Tradeoffs and complexities in new reconstruction methods

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Legalese

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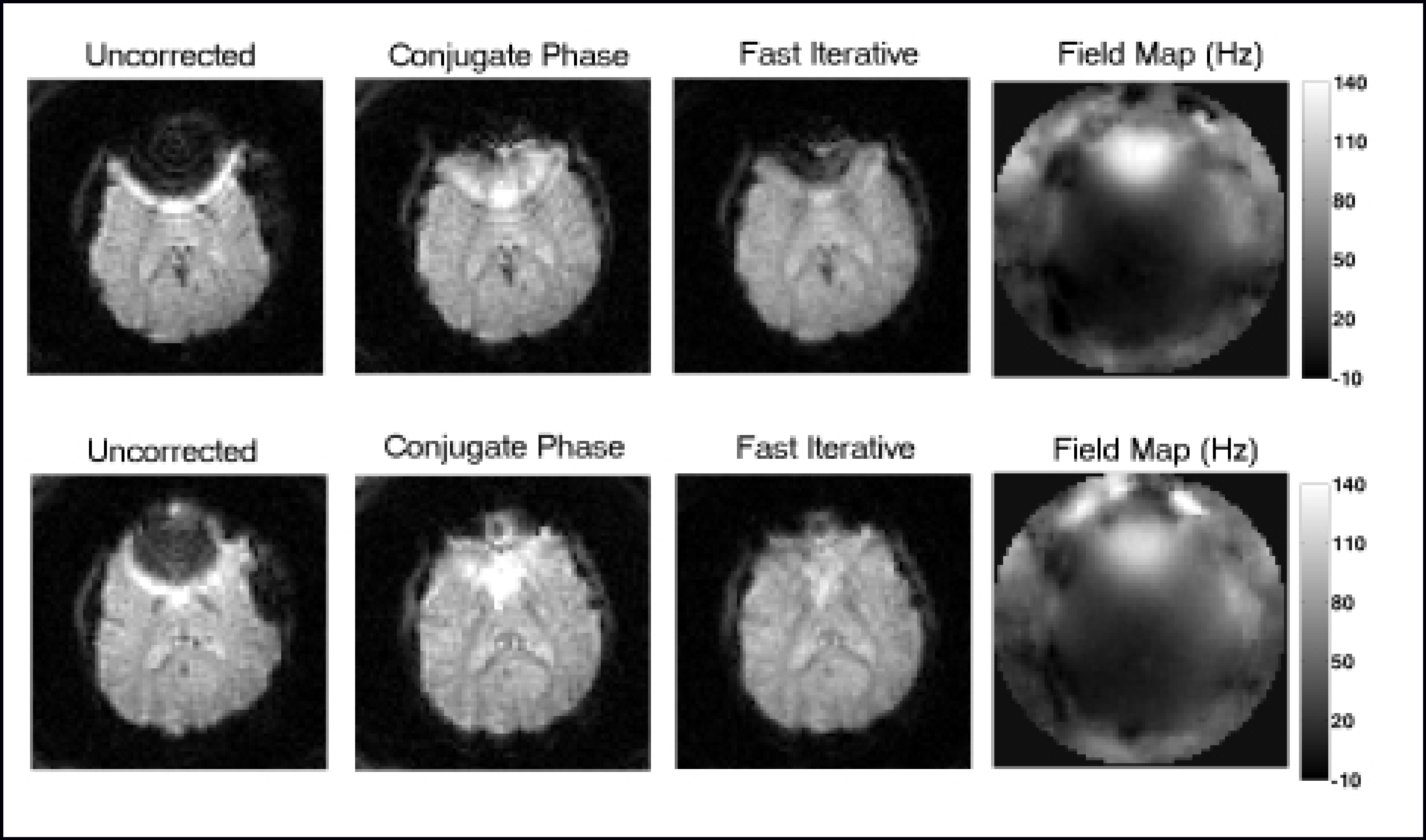
Outline

- MR image reconstruction problem description
- Overview of image reconstruction methods
- MR image reconstruction introduction
- Conventional reconstruction
- Model-based image reconstruction
- Iterations and computation (NUFFT etc.)
- Regularization
- Field inhomogeneity correction

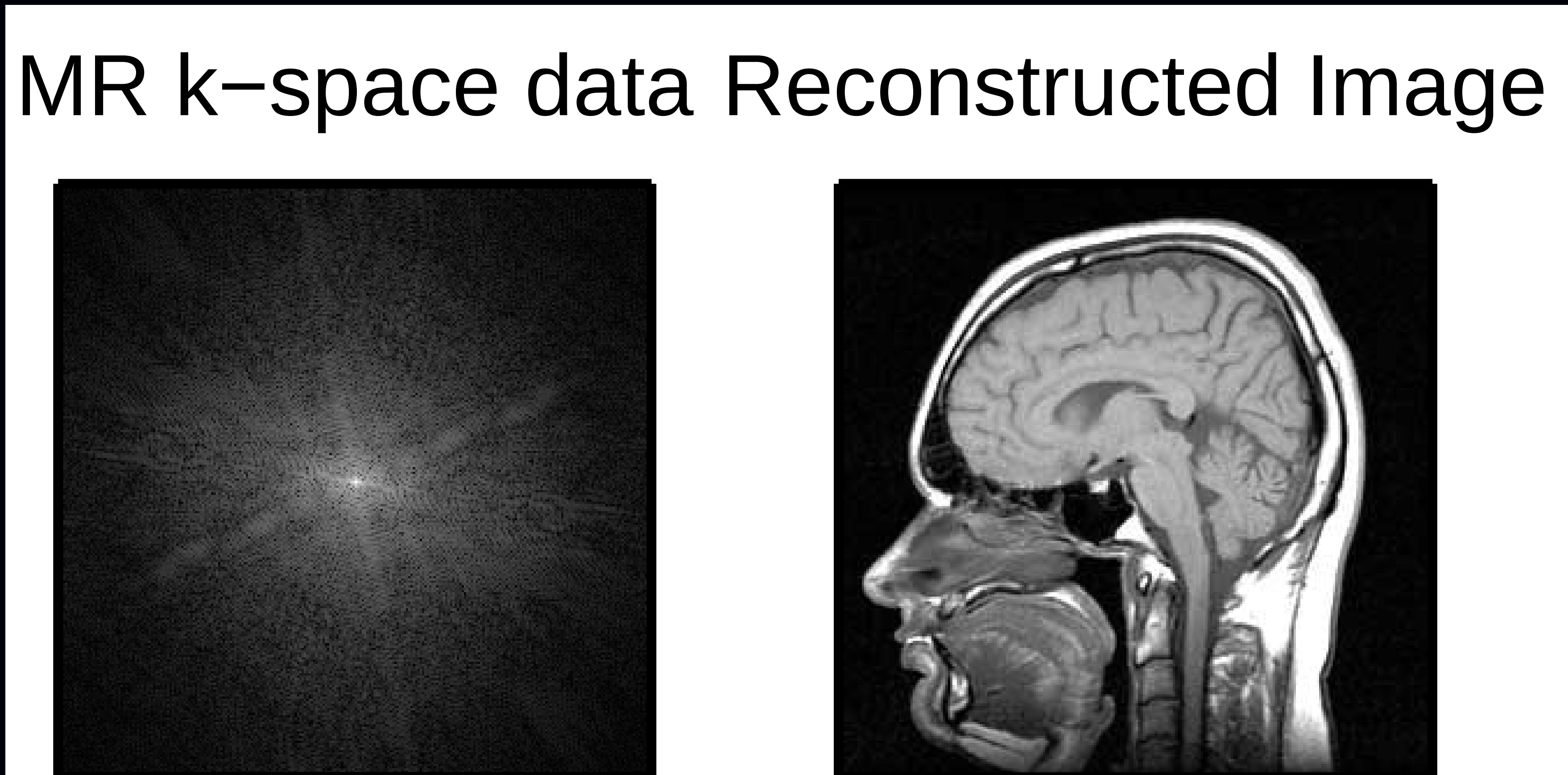
Image reconstruction toolbox:
`http://www.eecs.umich.edu/~fessler`

Example: Iterative Reconstruction under ΔB_0



Introduction to Reconstruction

Standard MR Image Reconstruction



Cartesian sampling in k-space. An inverse FFT. End of story.

Commercial MR system quotes 400 FFTs (256^2) per second.

