Sparsity in MRI parallel excitation

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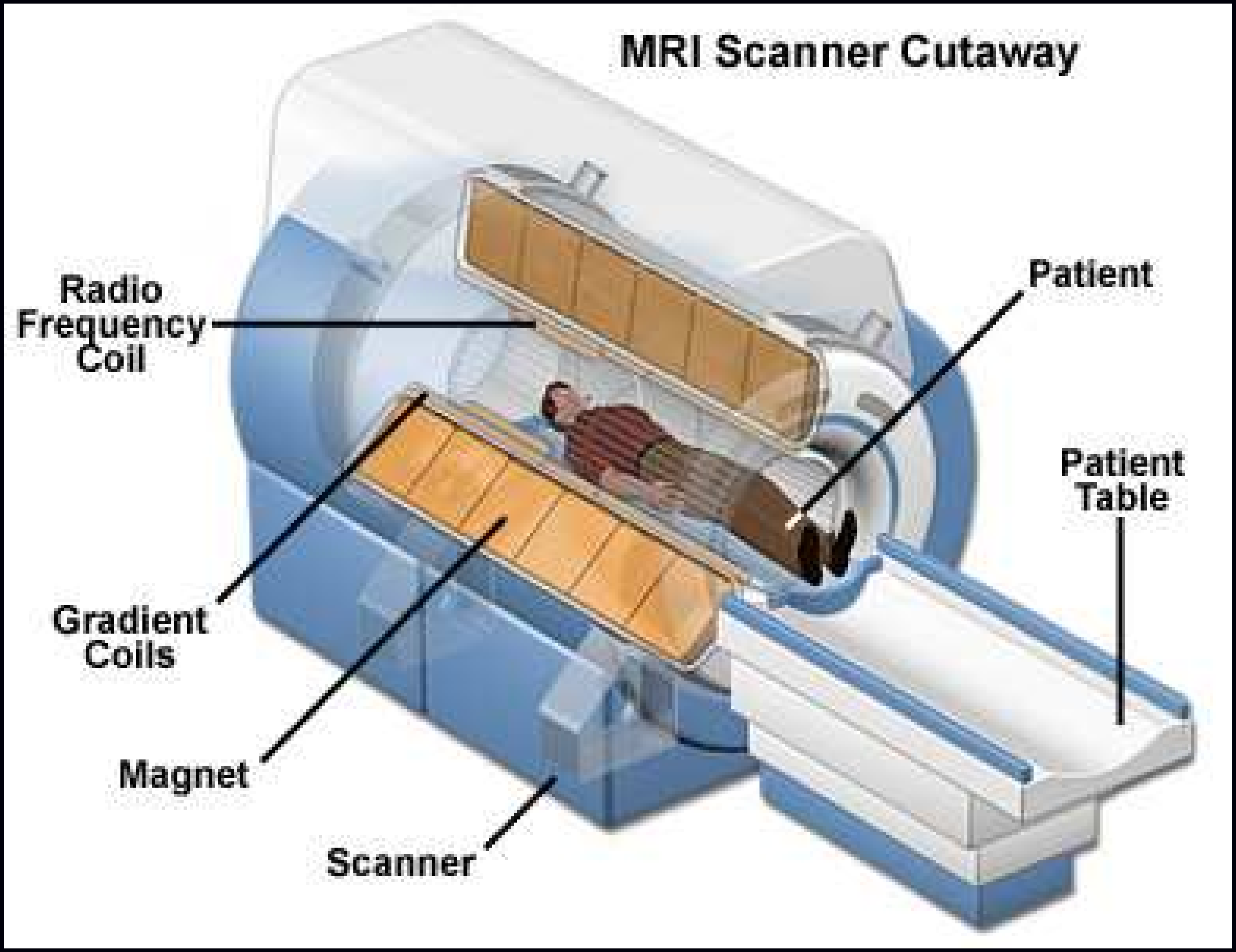
Outline

- Introduction to excitation in MRI
- Problems requiring sparsity
- Sparsity formulations
- Applications
- Summary

Image reconstruction toolbox:
`http://www.eecs.umich.edu/~fessler`

MRI Scans



www.magnet.fsu.edu

MRI scans alternate between excitation and readout (data acquisition)

RF Excitation: Overview

Forward model:

Applied field

$$\boldsymbol{B}(\boldsymbol{r},t) = \underbrace{\boldsymbol{B}_0}_{\text{main}} + \underbrace{\boldsymbol{B}_1(t)}_{\text{RF}} + \underbrace{\boldsymbol{r}\cdot\boldsymbol{G}(t)\,\boldsymbol{k}}_{\text{gradients}} \rightarrow \boxed{\text{Patient (Bloch equation)}} \rightarrow \text{magnetization pattern } \boldsymbol{M}(\boldsymbol{r},t)$$

Bloch equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{M}(\boldsymbol{r},t) = \boldsymbol{M}(\boldsymbol{r},t) \times \gamma \boldsymbol{B}(\boldsymbol{r},t) - \boldsymbol{T}\left[\boldsymbol{M}(\boldsymbol{r},t) - \boldsymbol{M}(\boldsymbol{r},0)\right]$$

where one often ignores the relaxation factors $\boldsymbol{T} = \begin{bmatrix} 1/T_2(\boldsymbol{r}) & 0 & 0 \\ 0 & 1/T_2(\boldsymbol{r}) & 0 \\ 0 & 0 & 1/T_1(\boldsymbol{r}) \end{bmatrix}$.

Excitation design goal:
find gradient waveforms $\boldsymbol{G}(t)$ and RF waveform $b_1(t),\ 0 \le t \le T$ that induce some desired magnetization pattern $\boldsymbol{M}_{\mathrm{d}}(\boldsymbol{r},T)$ at pulse end.
This is a “noiseless” inverse problem.

RF Excitation: Applications

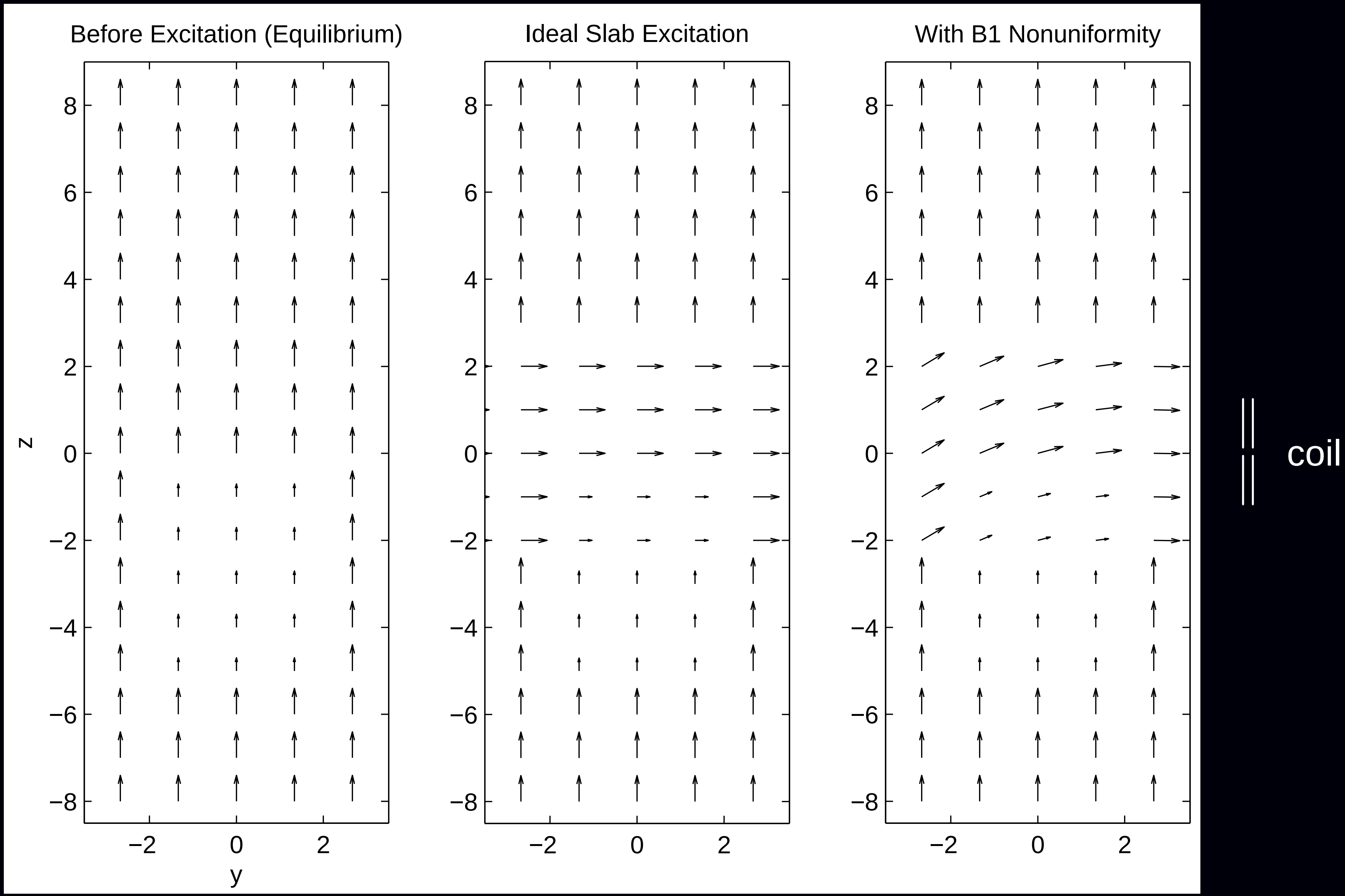
(Exciting all spins is relatively easy, *cf.* NMR spectroscopy)

- slice selection (1D)
- spatially selective excitation (2D and 3D)
  - imaging small regions
  - compensating for nonuniform coil sensitivity (high field)
  - compensating for undesired spin phase evolution (fMRI)

Constraints:

- RF amplitude, bandwidth (hardware)
- RF power deposition (patient safety)
- Gradient waveform amplitude, slew rate
- Excitation pulse duration

Standard Slice Selection with RF Inhomogeneity



Excitation coil inhomogeneity induces undesired image shading.

Solution: more sophisticated RF pulse design ... sparsity.

