Simulated noisy data from a $2048 \times 2048$ *interpolated* $T_2$-weighted image
  - Variable density spiral; 5 interleaves; reduction factor $\approx 5$
  - $L = 8$ coils with simulated sensitivity maps
  - SNR of data = 50 dB; $\text{SNR} = 10 \log_{10}(\|y_{\text{true}}\|_2^2/N\sigma^2)$
- Simulated $32 \times 32$ Cartesian low-resolution data for body and surface coils
  - Estimated smoothed sensitivity maps  
  Allison *et al.* IEEE TMI 2013
- Initialization: sum-of-squares (SoS) of conjugate phase (CP) reconstruction
- Regularization parameter adjusted manually
- Reconstructed $256 \times 256$ images using $\ell_1$-regularizer
- Ran 1000 iterations of *SB-10* to obtain a solution $x^*$
- Computed NRMSD w.r.t. $x^*$ as $\frac{\|x_{(j)} - x^*\|_2}{\|x^*\|_2}$
Simulation with a $T_2$-weighted Brainweb Image

Noisefree $T_2$-weighted test image

Variable density spiral 5 interleaves reduction factor $\approx 5$

Estimated sensitivity maps
Simulation with a $T_2$-weighted Brainweb Image

Noisefree $T_2$-weighted image

Regularized reconstruction $x^*$

SoS of CP Reconstruction

Absolute difference
Simulation with a $T_2$-weighted Brainweb Image

Plot of NRMSD as function of runtime
Summary & Conclusion

- Regularized SENSE reconstruction for non-Cartesian trajectories
  - Some existing variable-splitting methods can exploit inherent structures in the problem
  - These require repeated NUFFT-type computations for non-Cartesian MRI

- Our method combines majorize-minimize & variable-splitting concepts
  - Appropriate majorization can lead to an auxiliary cost function that is ”more amenable” to variable splitting
  - Proposed method: noniterative update steps, also amenable to two-step acceleration

- Preliminary results indicate faster convergence of the proposed method

- Useful for 3D non-Cartesian trajectories

- Future work: find suitably tight circulant majorizers
Simulation with Analytical Shepp-Logan Phantom

Plot of NRMSD as a function of iterations.

NRMSD to solution $x^*$ (dB) vs. Iterations.
Simulation with a $T_2$-weighted Brainweb Image

Plot of NRMSD as function of iterations

NRMSD to solution $x^*$ (dB)


Iterations

0  10  20  30  40  50  60  70  80  90  100

-3  -6  -9  -12  -15  -18  -21  -24  -27  -30

BOS
SB–2
AL–2
MAMAL–2
MALTS–2

Plot of NRMSD as function of iterations
Alternating Minimization of Augmented Lagrangian

- \( J_{\text{maj}}(x, x(j)) = 2\Re\{-x'S'[F'y + MSx(j)]\} + x'S'C'Cx + \Psi(Rx) \)

\[ M \triangleq C'C - F'F \]

- Augmented Lagrangian function:

\[ \mathcal{L}(u_0, u_1, u_2, x) \triangleq 2\Re\{-u_0'[F'y + MSx(j)]\} + u_0'C'Cu_0 + \Psi(u_1) + \mu\|u_0 - Sx - \eta_0\|_2^2 + \mu\nu_1\|u_1 - Ru_2 - \eta_1\|_2^2 + \mu\nu_2\|u_2 - x - \eta_2\|_2^2 \]

- Alternating minimization of \( \mathcal{L} \) at \( j \)th iteration of majorize-minimize step

\[ u_{0(k+1)} = (C'C + \mu I)^{-1}[F'y + MSx(j) + Sx(j,k) + \eta_0(k)] \]

\[ u_{1(k+1)} = \arg\min_{u_1} \Psi(u_1) + \mu\nu_1\|u_1 - Ru_{2(k)} - \eta_{1(k)}\|_2^2 \quad \text{(denoising problem)} \]

\[ u_{2(k+1)} = (R'R + \nu_2/\nu_1 I)^{-1}[R(u_{1(k+1)} - \eta_{1(k)}) + \nu_2/\nu_1(x(j,k) + \eta_{2(k)})] \]

\[ x_{(j,k+1)} = (S'S + \nu_2 I)^{-1}[S(u_{0(k+1)} - \eta_{0(k)}) + \nu_2(u_{2(k+1)} - \eta_{2(k)})] \]
Alternating Minimization of Augmented Lagrangian

\[ J_{\text{maj}}(x, x_{(j)}) = 2\Re\{-x'S'[F'y + MSx_{(j)}]\} + x'S'C'CSx + \Psi(Rx) \]

\[ M \triangleq C'C - F'F \]

Alternating minimization of \( \mathcal{L} \) at \( j \)th iteration of majorize-minimize step

\[ u_{0(k+1)} = (C'C + \mu I)^{-1}[F'y + MSx_{(j)} + Sx_{(j,k)} + \eta_{0(k)}] \]

\[ u_{1(k+1)} = \arg\min_{u_1} \Psi(u_1) + \mu \nu_1 \|u_1 - Ru_{2(k)} - \eta_{1(k)}\|^2 \quad \text{(denoising problem)} \]

\[ u_{2(k+1)} = (R'R + \nu_2/\nu_1 I)^{-1}[R(u_{1(k+1)} - \eta_{1(k)}) + \nu_2/\nu_1(x_{(j,k)} + \eta_{2(k)})] \]

\[ x_{(j,k+1)} = (S'S + \nu_2 I)^{-1}[S(u_{0(k+1)} - \eta_{0(k)}) + \nu_2(u_{2(k+1)} - \eta_{2(k)})] \]

Every \( k \)th iteration of alternating minimization involves

- Inversion of circulant matrices \((C'C + \mu I)\) & \((R'R + \nu_2/\nu_1 I)\)
- Inversion of diagonal matrix \((S'S + \nu_2 I)\)
- Denoising problem that admits closed-form solutions for many \( \Psi \)
- Trivial Lagrange-multiplier \( \eta_{0,1,2} \)-updates