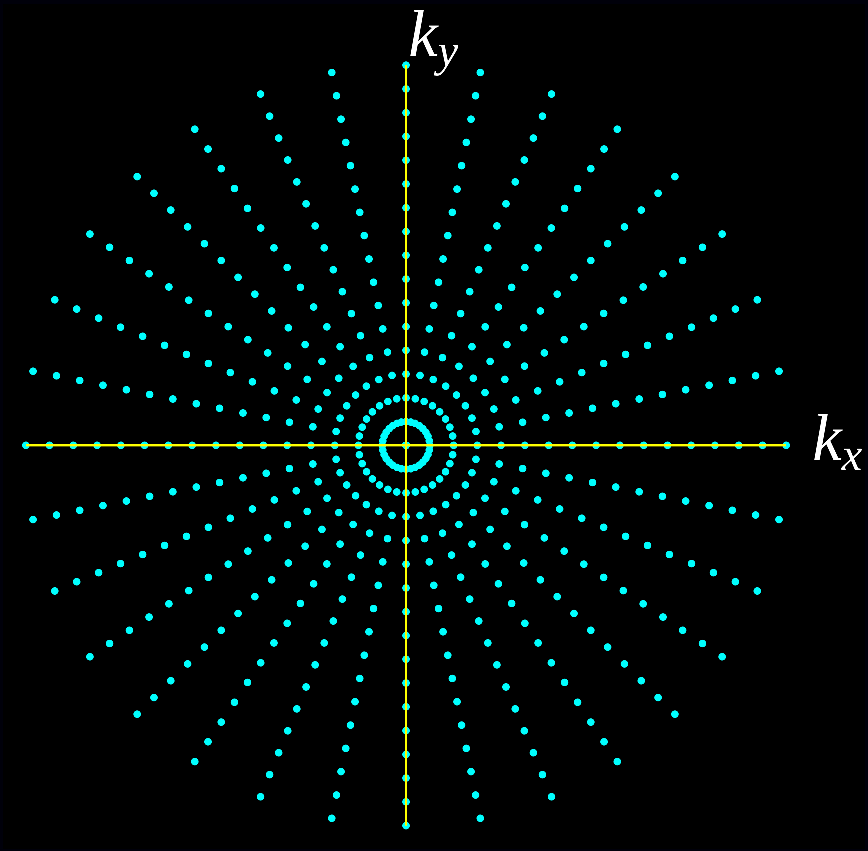
Non-Cartesian MR Image Reconstruction

"k-space" data

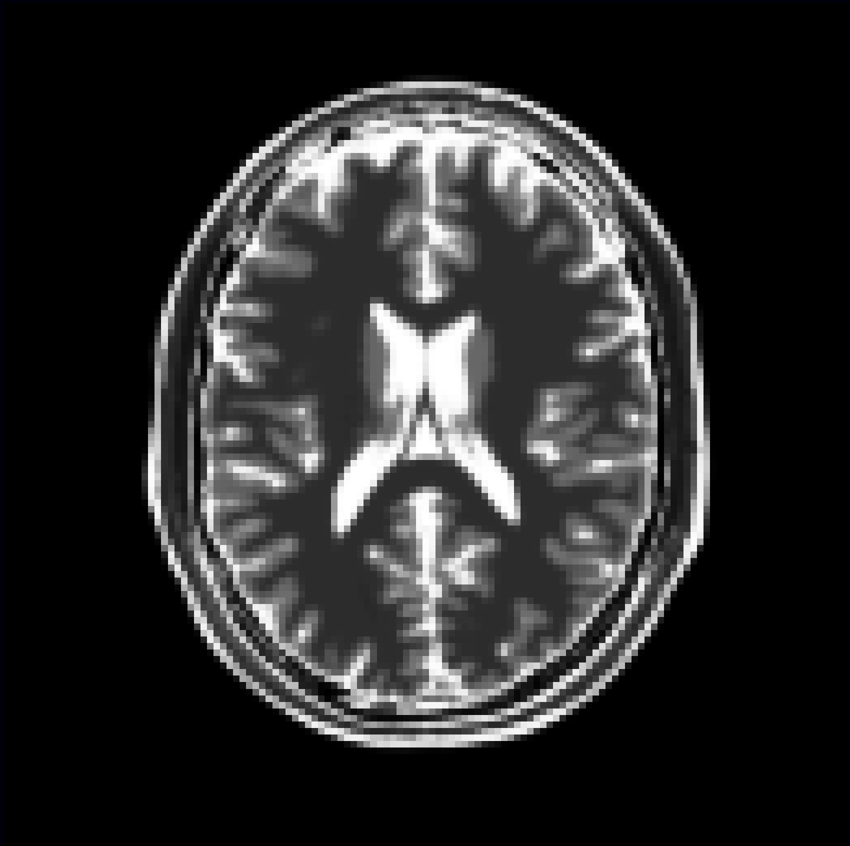
$\boldsymbol{y} = (y_1, \ldots, y_M)$

image

$f(\vec{r})$



$\Longrightarrow$



k-space trajectory:

$\vec{\kappa}(t) = (k_x(t), k_y(t))$

spatial coordinates:

$\vec{r} \in \mathbb{R}^{\bar{d}}$

Textbook MRI Measurement Model

Ignoring *lots* of things, the standard measurement model is:

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, M$$

$$s(t) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r} = F(\vec{\kappa}(t)).$$

$\vec{r}$: spatial coordinates
$\vec{\kappa}(t)$: k-space trajectory of the MR pulse sequence
$f(\vec{r})$: object's unknown transverse magnetization
$F(\vec{\kappa})$: Fourier transform of $f(\vec{r})$. We get noisy samples of this!
$e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}$ provides spatial information $\Longrightarrow$ Nobel Prize

Goal of image reconstruction: find $f(\vec{r})$ from measurements $\{y_i\}_{i=1}^{M}$.

The unknown object $f(\vec{r})$ is a continuous-space function,
but the recorded measurements $\boldsymbol{y} = (y_1, \ldots, y_M)$ are finite.

Under-determined (ill posed) problem $\Longrightarrow$ no canonical solution.

All MR scans provide only "partial" k-space data.

Image Reconstruction Strategies

- Continuous-continuous formulation

  Pretend that a continuum of measurements are available:

$$F(\vec{\kappa}) = \int f(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, \mathrm{d}\vec{r}\,.$$

  The “solution” is an inverse Fourier transform:

$$f(\vec{r}) = \int F(\vec{\kappa})\, \mathrm{e}^{\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, \mathrm{d}\vec{\kappa}\,.$$

  Now discretize the integral solution:

$$\hat{f}(\vec{r}) = \sum_{i=1}^{M} F(\vec{\kappa}_i)\, \mathrm{e}^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}}\, w_i \approx \sum_{i=1}^{M} y_i w_i\, \mathrm{e}^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}},$$

  where w_i values are “sampling density compensation factors.”
  Numerous methods for choosing w_i values in the literature.
  For Cartesian sampling, using $w_i = 1/N$ suffices,
  and the summation is an inverse FFT.
  For non-Cartesian sampling, replace summation with gridding.

- Continuous-discrete formulation

  Use many-to-one linear model:

  $$\boldsymbol{y} = \mathcal{A}\, f + \boldsymbol{\varepsilon}, \text{ where } \mathcal{A} : \mathcal{L}_2(\mathbb{R}^{\bar{d}}) \to \mathbb{C}^M.$$

  Minimum norm solution (*cf.* “natural pixels”):

  $$\min_{\hat{f}} \|\hat{f}\|_2 \text{ subject to } \boldsymbol{y} = \mathcal{A}\, \hat{f}$$

  $$\hat{f} = \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\boldsymbol{y} = \textstyle\sum_{i=1}^{M} c_i \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}, \text{ where } \mathcal{A}\mathcal{A}^* \boldsymbol{c} = \boldsymbol{y}.$$

- Discrete-discrete formulation

  Assume parametric model for object:

  $$f(\vec{r}) = \sum_{j=1}^{N} f_j\, p_j(\vec{r}).$$

  Estimate parameter vector $\boldsymbol{f} = (f_1, \ldots, f_N)$ from data vector $\boldsymbol{y}$.

Model-Based Image Reconstruction: Overview

Model-Based Image Reconstruction

MR signal equation with more complete physics:

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\,\omega(\vec{r})\,t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, M$$

- $s^{\text{coil}}(\vec{r})$ Receive-coil sensitivity pattern(s) (for SENSE)
- $\omega(\vec{r})$ Off-resonance frequency map
  (due to field inhomogeneity / magnetic susceptibility)
- $R_2^*(\vec{r})$ Relaxation map

Other physical factors (?)

- Eddy current effects; in $\vec{\kappa}(t)$
- Concomitant gradient terms
- Chemical shift
- Motion

Goal?

(it depends)

Field Inhomogeneity-Corrected Reconstruction

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r})\, t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given field map $\omega(\vec{r})$.
(Assume all other terms are known or unimportant.)

Motivation

Essential for functional MRI of brain regions near sinus cavities!

(Sutton *et al.*, ISMRM 2001; T-MI 2003)

Sensitivity-Encoded (SENSE) Reconstruction

$$s(t) = \int f(\vec{r}) s^{\text{coil}}(\vec{r})\, e^{-\imath\omega(\vec{r})\,t}\, e^{-R_2^*(\vec{r})\,t}\, e^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given sensitivity maps $s^{\text{coil}}(\vec{r})$.
(Assume all other terms are known or unimportant.)

Can combine SENSE with field inhomogeneity correction "easily."

(Sutton *et al.*, ISMRM 2001, Olafsson *et al.*, ISBI 2006)

Joint Estimation of Image and Field-Map

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r})\, t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate *both* the image $f(\vec{r})$ and the field map $\omega(\vec{r})$
(Assume all other terms are known or unimportant.)

Analogy:
joint estimation of emission image and attenuation map in PET.

(Sutton *et al.*, ISMRM Workshop, 2001; ISBI 2002; ISMRM 2002; ISMRM 2003; MRM 2004)

The Kitchen Sink

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate image $f(\vec{r})$, field map $\omega(\vec{r})$, *and* relaxation map $R_2^*(\vec{r})$

Requires “suitable” k-space trajectory.

(Sutton *et al.*, ISMRM 2002; Twieg, MRM, 2003)

Estimation of Dynamic Rate Maps

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate dynamic field map $\omega(\vec{r})$ and “BOLD effect” $R_2^*(\vec{r})$ given baseline image $f(\vec{r})$ in fMRI.

Motion...

(Olafsson *et al.*, IEEE T-MI 2008)

Model-Based Image Reconstruction: Details

Basic Signal Model

$$y_i = s(t_i) + \varepsilon_i, \qquad \mathsf{E}[y_i] = s(t_i) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ from $\boldsymbol{y} = (y_1, \ldots, y_M)$.

Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{N} f_j\, p(\vec{r} - \vec{r}_j) \longleftarrow \text{usually 2D rect functions.}$$

Substituting into signal model yields

$$\mathsf{E}[y_i] = \int \left[\sum_{j=1}^{N} f_j\, p(\vec{r} - \vec{r}_j) \right] e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, d\vec{r} = \sum_{j=1}^{N} \left[\int p(\vec{r} - \vec{r}_j)\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, d\vec{r} \right] f_j$$

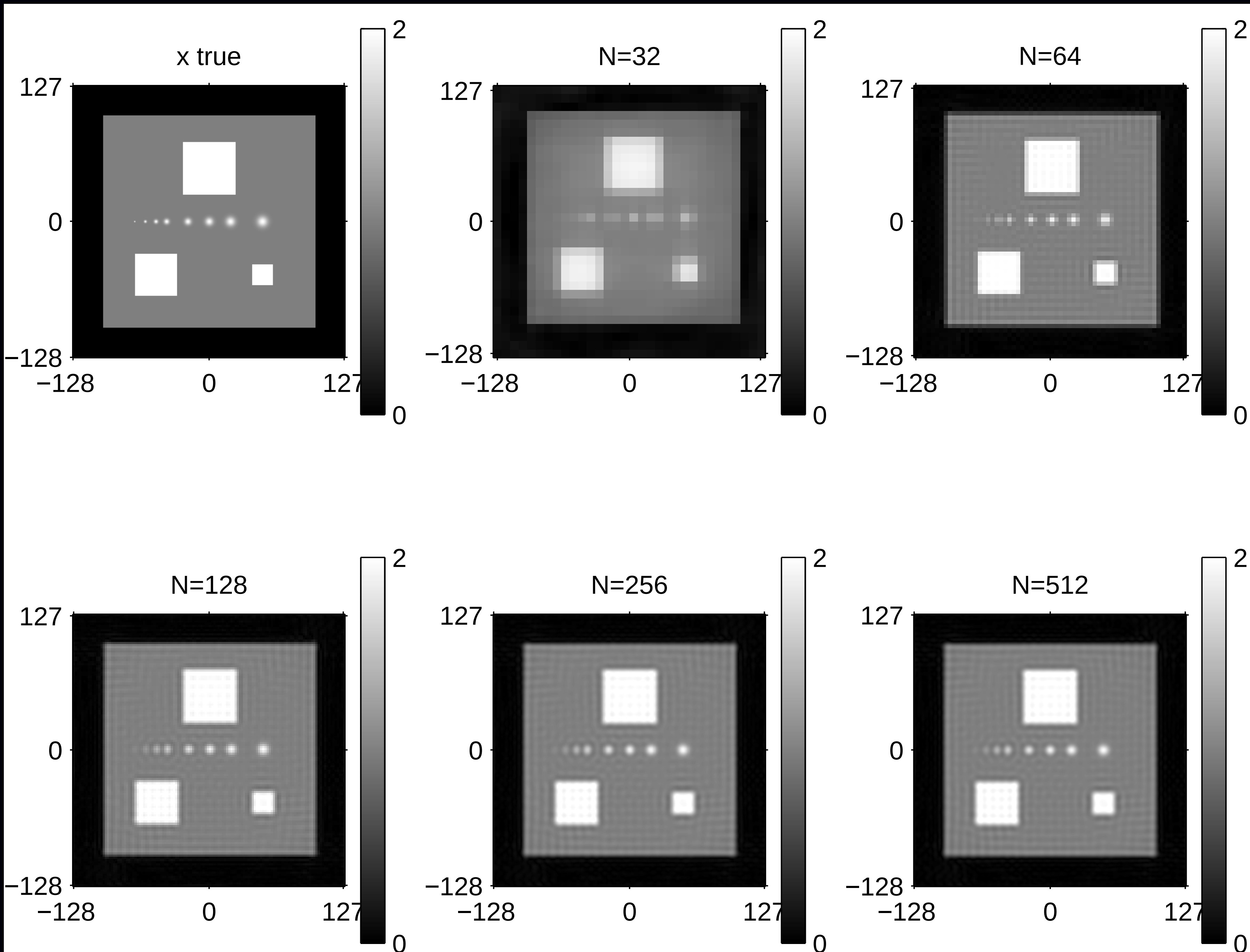
$$= \sum_{j=1}^{N} a_{ij} f_j, \qquad a_{ij} = P(\vec{\kappa}_i)\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}, \qquad p(\vec{r}) \overset{\text{FT}}{\Longleftrightarrow} P(\vec{\kappa}).$$

Discrete-discrete measurement model with system matrix $\boldsymbol{A} = \{a_{ij}\}$:

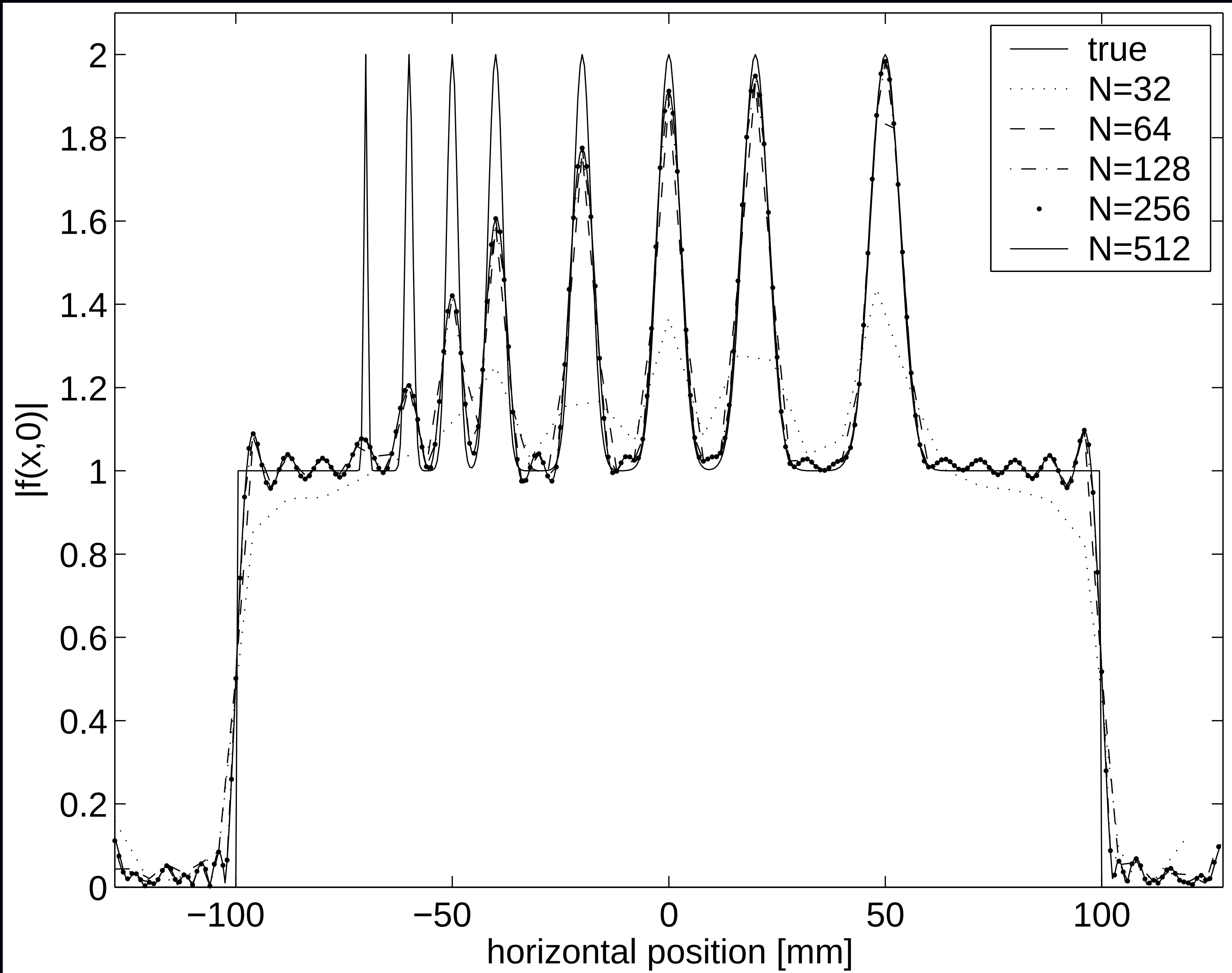
$$\boxed{\boldsymbol{y} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\varepsilon}.}$$

Goal: estimate coefficients (pixel values) $\boldsymbol{f} = (f_1, \ldots, f_N)$ from $\boldsymbol{y}$.