Small-tip Solution to Bloch Equation

Relate RF pulse envelope and induced field:

$$\boldsymbol{B}_1(\boldsymbol{r},t) = \underbrace{\mathrm{coil}(\boldsymbol{r})}_{\text{coil response}} \quad \underbrace{b_1(t)}_{\text{RF pulse envelope}} \begin{bmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \\ 0 \end{bmatrix}, \qquad \underbrace{\omega_0 = \gamma B_0}_{\text{Larmor frequency}} .$$

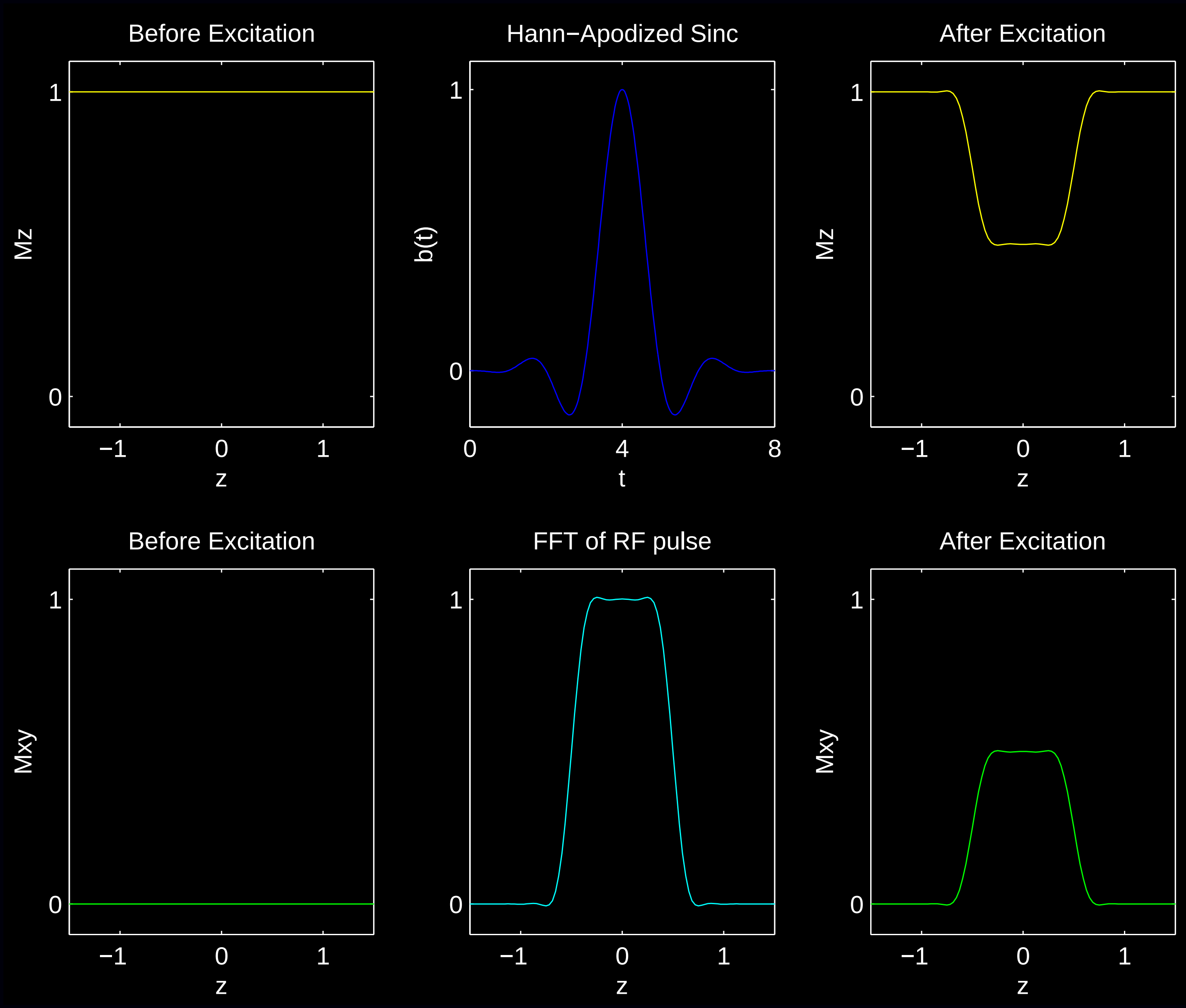
Small-tip approximation to Bloch solution (Pauly, 1989):

$$\begin{aligned} M(\boldsymbol{r},T) &\triangleq M_{\mathrm{x}}(\boldsymbol{r},T) + \imath M_{\mathrm{y}}(\boldsymbol{r},T) \\ &\approx \imath\gamma \underbrace{M_0(\boldsymbol{r})\,\mathrm{coil}(\boldsymbol{r})}_{\text{shading}} \underbrace{\int_0^T b_1(t)\, \mathrm{e}^{\imath 2\pi \boldsymbol{r}\cdot\boldsymbol{k}(t)(t-T)}\, \mathrm{d}t}_{\text{Fourier-like}} \end{aligned}$$

where the excitation k-space trajectory is:

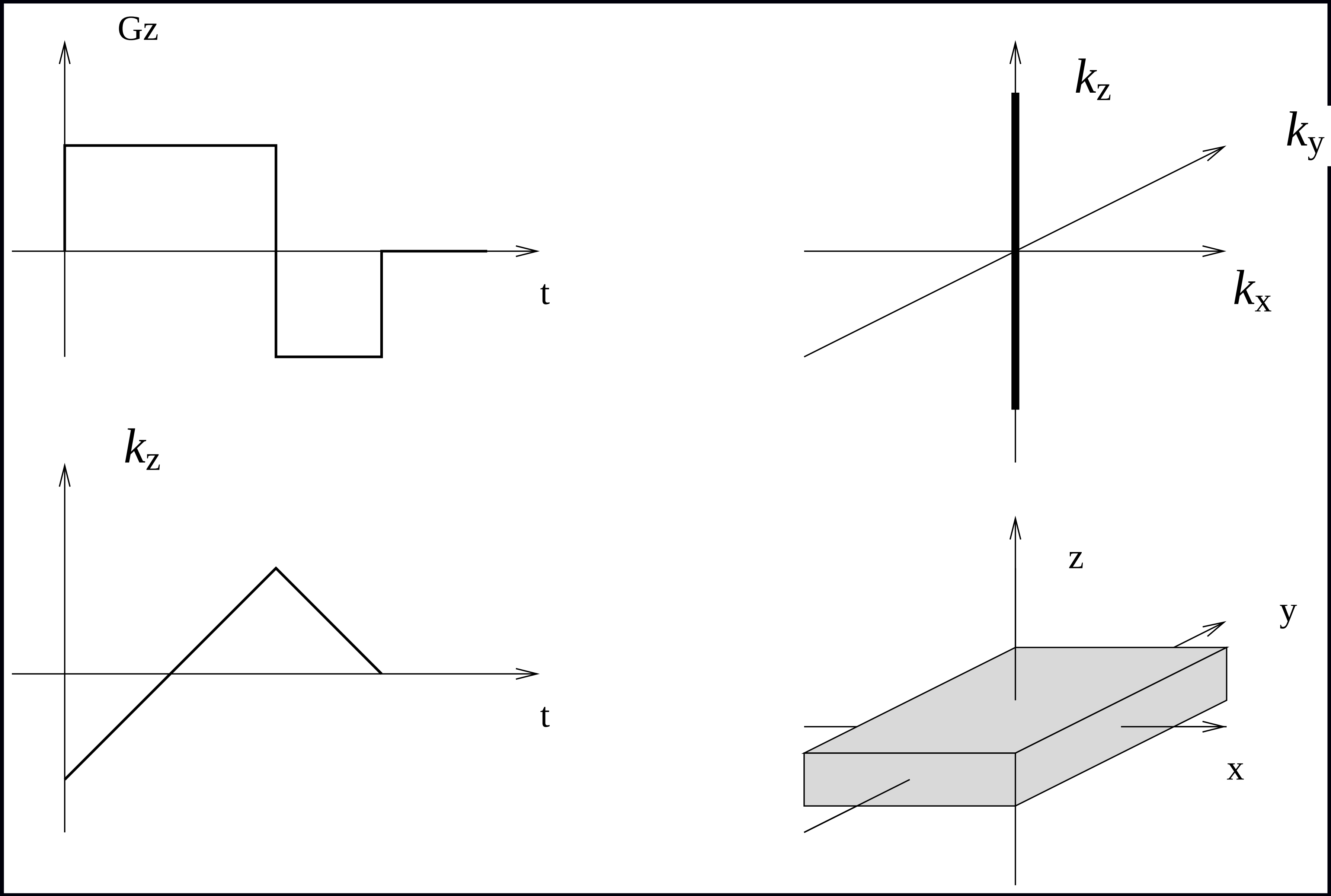
$$\boldsymbol{k}(t) \triangleq -\frac{\gamma}{2\pi} \int_t^T \boldsymbol{G}(t')\, \mathrm{d}t' .$$