Small Pixel Size Need Not Matter



Profiles


true
N=32
N=64
N=128
N=256
N=512
|f(x,0)|
horizontal position [mm]
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−100
−50
0
50
100

Least-Squares Estimation

Estimate object by minimizing a simple cost function:

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f} \in \mathbb{C}^N} \Psi(\boldsymbol{f}), \qquad \Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$$

- data fit term $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$
  corresponds to negative log-likelihood of Gaussian distribution
- Equivalent to maximum-likelihood (ML) estimation

Issues:

- computing minimizer rapidly
- stopping iteration (?)
- image quality

Iterative Minimization by Conjugate Gradients

Choose initial guess $\boldsymbol{f}^{(0)}$ (*e.g.*, fast conjugate phase / gridding).
Iteration (unregularized):

$$\boldsymbol{g}^{(n)} = \nabla\Psi\left(\boldsymbol{f}^{(n)}\right) = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \quad \text{gradient}$$

$$\boldsymbol{p}^{(n)} = \boldsymbol{P}\boldsymbol{g}^{(n)} \quad \text{precondition}$$

$$\gamma_n = \begin{cases} 0, & n = 0 \\ \dfrac{\langle \boldsymbol{g}^{(n)}, \boldsymbol{p}^{(n)} \rangle}{\langle \boldsymbol{g}^{(n-1)}, \boldsymbol{p}^{(n-1)} \rangle}, & n > 0 \end{cases}$$

$$\boldsymbol{d}^{(n)} = -\boldsymbol{p}^{(n)} + \gamma_n \boldsymbol{d}^{(n-1)} \quad \text{search direction}$$

$$\boldsymbol{v}^{(n)} = \boldsymbol{A}\boldsymbol{d}^{(n)}$$

$$\alpha_n = \langle \boldsymbol{d}^{(n)}, -\boldsymbol{g}^{(n)} \rangle / \|\boldsymbol{v}^{(n)}\|^2 \quad \text{step size}$$

$$\boldsymbol{f}^{(n+1)} = \boldsymbol{f}^{(n)} + \alpha_n \boldsymbol{d}^{(n)} \quad \text{update}$$

Bottlenecks: computing $\boldsymbol{A}\boldsymbol{f}^{(n)}$ and $\boldsymbol{A}'\boldsymbol{r}$.

- $\boldsymbol{A}$ is too large to store explicitly (not sparse)
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{f}$ is $O(MN)$ per iteration, whereas FFT is only $O(M \log M)$.

Computing $\boldsymbol{A f}$ Rapidly

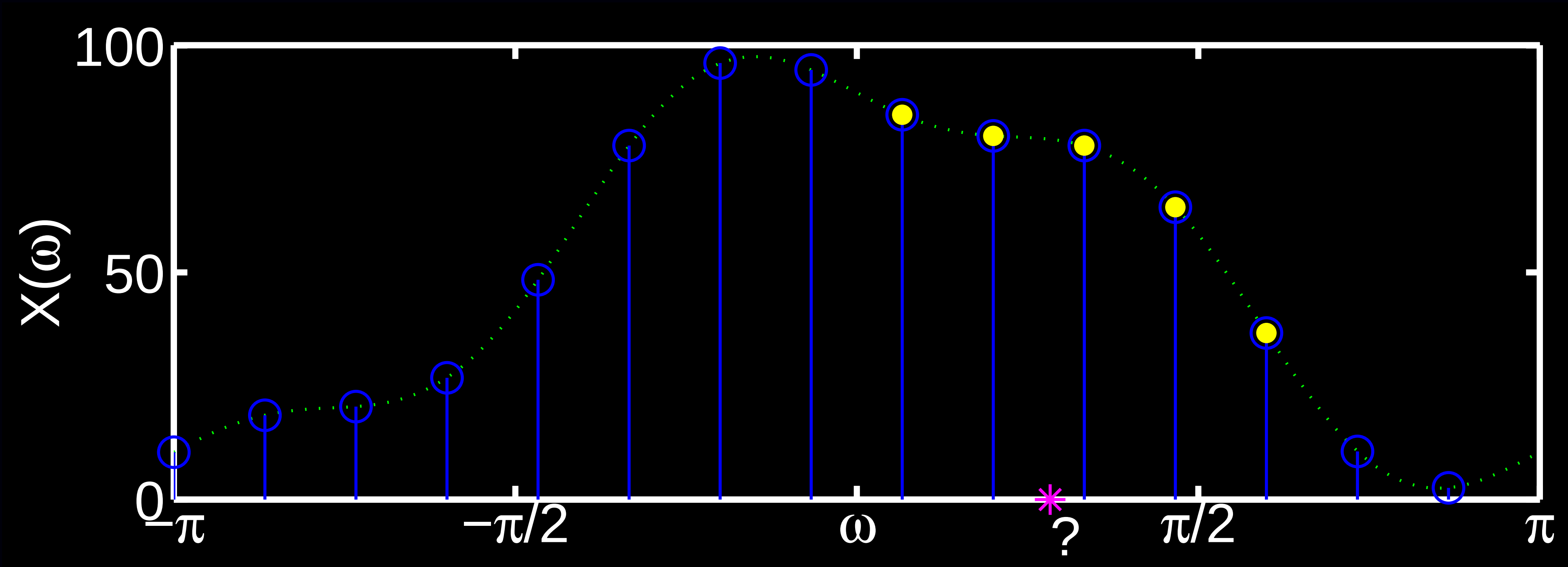
$$[\boldsymbol{A f}]_i = \sum_{j=1}^{N} a_{ij} f_j = P(\vec{\kappa}_i) \sum_{j=1}^{N} \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j} f_j, \qquad i = 1, \ldots, M$$

- Pixel locations $\{\vec{r}_j\}$ are uniformly spaced
- k-space locations $\{\vec{\kappa}_i\}$ are unequally spaced

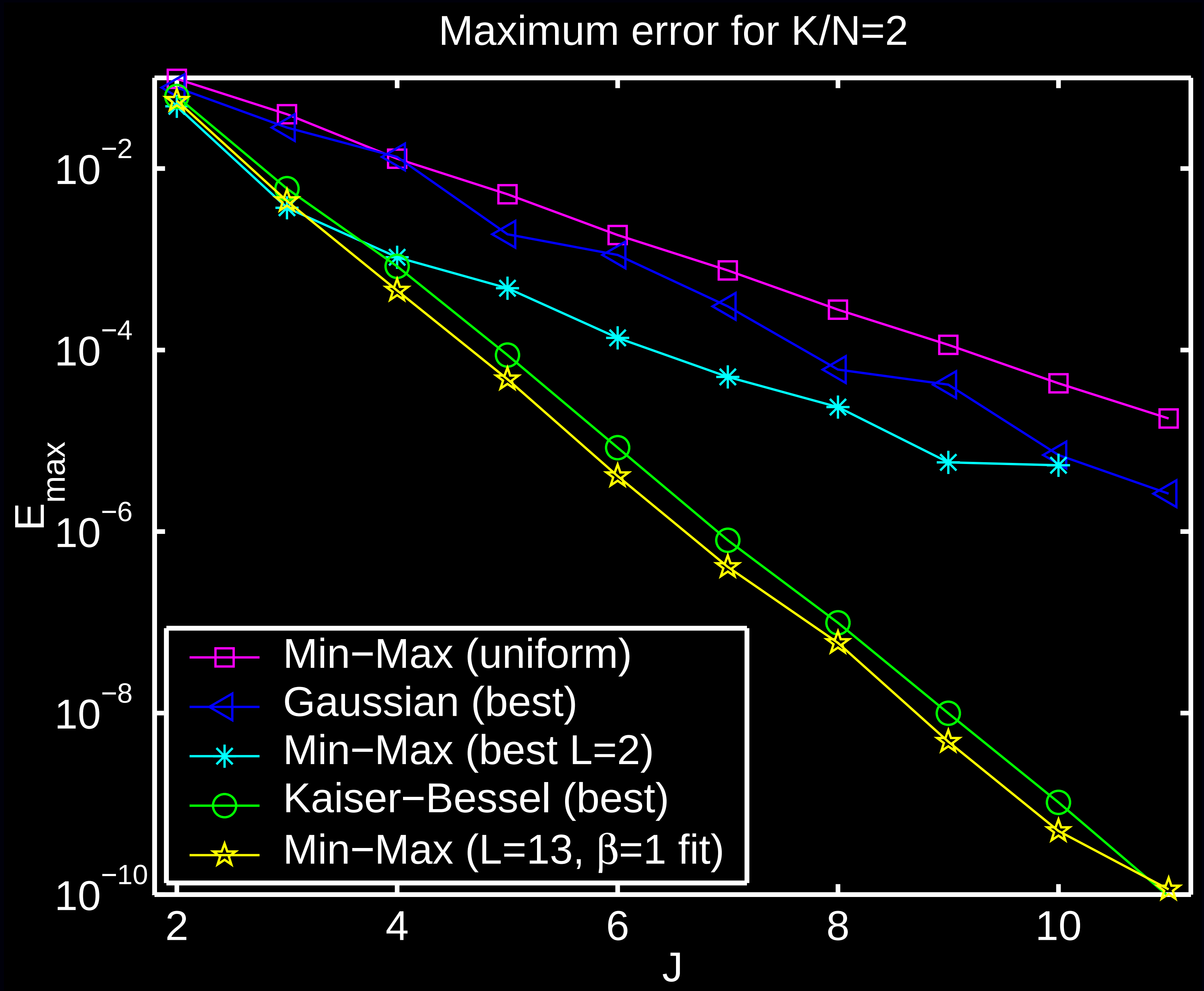
$\Longrightarrow$ needs nonuniform fast Fourier transform (NUFFT)

NUFFT (Type 2)

- Compute over-sampled FFT of equally-spaced signal samples
- Interpolate onto desired unequally-spaced frequency locations
- Dutt & Rokhlin, SIAM JSC, 1993, Gaussian bell interpolator
- Fessler & Sutton, IEEE T-SP, 2003, min-max interpolator and min-max optimized Kaiser-Bessel interpolator.
  NUFFT toolbox: `http://www.eecs.umich.edu/~fessler/code`



Worst-Case NUFFT Interpolation Error



Further Acceleration using Toeplitz Matrices

Cost-function gradient:

$$\begin{aligned} \boldsymbol{g}^{(n)} &= \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \\ &= \boldsymbol{T}\boldsymbol{f}^{(n)} - \boldsymbol{b}, \end{aligned}$$

where

$$\boldsymbol{T} \triangleq \boldsymbol{A}'\boldsymbol{A}, \qquad \boldsymbol{b} \triangleq \boldsymbol{A}'\boldsymbol{y}.$$

In the absence of field inhomogeneity, the Gram matrix $\boldsymbol{T}$ is Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2 \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}.$$

Computing $\boldsymbol{T}\boldsymbol{f}^{(n)}$ requires an ordinary ($2\times$ over-sampled) FFT.