1D Example: Slice Selection

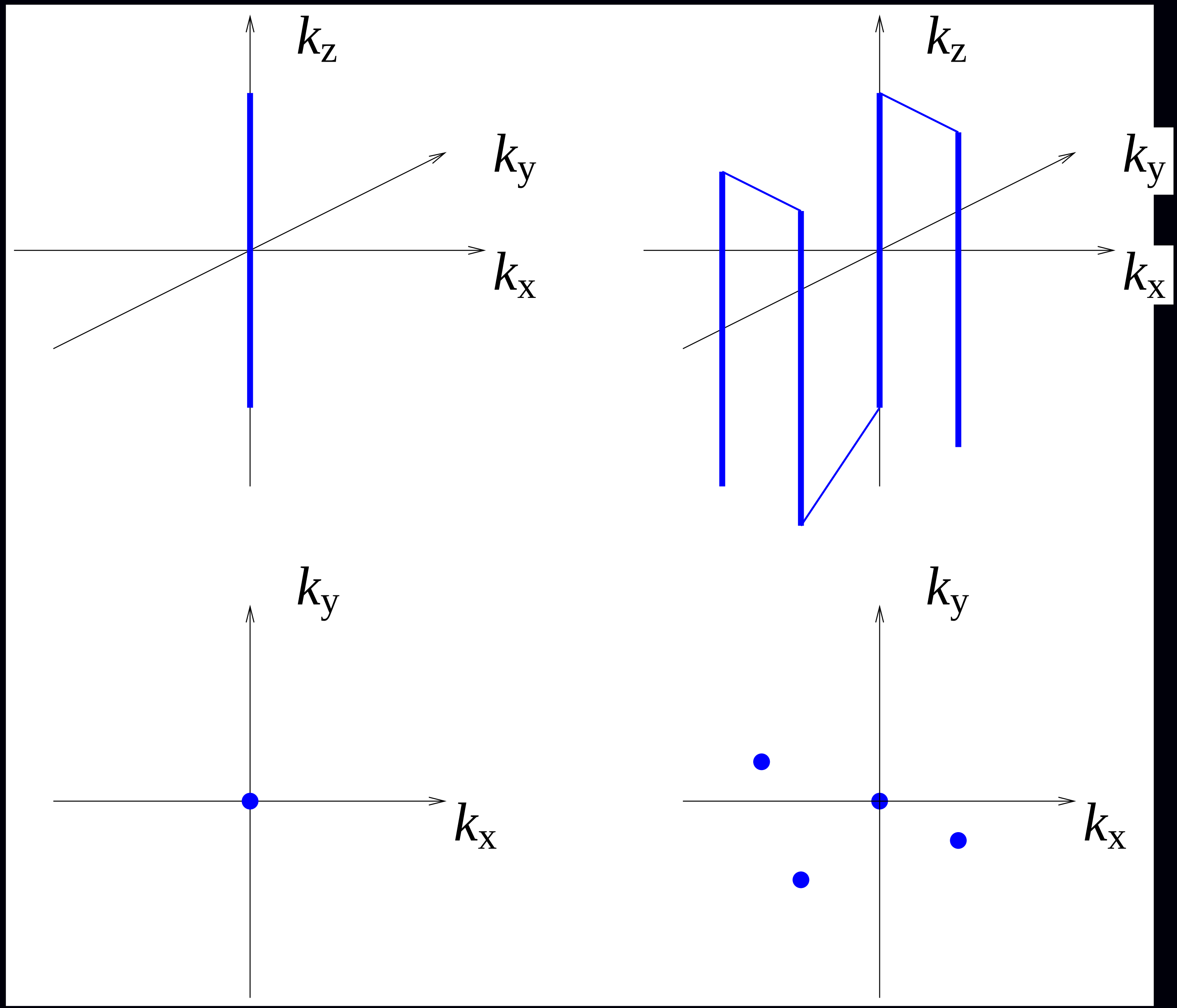


(for uniform coil, with 30° tip)

1D Example: k-space Perspective



From slice-selection to spatially selective excitation



Choosing phase-encode locations?

- Select desired excitation pattern $\boldsymbol{d}$
- Select number of $(k_{\mathrm{x}}, k_{\mathrm{y}})$-space phase encodes N_{e}
- Optimize jointly the RF pulse parameters $\boldsymbol{b}$ and the $(k_{\mathrm{x}}, k_{\mathrm{y}})$-space phase encode locations $\boldsymbol{\phi} \in \mathbb{R}^{N_{\mathrm{e}} \times 2}$:

$$\arg\min_{\boldsymbol{b},\boldsymbol{\phi}} \underbrace{\|\boldsymbol{d} - \boldsymbol{A}(\boldsymbol{\phi})\boldsymbol{b}\|_{\boldsymbol{W}}^2}_{\text{excitation fidelty}} + \beta \underbrace{\|\boldsymbol{b}\|^2}_{\text{RF energy}}$$

where, from discretization of small-tip approximation:

$$[\boldsymbol{A}(\phi)]_{nm} = \imath\gamma \operatorname{coil}(\boldsymbol{r}_n)\, e^{\imath 2\pi \boldsymbol{r}_n \cdot \boldsymbol{k}(t_m;\boldsymbol{\phi})(t_m - T)}.$$

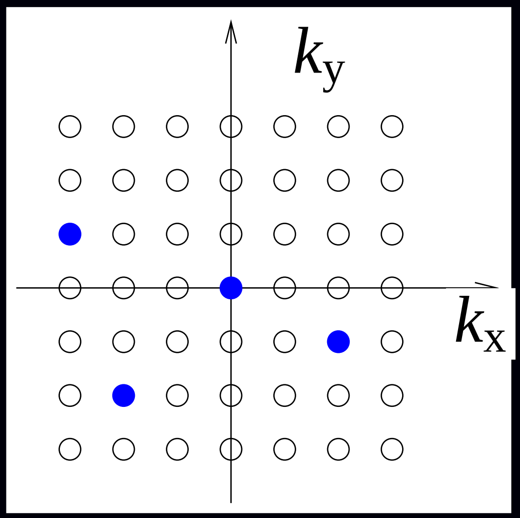
$\boldsymbol{W}$ allows ROI specification, a key benefit of model-based formulations

Alternating minimization.
Minimizing over $\boldsymbol{b}$ is easy via CG. (Yip *et al.*, MRM 54(4), Oct. 2005)
Minimizing over $\boldsymbol{\phi}$ is challenging. (Yip *et al.*, MRM 58(3), Sep. 2007)

Sparsity-Constrained Formulation

Allow one (or more) pulse parameters $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_\mathrm{g}})$ for *every* $(k_\mathrm{x}, k_\mathrm{y})$ phase-encode location (on a discrete grid of N_g points).
Choose a sparse subset of possible phase-encode locations:

$$\arg\min_{\boldsymbol{x}} \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{x}\|_{\boldsymbol{W}}^2 \text{ subject to } \underbrace{\|\boldsymbol{x}\|_0 \le N_\mathrm{e}}_{\text{sparsity}}$$



Alternate "sparse approximation" formulation:

$$\arg\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_0 \text{ subject to } \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{x}\|_{\boldsymbol{W}}^2 \le \delta.$$

Both formulations are non-convex and challenging (NP-complete).

Convex Formulation (Single Coil)

Convex relaxation.

Zelinski *et al.*: MRM 59(6), June 2008; T-MI Sep. 2008

For sparse approximation, replacing $\|\boldsymbol{x}\|_0$ with $\|\boldsymbol{x}\|_1$ usually works:

Tropp: IEEE T-IT, Mar. 2006

$$\arg\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1 \text{ subject to } \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{x}\|_{\boldsymbol{W}}^2 \leq \delta.$$

Lagrange multiplier or regularization approach:

$$\arg\min_{\boldsymbol{x}} \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{x}\|_{\boldsymbol{W}}^2 + \beta \|\boldsymbol{x}\|_1,$$

where one adjusts β to trade off sparsity (pulse length) and approximation error (excitation accuracy).

- Solving second-order cone program (SOCP) can be slow
- Many minutes, depending on sampling, number of coils, etc.
- Want fast methods for on-line use!
- $\boldsymbol{A}$ and/or $\boldsymbol{d}$ are often patient dependent

Greedy Approach: Orthogonal Matching Pursuit

OMP method attempts sparse signal approximation:

$$\min_{\boldsymbol{x}} \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{x}\|_{\boldsymbol{W}}^2 \text{ sub. to } \|\boldsymbol{x}\|_0 \le N_{\mathrm{e}}.$$

- Set $\Lambda = \{\}$ (initial index set)
- Set $\boldsymbol{r} = \boldsymbol{d}$ (initial residual vector)
- For each iteration (until desired sparsity):
  - add column of $\boldsymbol{A}$ most correlated with residual:

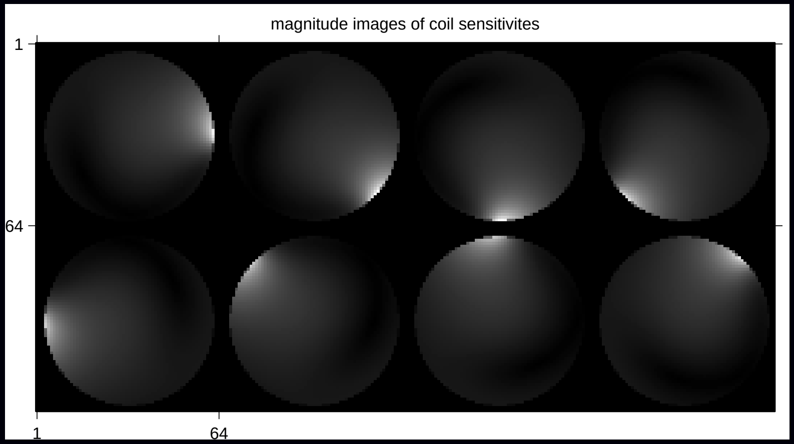
$$\Lambda := \Lambda \cup \left\{ \arg\max_j |\langle A(:,j), \boldsymbol{r}\rangle_{\boldsymbol{W}}| \right\}$$

  - Project residual onto columns of $\boldsymbol{A}$ indexed by Λ
  - Update residual: $\boldsymbol{r} := \boldsymbol{r} - \mathcal{P}_{\boldsymbol{A}_\Lambda} \boldsymbol{r} = \mathcal{P}^{\perp}_{\boldsymbol{A}_\Lambda} \boldsymbol{r}$
- Finally, solve for selected elements of $\boldsymbol{x}$:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x} \in X_\Lambda} \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{x}\|_{\boldsymbol{W}}^2$$

OMP is fast (FFT). It is nearly optimal under coherence conditions on $\boldsymbol{A}$ that may not hold in MRI excitation. (Tropp, T-IT, Oct. 2004)

Parallel Transmit: Multiple Coils



- Magnetic field is superposition of contribution of each coil
- Induced magnetization is too (under small-tip approximation):

$$\boldsymbol{d} \approx \boldsymbol{A}_1 \boldsymbol{x}_1 + \cdots + \boldsymbol{A}_K \boldsymbol{x}_K$$

- $\boldsymbol{A}_k$ includes excitation response (B1+ map) of kth coil
- $\boldsymbol{x}_k$ parameterizes the RF signal into the kth coil
- Coil have individual RF signals *but share the k-space trajectory*

Parallel sparsity problem:

$$\min \left\| \boldsymbol{d} - \sum_{k=1}^{K} \boldsymbol{A}_k \boldsymbol{x}_k \right\|_{\boldsymbol{W}}^2 \text{ sub. to "joint sparsity" of } \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K\}$$

Simultaneous Sparsity Problems

- Conventional sparse approximation problem:

$$\min \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\| \text{ sub. to } \|\boldsymbol{x}\|_0 \leq N_{\mathrm{e}}$$

- Conventional simultaneous sparsity problem: Tropp *et al.*, SigPro, 2006

$$\min \sum_{k=1}^{K} \|\boldsymbol{y}_k - \boldsymbol{A}\boldsymbol{x}_k\| \text{ sub. to } \|[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K]\|_{\infty,0} \leq N_{\mathrm{e}}$$

where $\|X\|_{\infty,0}$ counts the number of *rows* of X with nonzero entries.
(Use the same dictionary elements to approximate several signals.)

- ○ Greedy algorithms (S-OMP) Tropp *et al.*, SigPro, Mar. 2006
- ○ Convex relaxation: Tropp, SigPro, Mar. 2006
  replace $\|X\|_{\infty,0}$ with $\|X\|_{2,1}$, the sum of ℓ_2 norm of each row.

- MRI "parallel sparsity" approximation problem:

$$\min \left\|\boldsymbol{y} - \textstyle\sum_{k=1}^{K} \boldsymbol{A}_k \boldsymbol{x}_k\right\| \text{ sub. to } \|[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K]\|_{\infty,0} \leq N_{\mathrm{e}}$$

SOCP slow...