(Chan & Ng, SIAM Review, 1996)

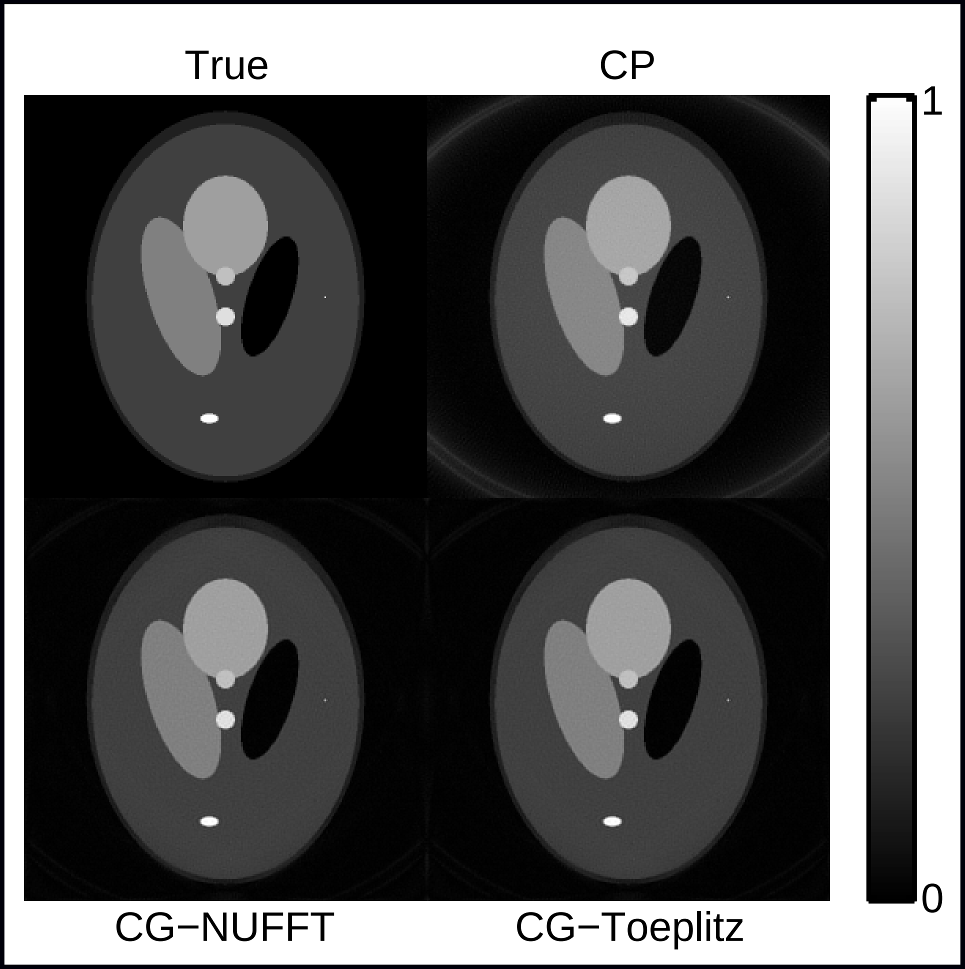
In 2D: block Toeplitz with Toeplitz blocks (BTTB).

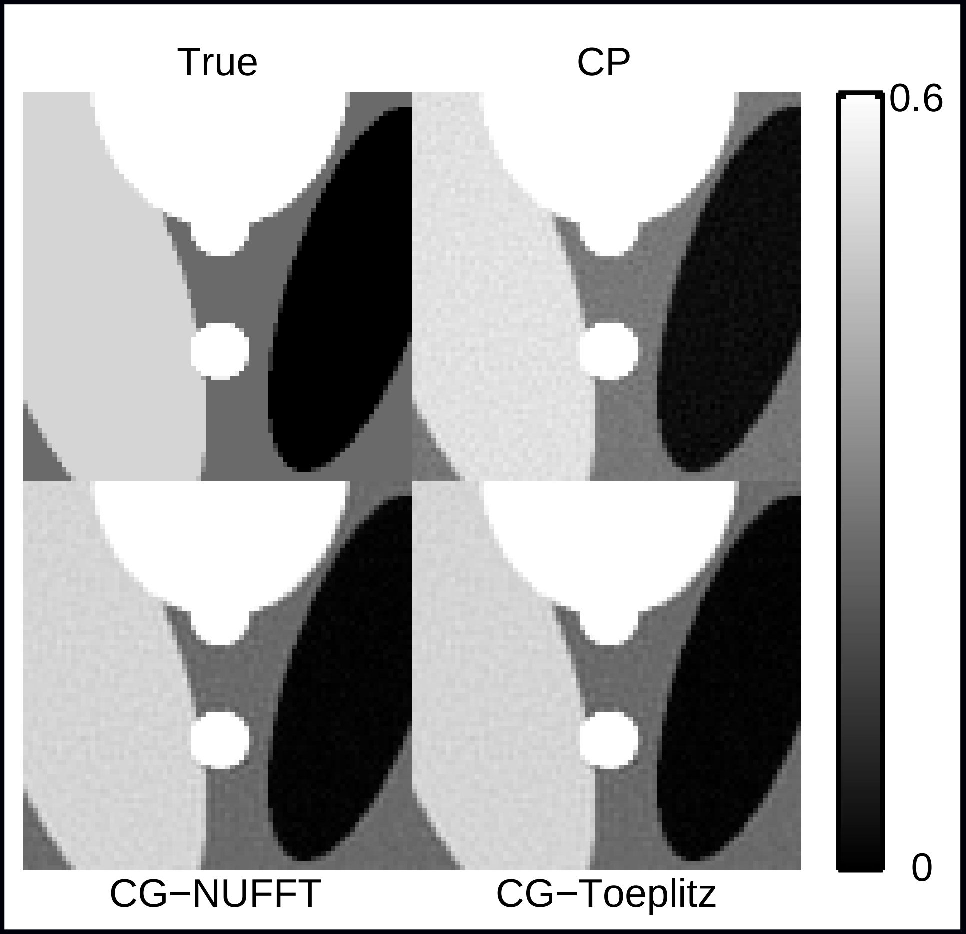
Precomputing the first column of $\boldsymbol{T}$ and $\boldsymbol{b}$ requires a couple NUFFTs.
(Wajer, ISMRM 2001, Eggers ISMRM 2002, Liu ISMRM 2005)

This formulation seems ideal for “hardware” FFT systems.

Toeplitz Acceleration

Example: 256^2 image. radial trajectory, $2\times$ angular under-sampling.





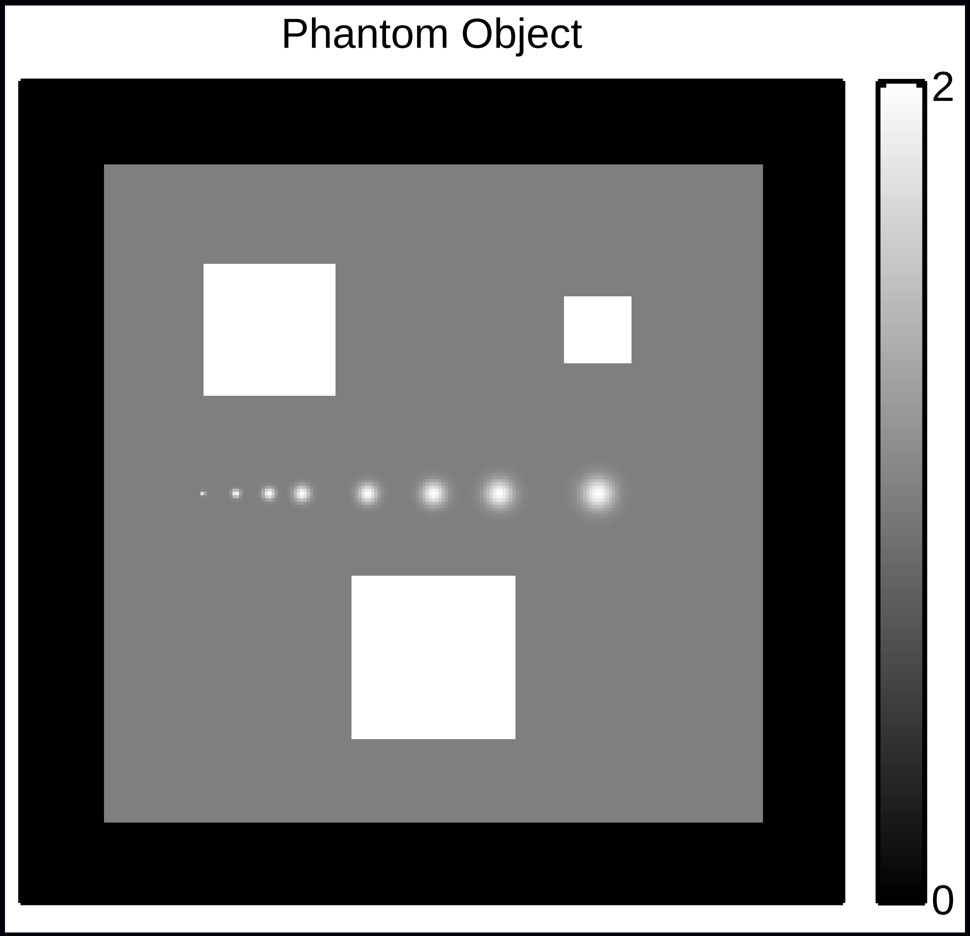
(Iterative provides reduced aliasing energy.)