Proposed Parallel OMP (P-OMP)

Theory: Maleh *et al.*, SPARS, Apr. 2009

MRI application: Yoon *et al.*, ISMRM, Apr. 2009

- Set $\Lambda = \{\}$ (initial index set)
- Set $\boldsymbol{r} = \boldsymbol{d}$ (initial residual vector)
- For each iteration (until desired sparsity):
  - add column index of $\{\boldsymbol{A}_k\}$ "most correlated" with residual:

$$\Lambda := \Lambda \cup \left\{ \arg\max_j \sum_{k=1}^{K} \left| \langle A_k(:,j), \boldsymbol{r} \rangle_{\boldsymbol{W}} \right| \right\}$$

  - Project residual onto columns of $\{\boldsymbol{A}_k\}$ indexed by Λ
  - Update residual:

$$\boldsymbol{r} := \boldsymbol{r} - \mathcal{P}_S \boldsymbol{r} = \mathcal{P}_S^{\perp} \boldsymbol{r}, \quad S = \{A_k(:,j),\ k = 1,\ldots,N_{\mathrm{e}},\ j \in \Lambda\}$$

- Finally, solve for selected elements of $\boldsymbol{x}$:

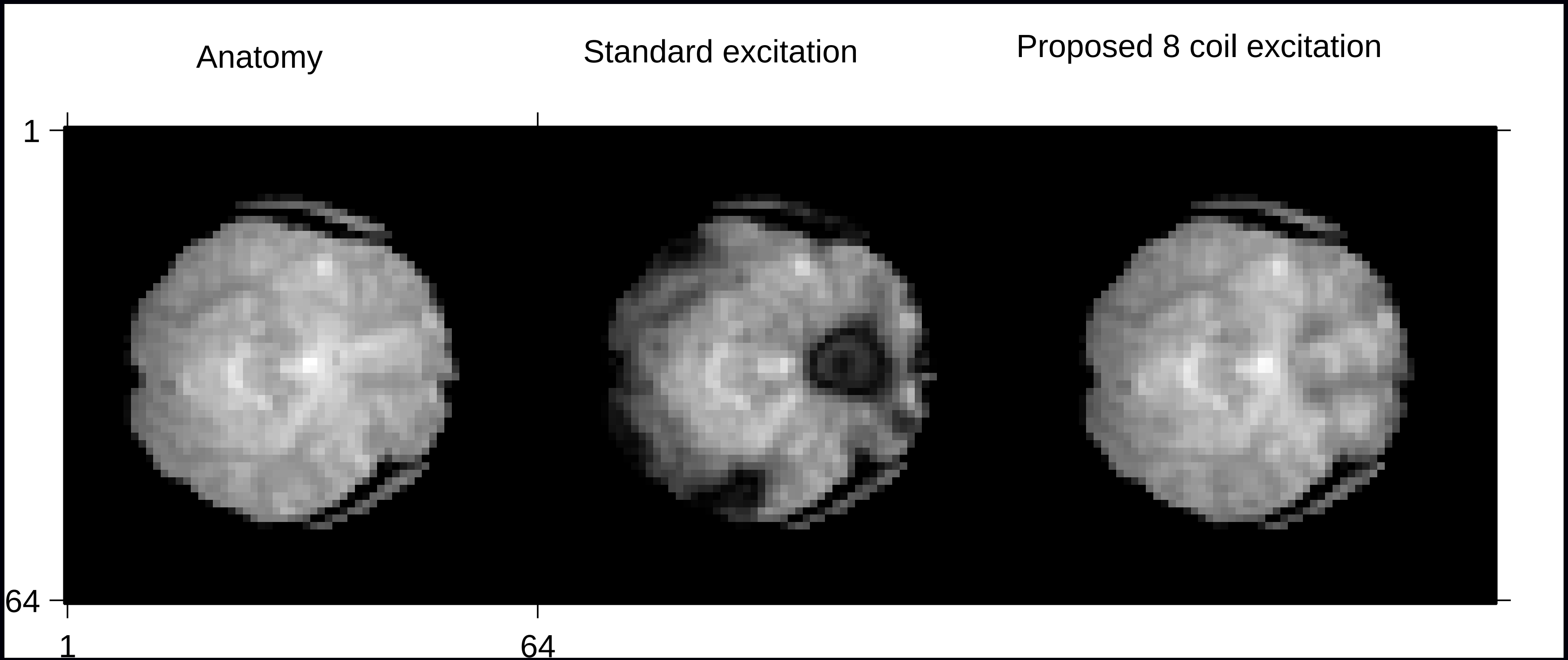
$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x} \in X_\Lambda} \left\| \boldsymbol{d} - \textstyle\sum_{k=1}^{K} \boldsymbol{A}_k \boldsymbol{x}_k \right\|_{\boldsymbol{W}}^2 .$$

Variations: $\sum_{k=1}^{K} |\cdot|^p$ for $p = 2$ or $p \to \infty$, or projection onto span of $\{A_1(:,j),\ldots,A_K(:,j)\}$.

Theoretical correctness guarantees for certain conditions, not quite satisfied in MRI...

Application:
B_0 Inhomogeneity "Precompensation" in BOLD fMRI imaging

B_0 inhomogeneity compensation: Overview

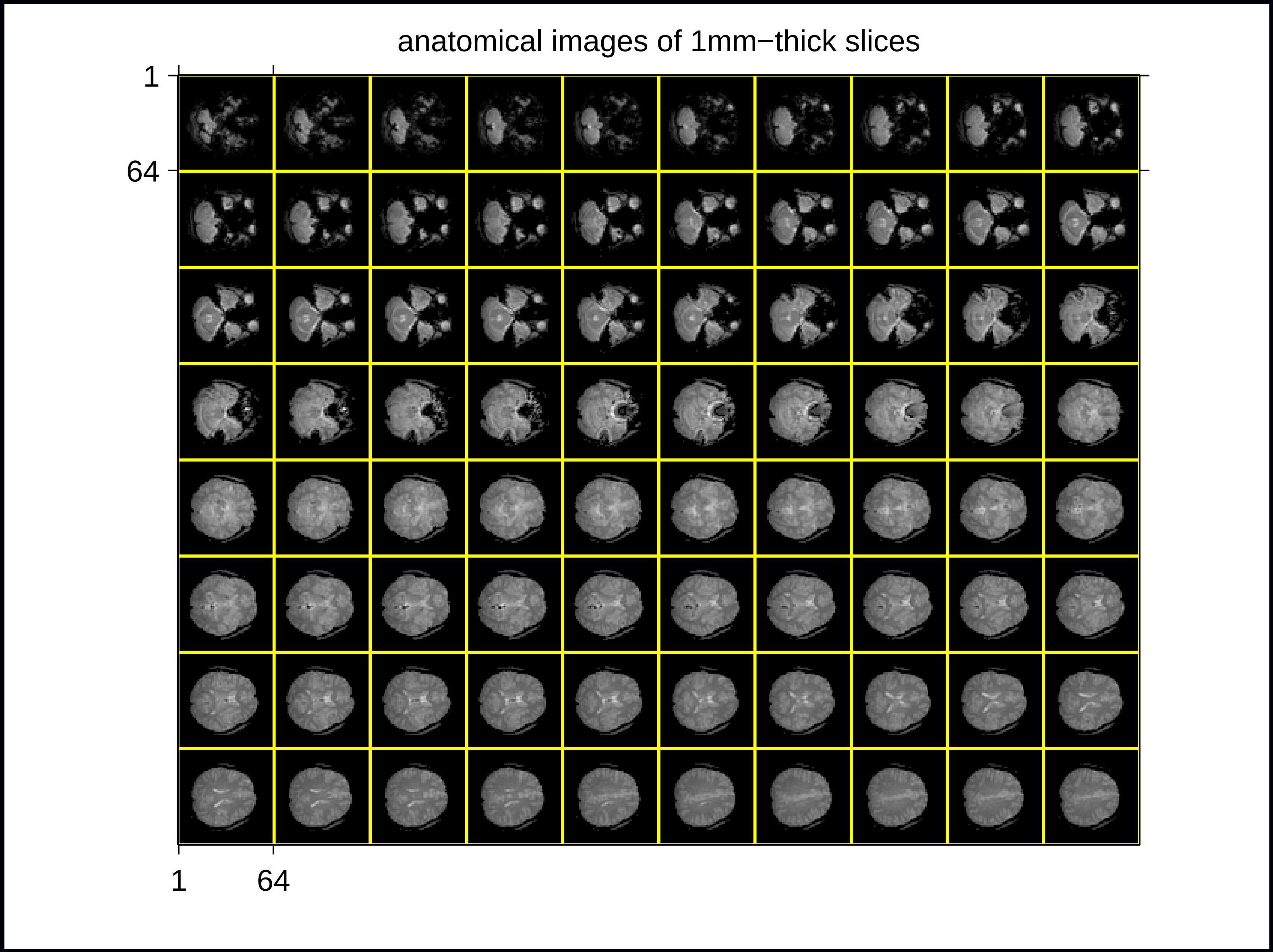


short TE | T_2^* weighted | T_2^* weighted

- Signal loss due to through-plane B_0 inhomogeneity
- Severe near susceptibility gradients in BOLD fMRI
- Solution: iteratively designed, tailored RF pulses that precompensate for through-plane field variations