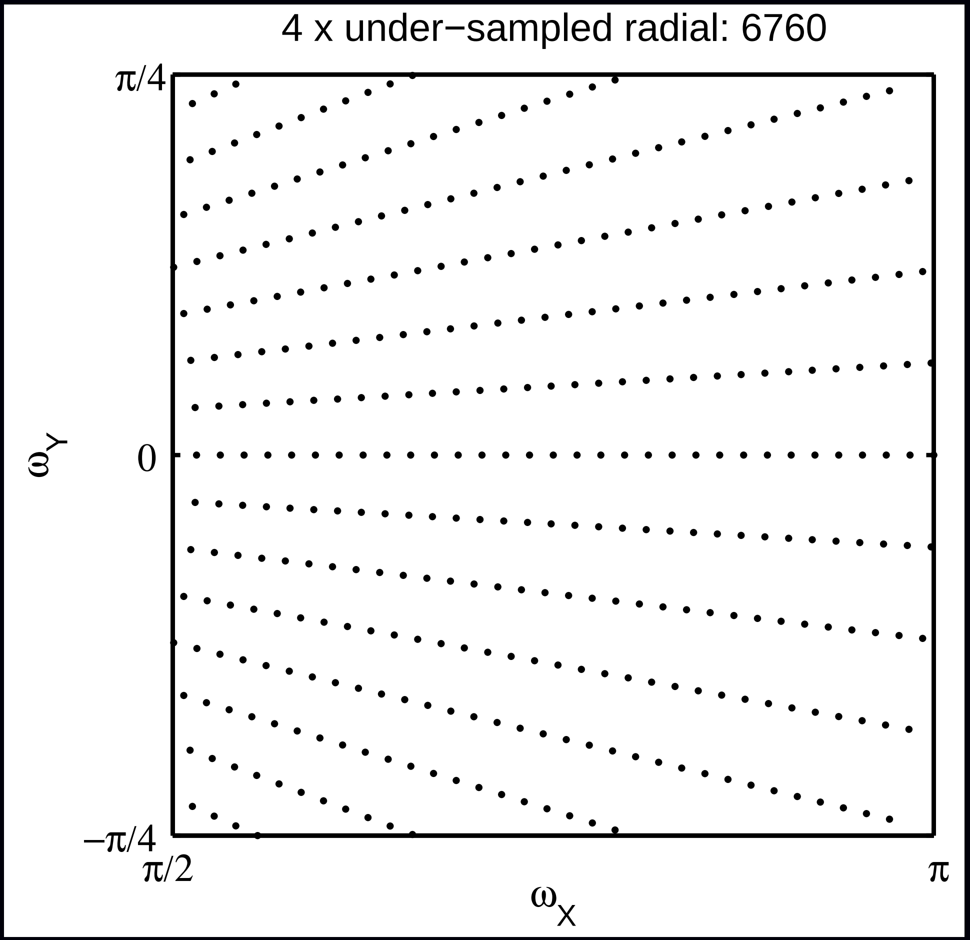
Toeplitz Acceleration

Method	$A'Dy$	$b = A'y$	T	20 iter	Total Time	NRMS (50dB)
Conj. Phase	0.3				0.3	7.8%
CG-NUFFT				12.5	12.5	4.1%
CG-Toeplitz		0.3	0.8	3.5	4.6	4.1%

- Toeplitz aproach reduces CPU time by more than $2\times$ on conventional workstation (Xeon 3.4GHz)
- Eliminates k-space interpolations $\Longrightarrow$ ideal for FFT hardware
- No SNR compromise
- CG reduces NRMS error relative to CP, but $15\times$ slower... (More dramatic improvements seen in fMRI when correcting for field inhomogeneity.)

Unregularized Example: Simulated Data

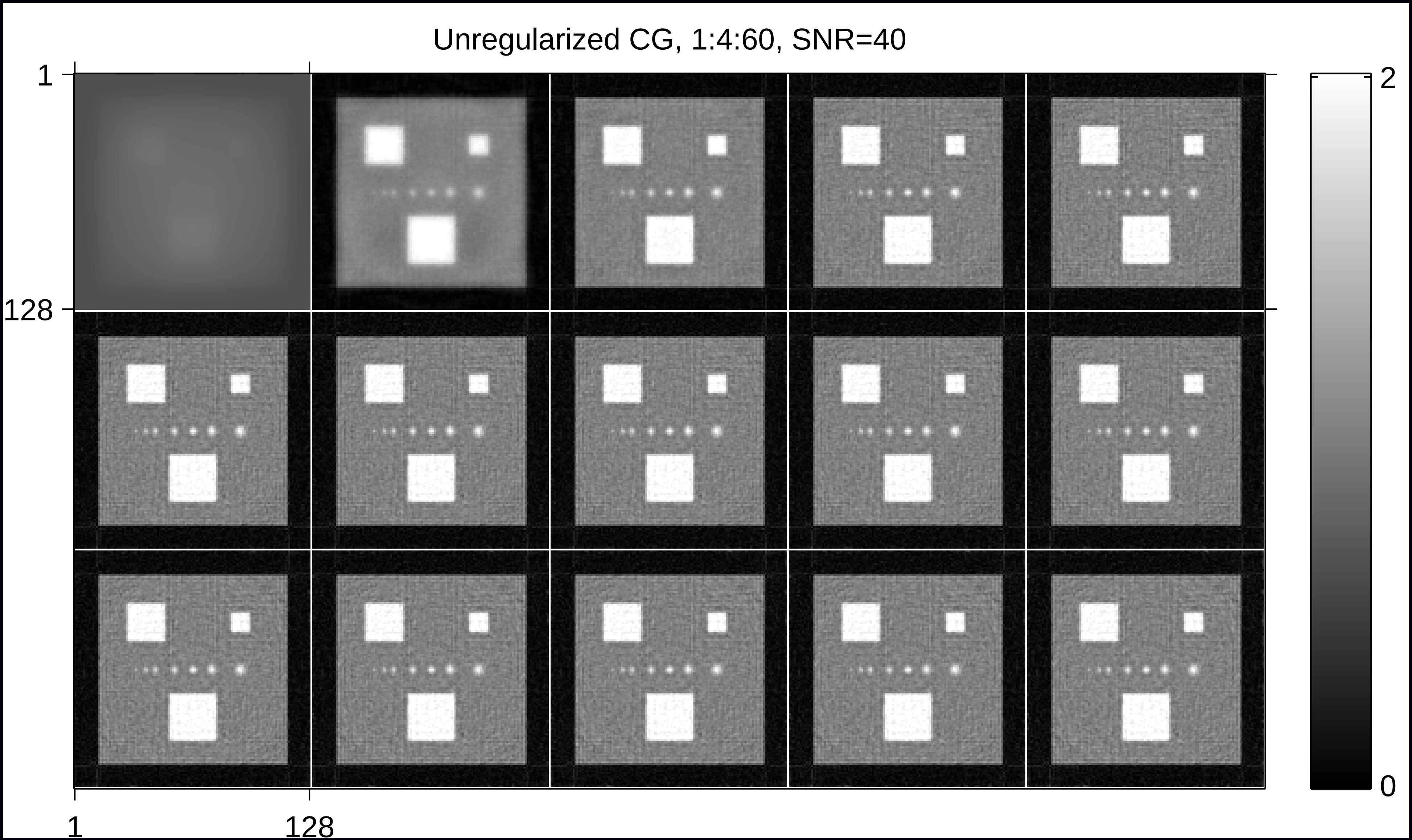




4× under-sampled radial k-space data

Analytical k-space data generation

Unregularized Example: Images

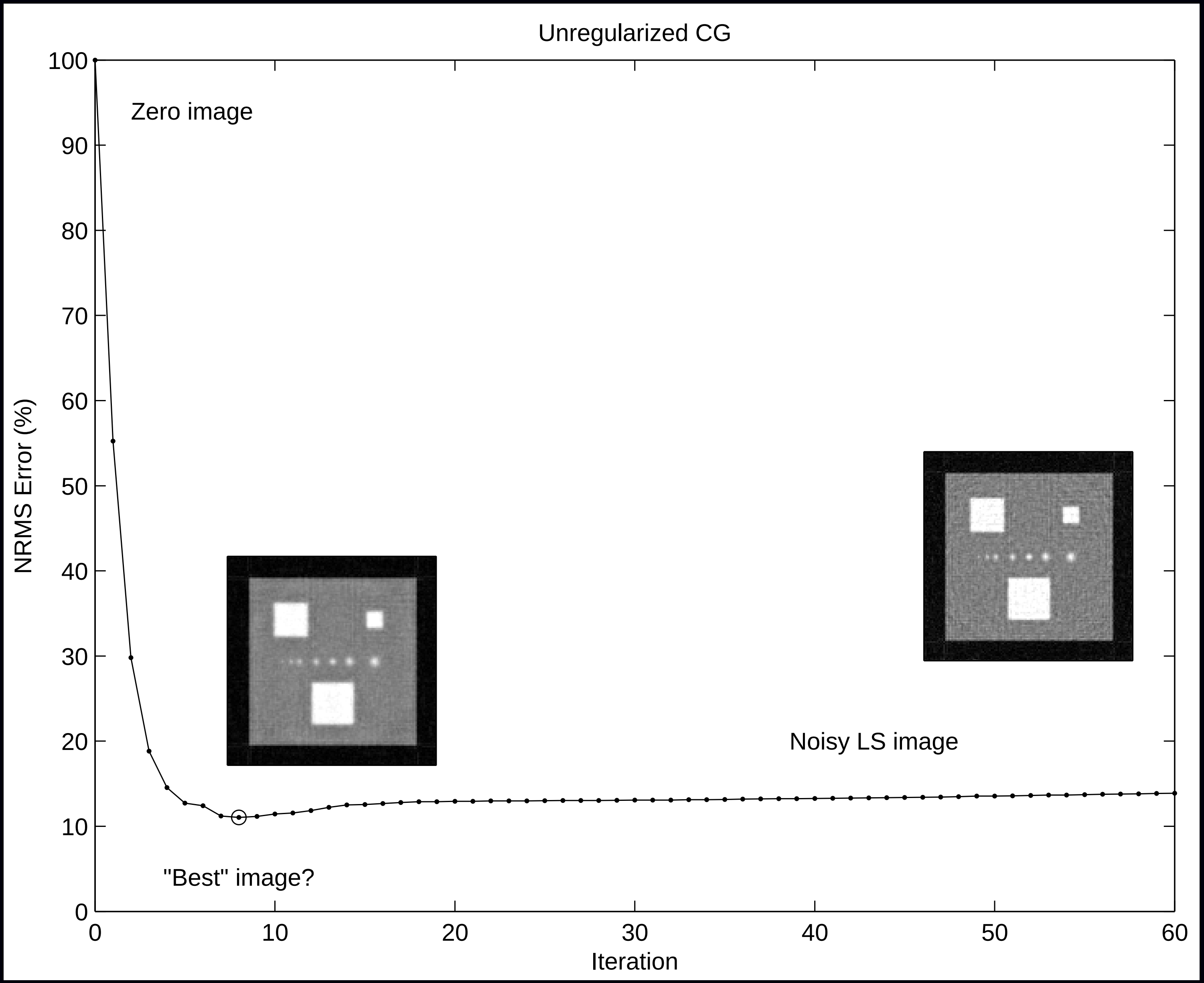


Iterations 1:4:60 of unregularized CG reconstruction

Unregularized Example: Movie

(movie in pdf)

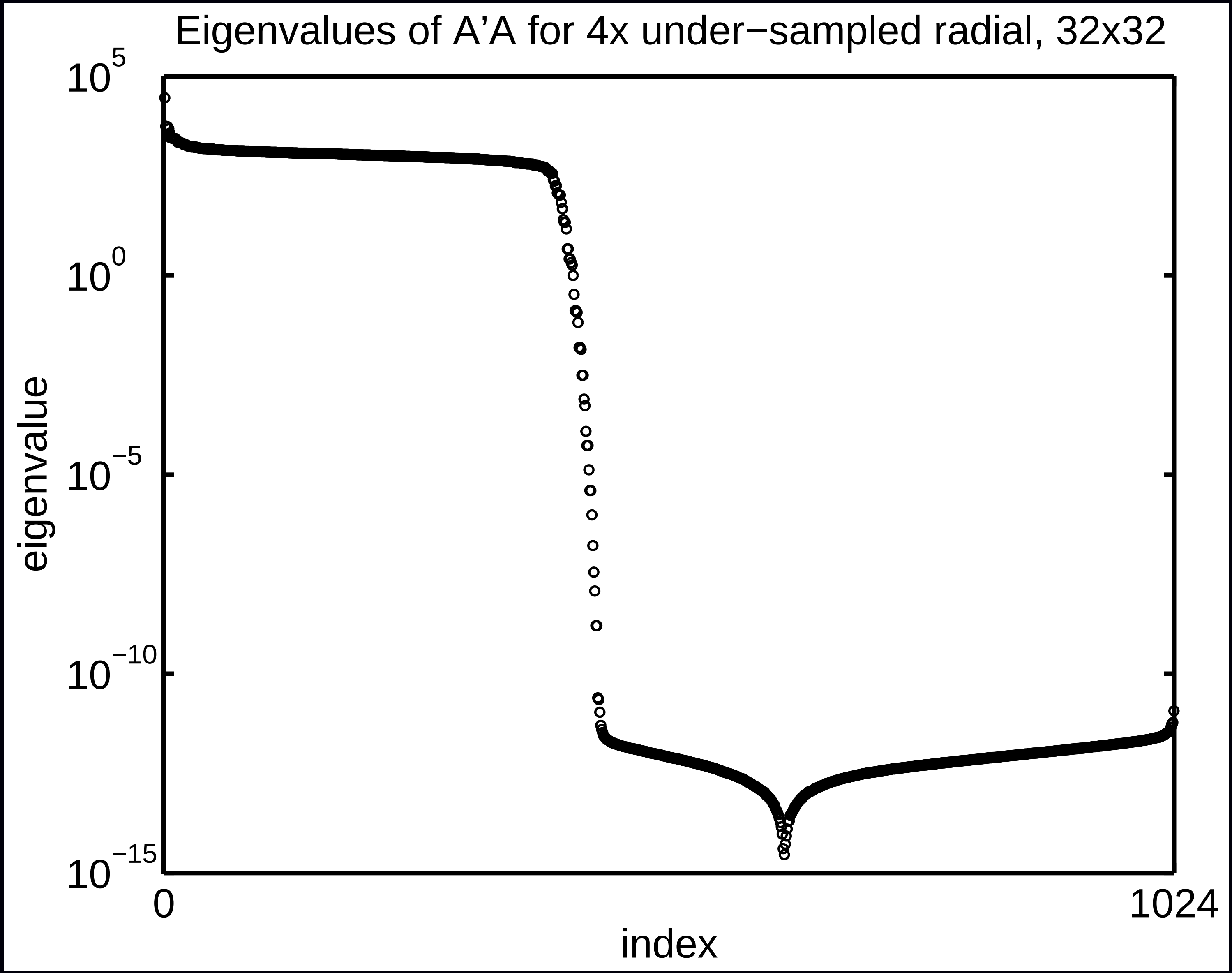
Unregularized Example: RMS Error



Complexity: when to stop?

A solution: regularization.