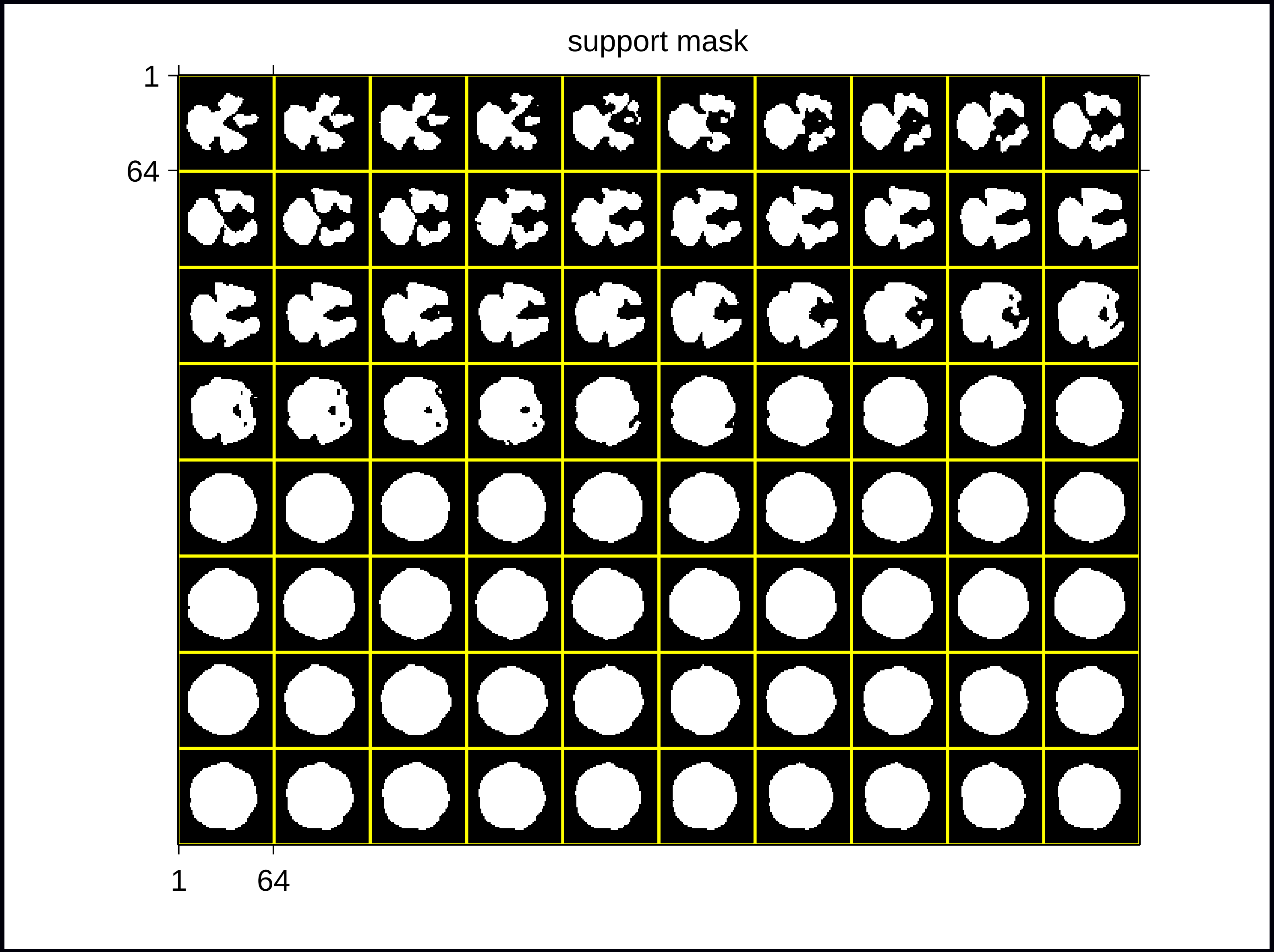
Glover & Lai, ISMRM 1998; Yip *et al.*, MRM 56(5), Nov. 2006

B_0 inhomogeneity compensation: Sub. 1 Anatomy



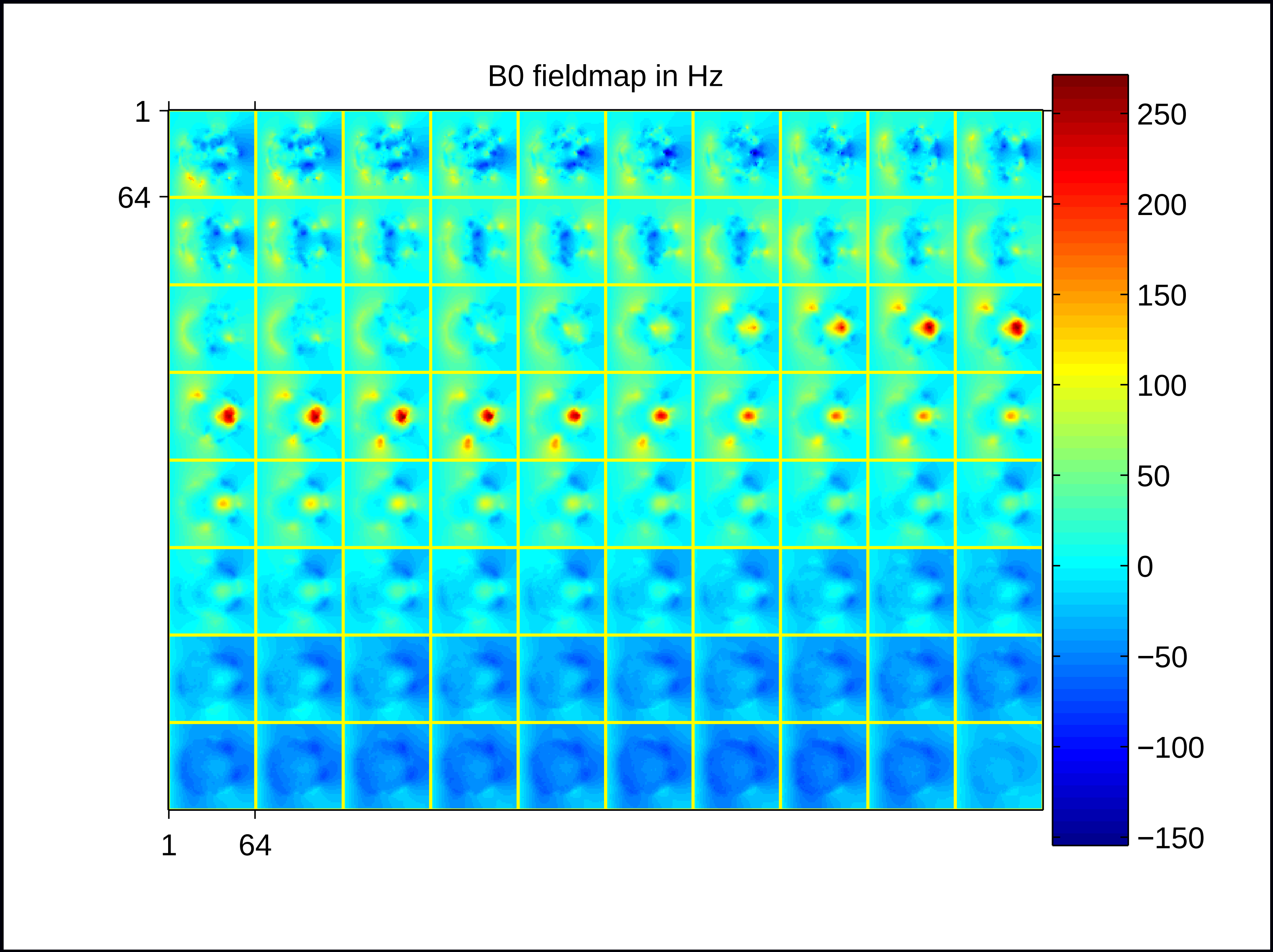
1 mm slices

B_0 inhomogeneity compensation: Sub. 1 Mask



"do not care" outside mask

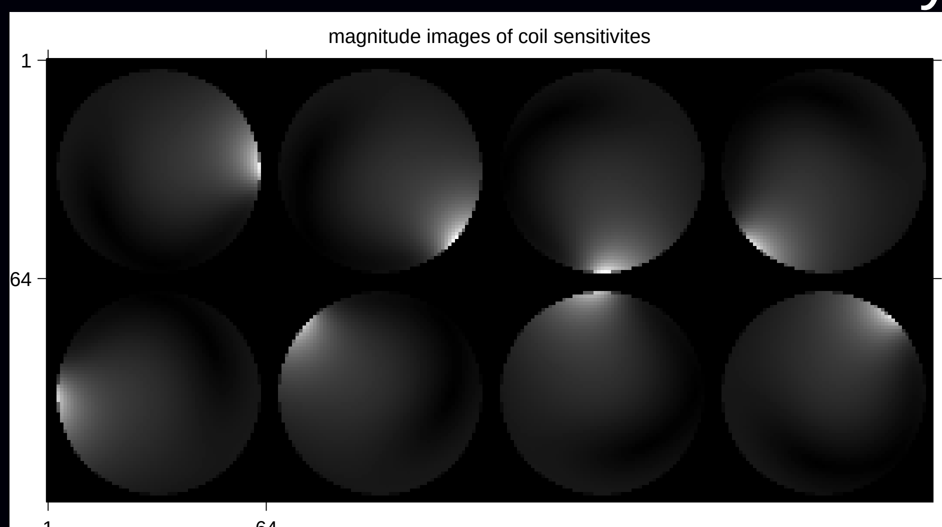
B_0 inhomogeneity compensation: Sub. 1 Fieldmap



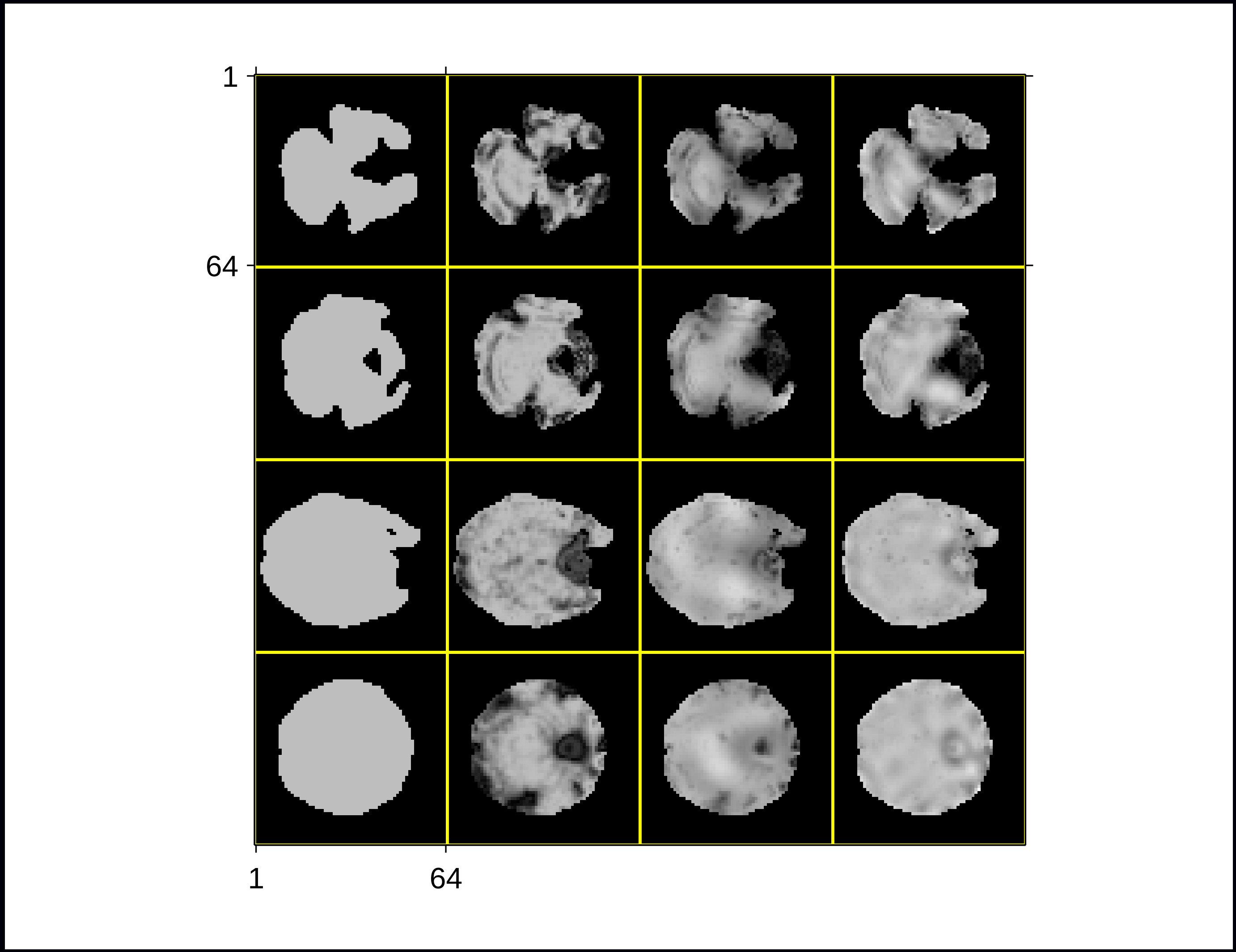
These 1 mm slices determine phase of desired pattern $\boldsymbol{d}$

B_0 inhomogeneity compensation: Design

- 5 mm thick slice to be excited
- 3D desired excitation pattern over five 1 mm slices: uniform magnitude, phase from B_0 map: $d(\boldsymbol{r}) = e^{-\imath\omega(\boldsymbol{r})T_E}$
- $1 \leq N_e \leq 30$ phase-encode locations in (k_x, k_y) plane
- P-OMP algorithm for joint trajectory / RF pulse design
- Simulation based on acquired images and field maps.
- Two cases:
  - Single uniform transmit coil
  - 8-coil array with calculated transmit sensitivity patterns:



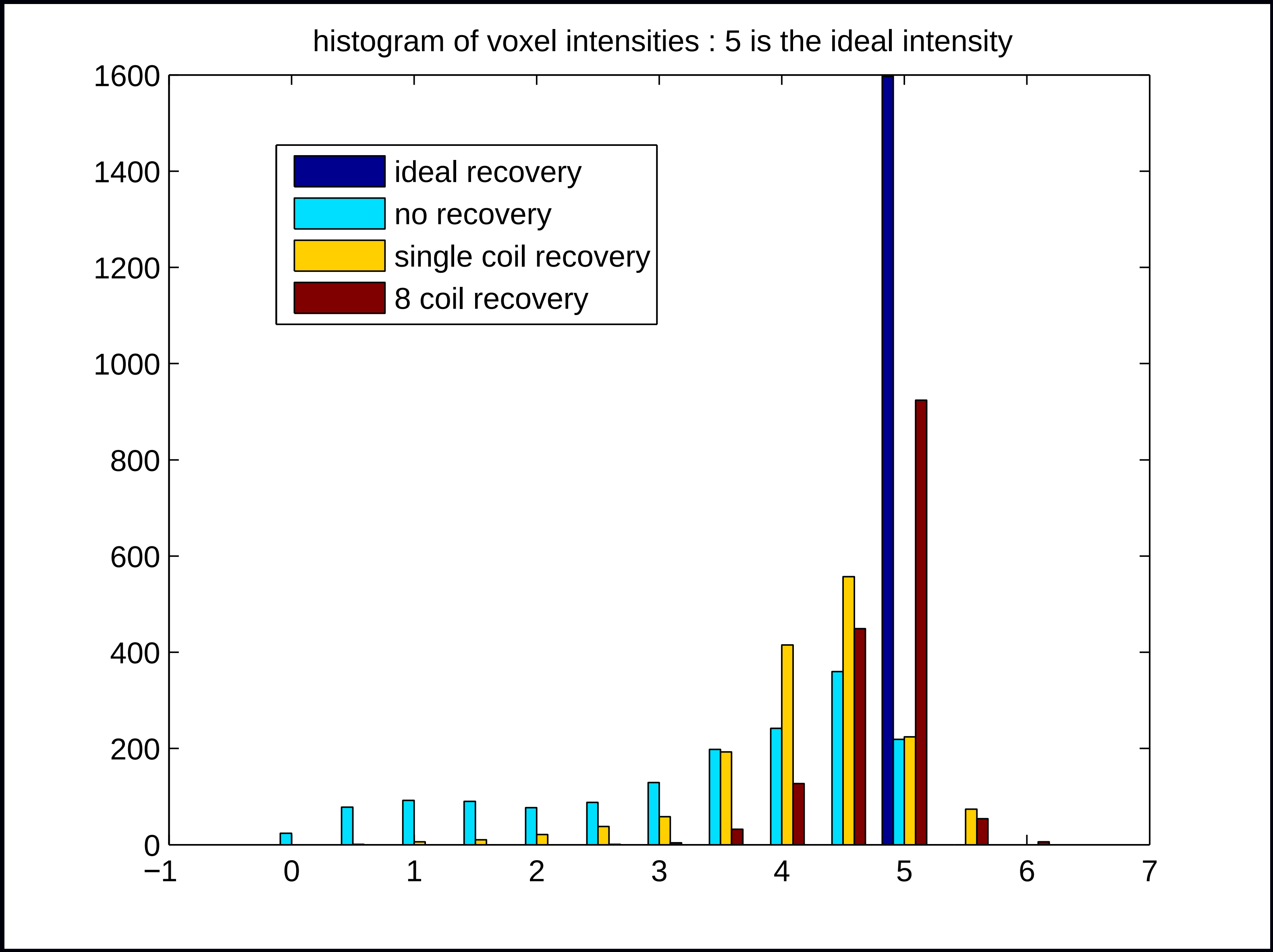
B_0 inhomogeneity compensation: Results



ideal standard 1-coil correction 8-coil correction

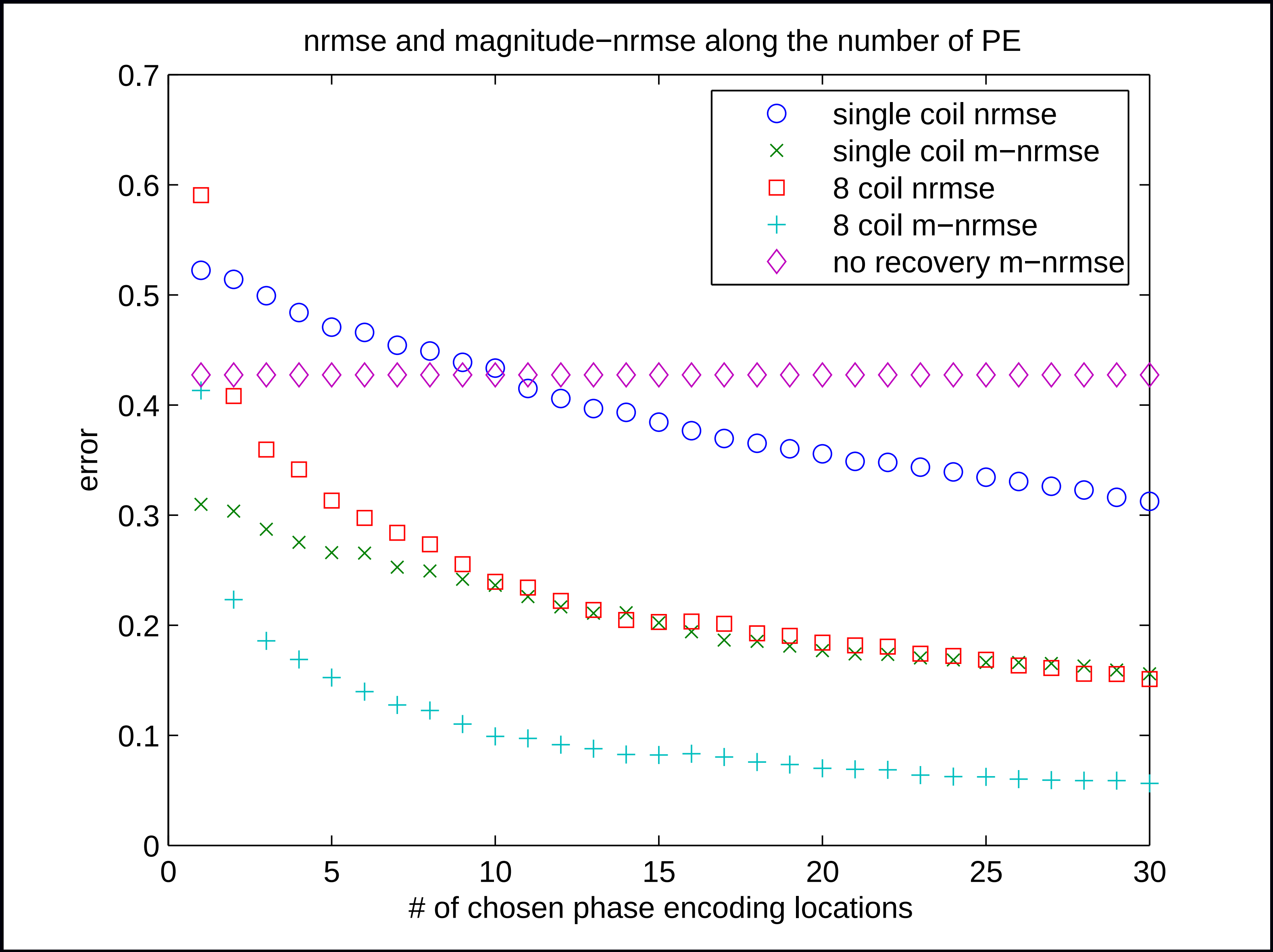
12 phase encoding locations. Total pulse length 7-9 msec. 8-coil design time ≈ 2 min.

B_0 inhomogeneity compensation: Results



(From 4th row of preceding slide)

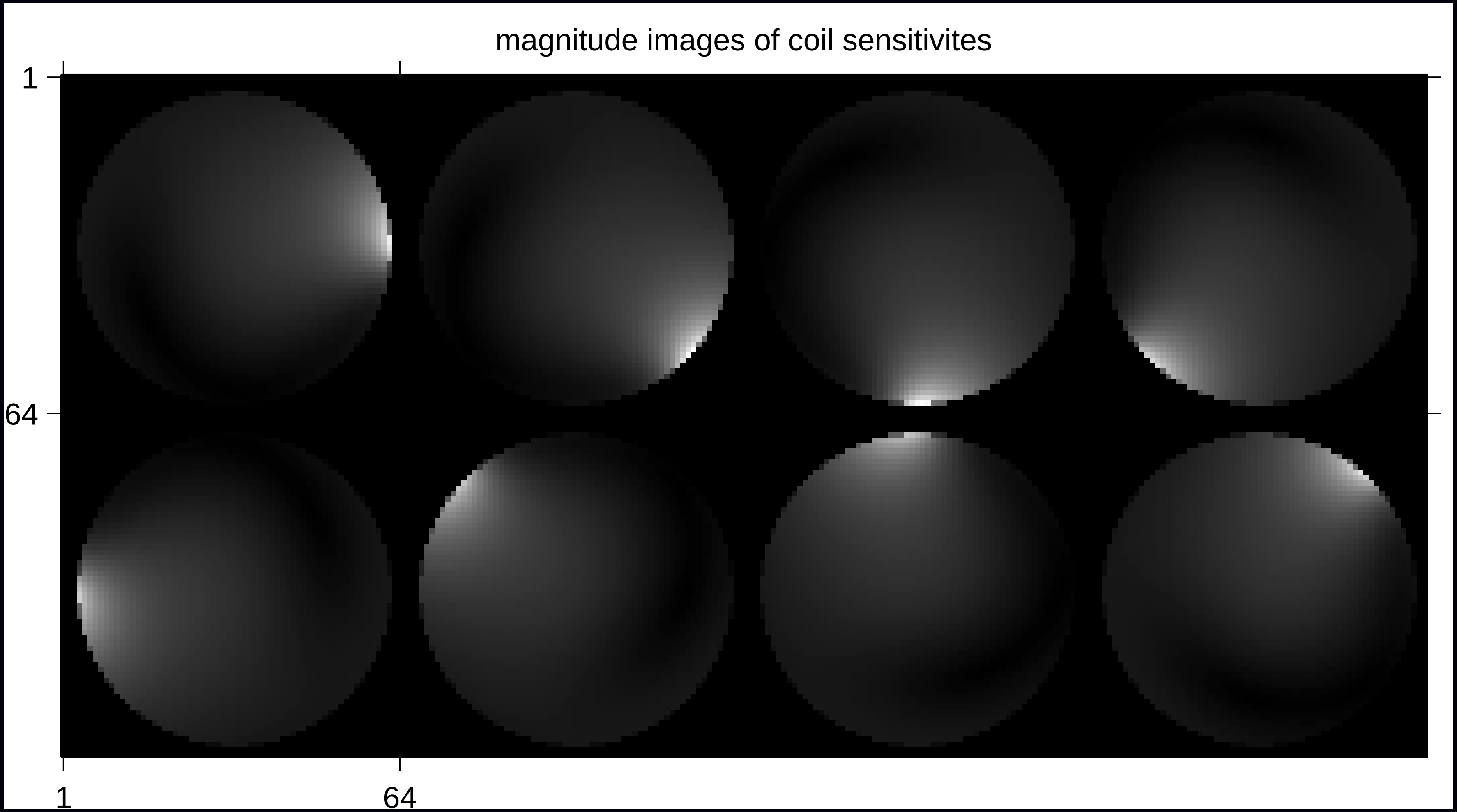
B_0 inhomogeneity compensation: NRMSE



(magnitude-nrmse is at echo time)

Application:
B_1 Inhomogeneity Correction

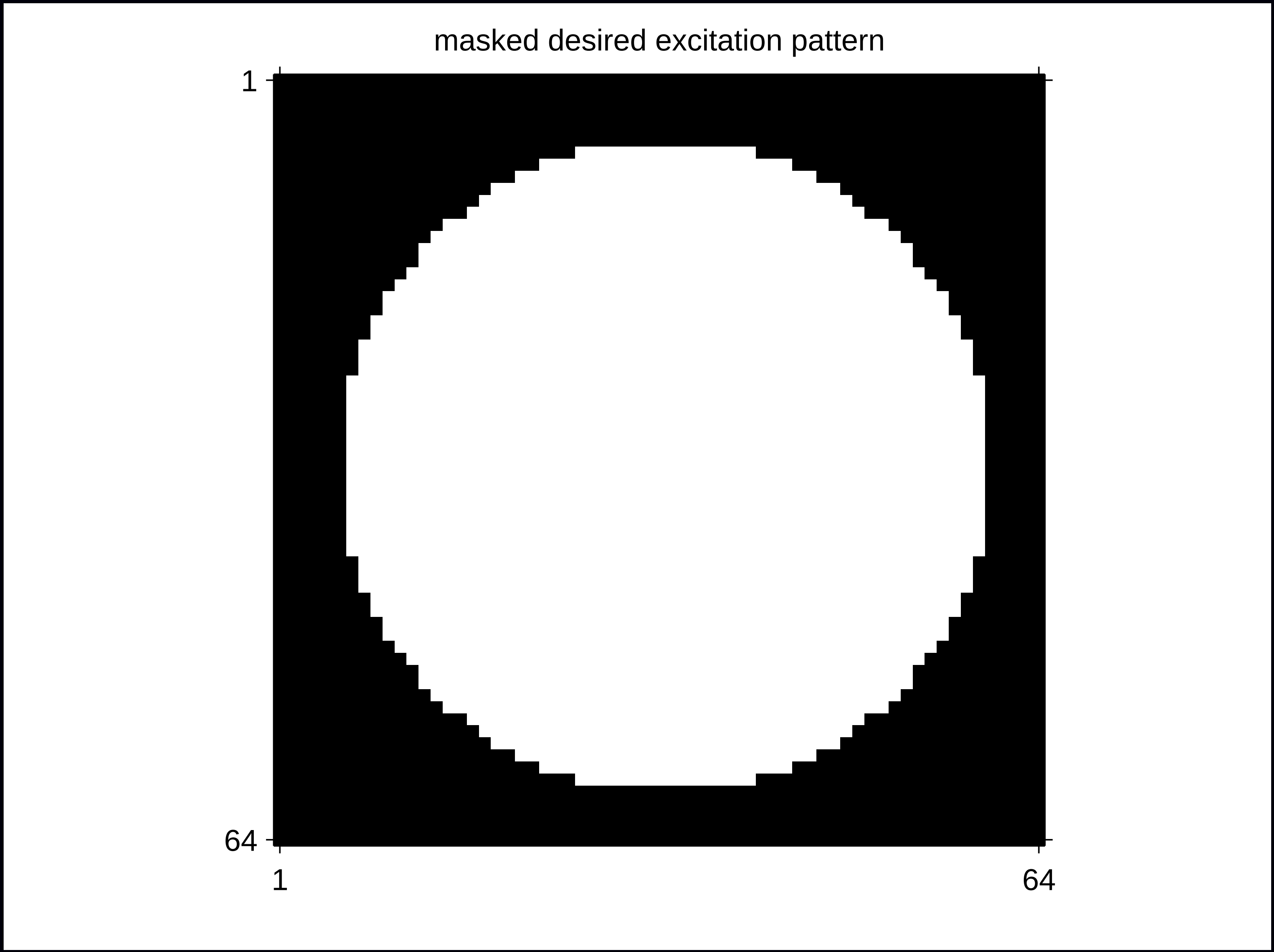
B1 Nonuniformity Correction: Coil Sensitivities



8 head coils

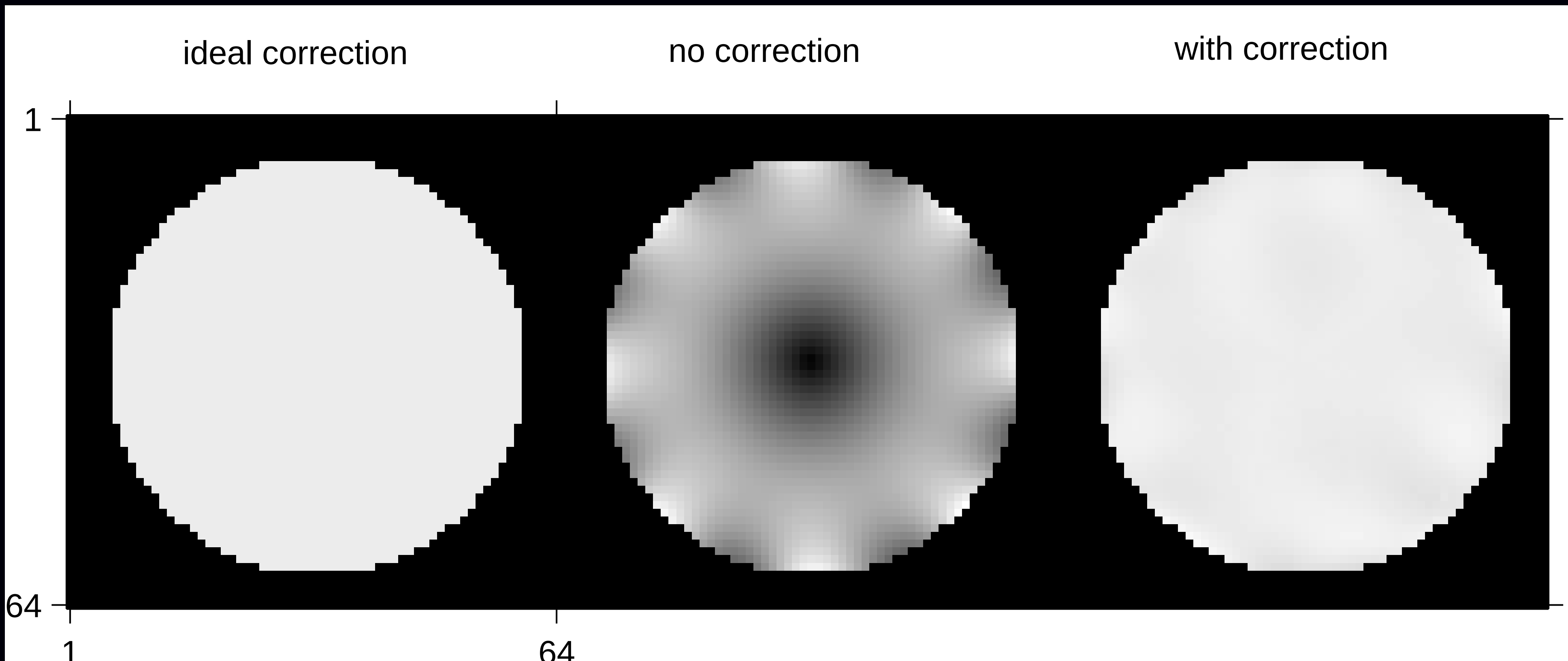
Simulation with 24 cm FOV, 64×64 sampling grid

B1 Nonuniformity Correction: Desired Pattern



2D disk with diameter 20.25 cm. “Don’t care” outside mask.

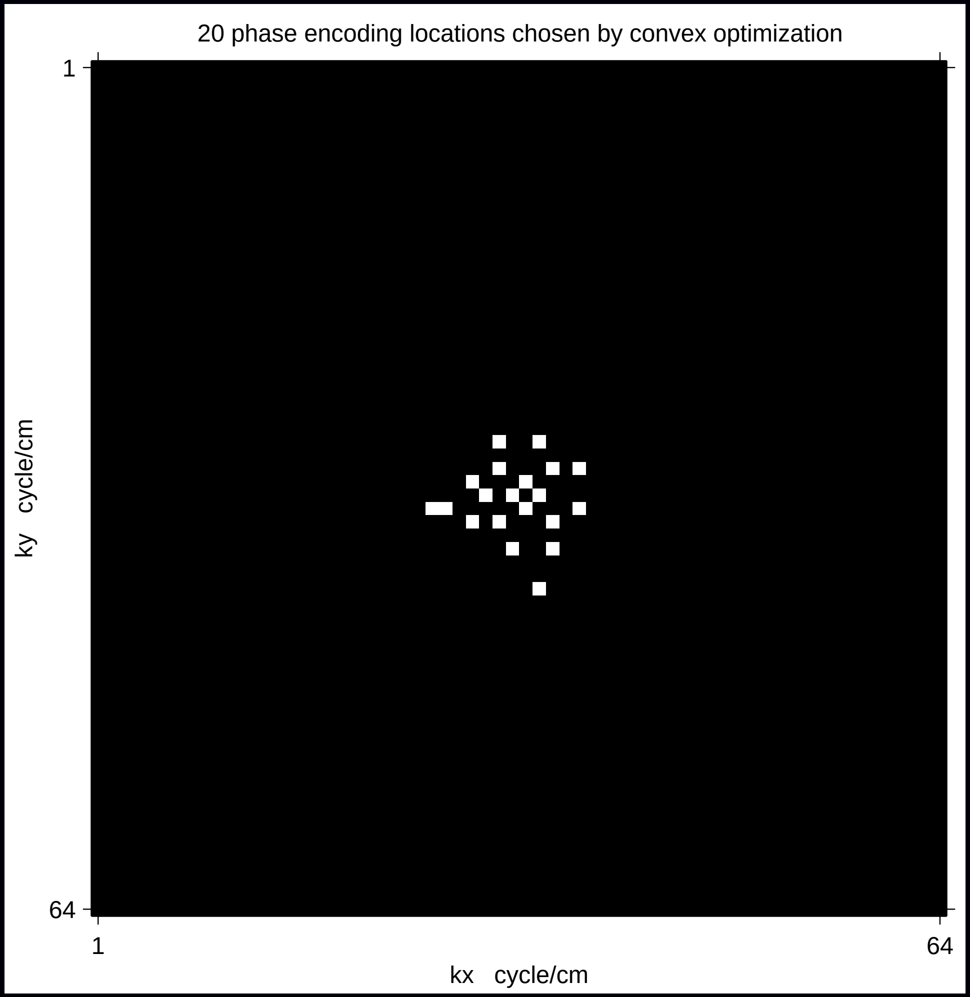
B1 Nonuniformity Correction: Excitation Results

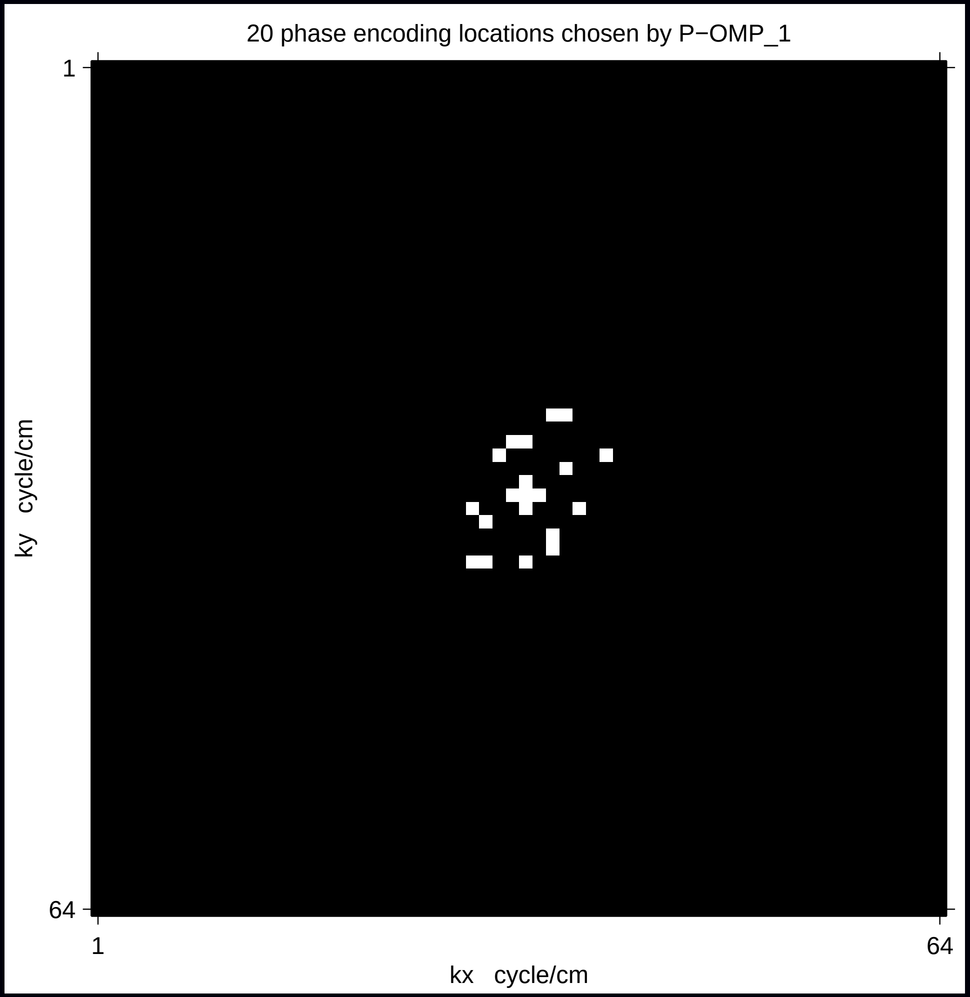


$N_e = 4$ phase encoding locations.

P-OMP
(2.2 sec)

Phase-Encode Locations



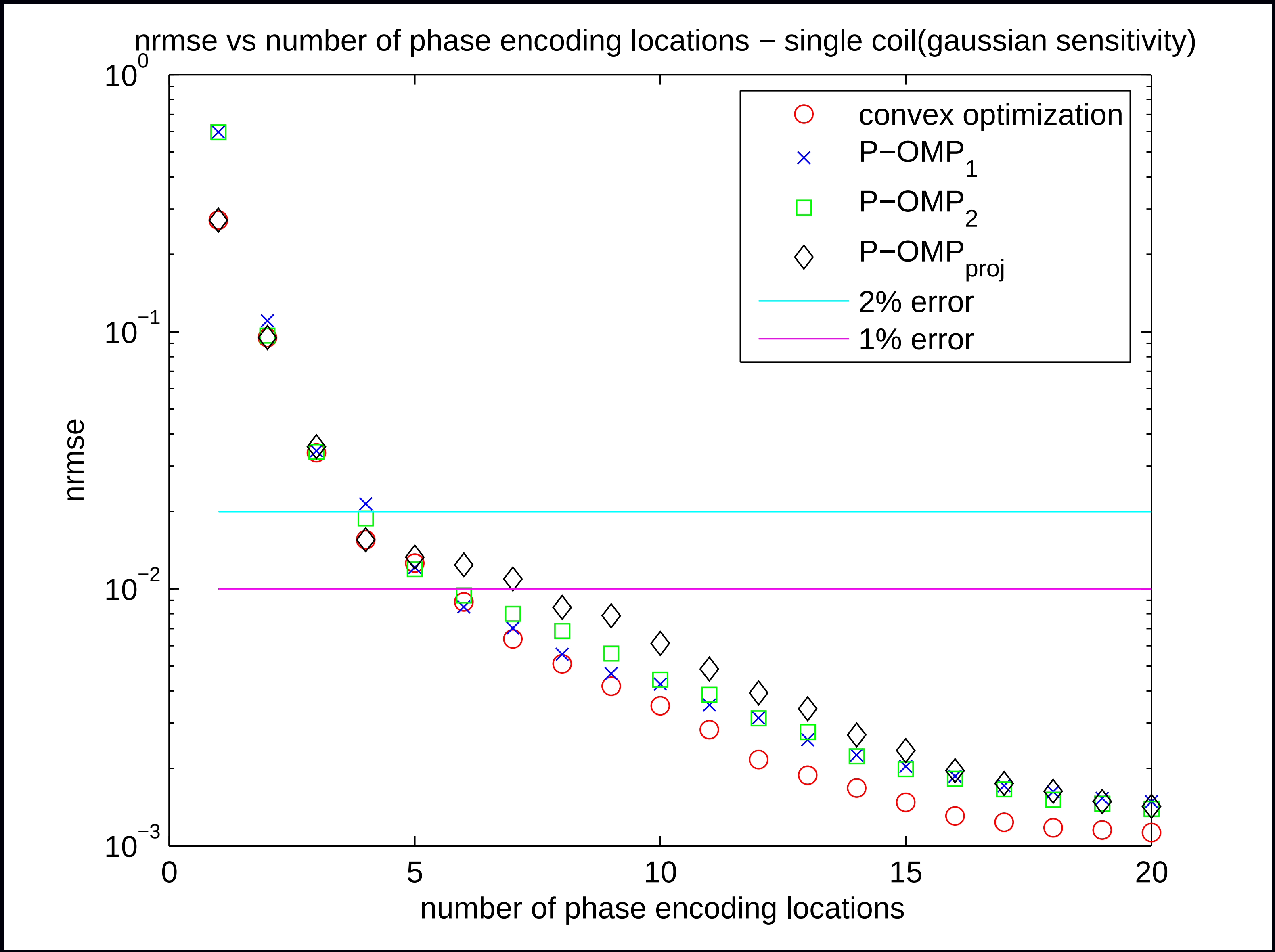


Convex optimization

P-OMP

$N_e = 20$

B1 Nonuniformity Correction Results: Accuracy



Tradeoff between pulse length and excitation error (residual nonuniformity).

P-OMP provides reasonable uniformity (1-2%) with 4-5 phase encodes, quickly.

Summary

- Sparsity everywhere, even in MRI excitation
- Applications:
  - B_0 correction
  - B_1 correction
  - Cases requiring slice-selection and within-plane variations
- Using greedy algorithms like OMP accelerate computation
- P-OMP extends OMP to the problem of “parallel sparsity”