Unregularized Eigenspectrum

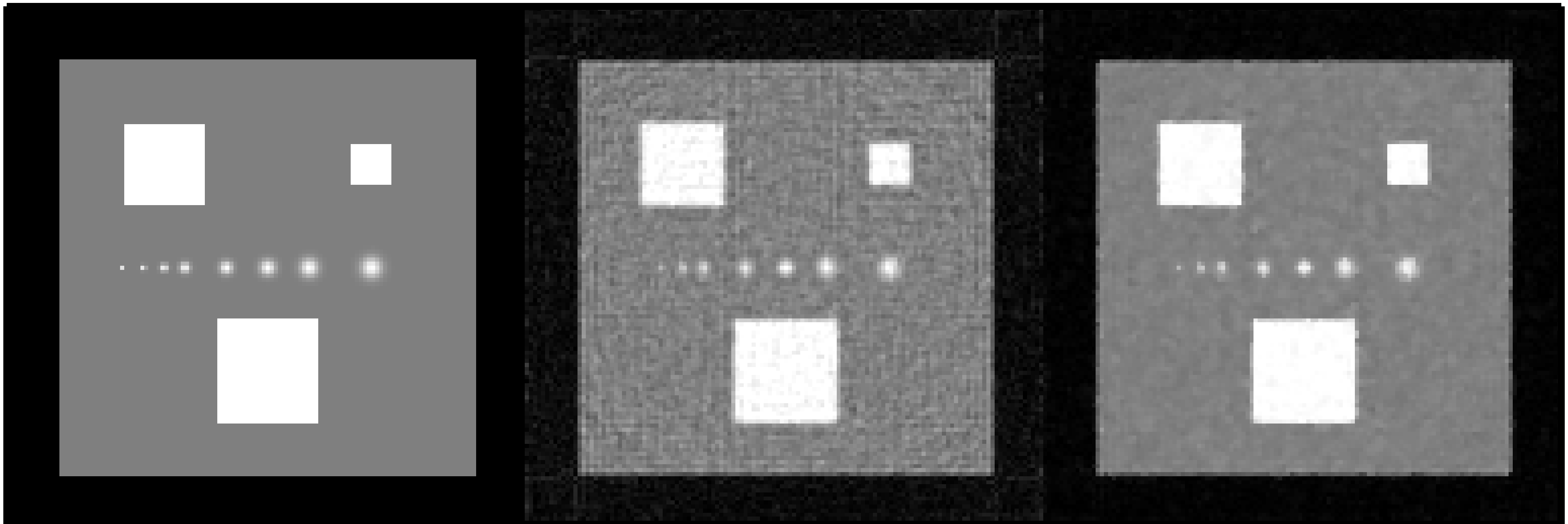


Regularized Example: Movie

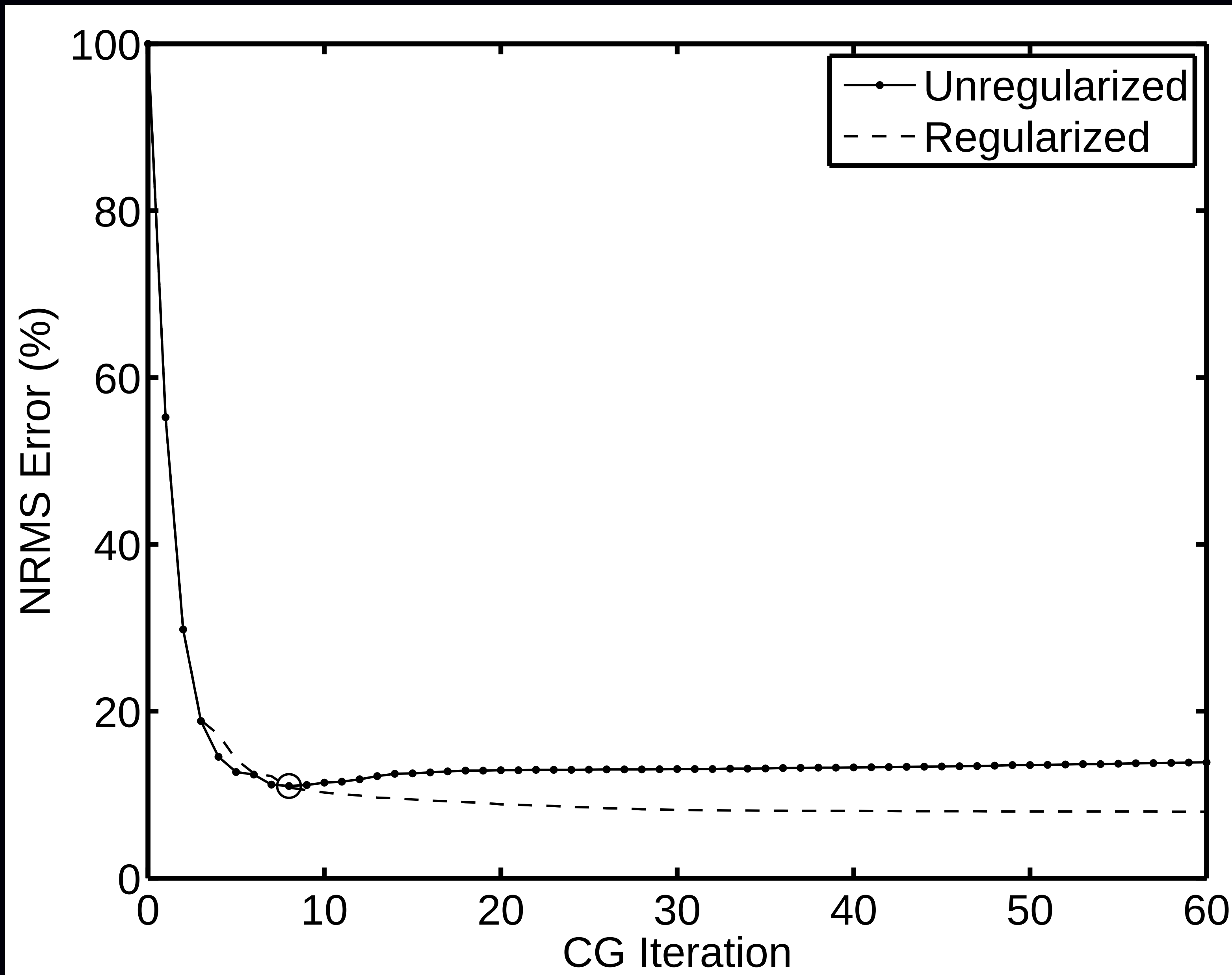
(movie in pdf)

Regularized Example: Image Comparison

True | Unregularized | Edge preserving regularization



Regularized Example: RMS Error



Regularized Least-Squares Estimation

Estimate object by minimizing a *regularized* cost function:

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f} \in \mathbb{C}^N} \Psi(\boldsymbol{f}), \qquad \Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha \mathsf{R}(\boldsymbol{f})$$

- data fit term $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$
  corresponds to negative log-likelihood of Gaussian distribution
- regularizing term $\mathsf{R}(\boldsymbol{f})$ controls noise by penalizing roughness,

$$\textit{e.g.}: \quad \mathsf{R}(\boldsymbol{f}) \approx \int \|\nabla f\|^2 \, \mathrm{d}\vec{r}$$

- regularization parameter $\alpha > 0$
  controls tradeoff between spatial resolution and noise
- Equivalent to Bayesian MAP estimation with prior $\propto \mathrm{e}^{-\alpha \mathsf{R}(\boldsymbol{f})}$

Complexities:

- choosing $\mathsf{R}(f)$
- choosing α
- computing minimizer rapidly.

Quadratic regularization

1D example: squared differences between neighboring pixel values:

$$\mathsf{R}(f) = \sum_{j=2}^{N} \frac{1}{2} |f_j - f_{j-1}|^2 .$$

In matrix-vector notation, $\mathsf{R}(\boldsymbol{f}) = \frac{1}{2} \|\boldsymbol{C}\boldsymbol{f}\|^2$ where

$$\boldsymbol{C} = \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ & & & \ddots & \ddots & \\ 0 & \dots & 0 & 0 & -1 & 1 \end{bmatrix}, \text{ so } \boldsymbol{C}\boldsymbol{f} = \begin{bmatrix} f_2 - f_1 \\ \vdots \\ f_N - f_{N-1} \end{bmatrix}.$$

For 2D and higher-order differences, modify differencing matrix $\boldsymbol{C}$.

Leads to closed-form solution:

$$\begin{aligned} \hat{\boldsymbol{f}} &= \arg\min_{\boldsymbol{f}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha \|\boldsymbol{C}\boldsymbol{f}\|^2 \\ &= \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}. \end{aligned}$$

(a formula of limited practical use for computing $\hat{\boldsymbol{f}}$)

Choosing the Regularization Parameter

Spatial resolution analysis (Fessler & Rogers, IEEE T-IP, 1996):

$$\hat{f} = [A'A + \alpha C'C]^{-1} A'y$$

$$\mathsf{E}\left[\hat{f}\right] = [A'A + \alpha C'C]^{-1} A' \,\mathsf{E}[y]$$

$$\mathsf{E}\left[\hat{f}\right] = \underbrace{[A'A + \alpha C'C]^{-1} A'A}_{\text{blur}}\, f$$

$A'A$ and $C'C$ are Toeplitz $\Longrightarrow$ blur is approximately shift-invariant.

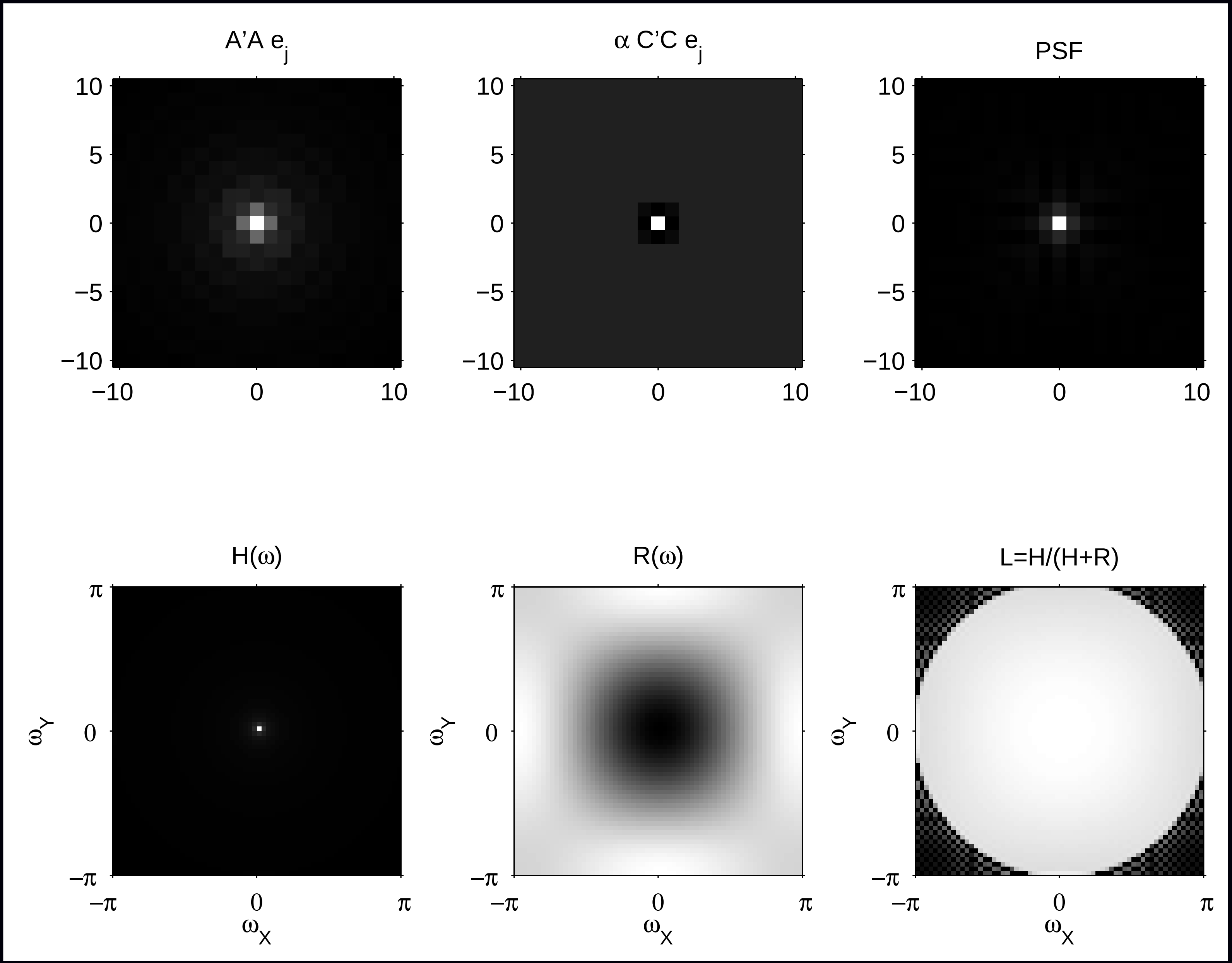
Frequency response of blur:

$$L(\omega) = \frac{H(\omega)}{H(\omega) + \alpha R(\omega)}$$

- $H(\omega_k) = \mathrm{FFT}(A'A\, e_j)$ (lowpass)
- $R(\omega_k) = \mathrm{FFT}(C'C\, e_j)$ (highpass)

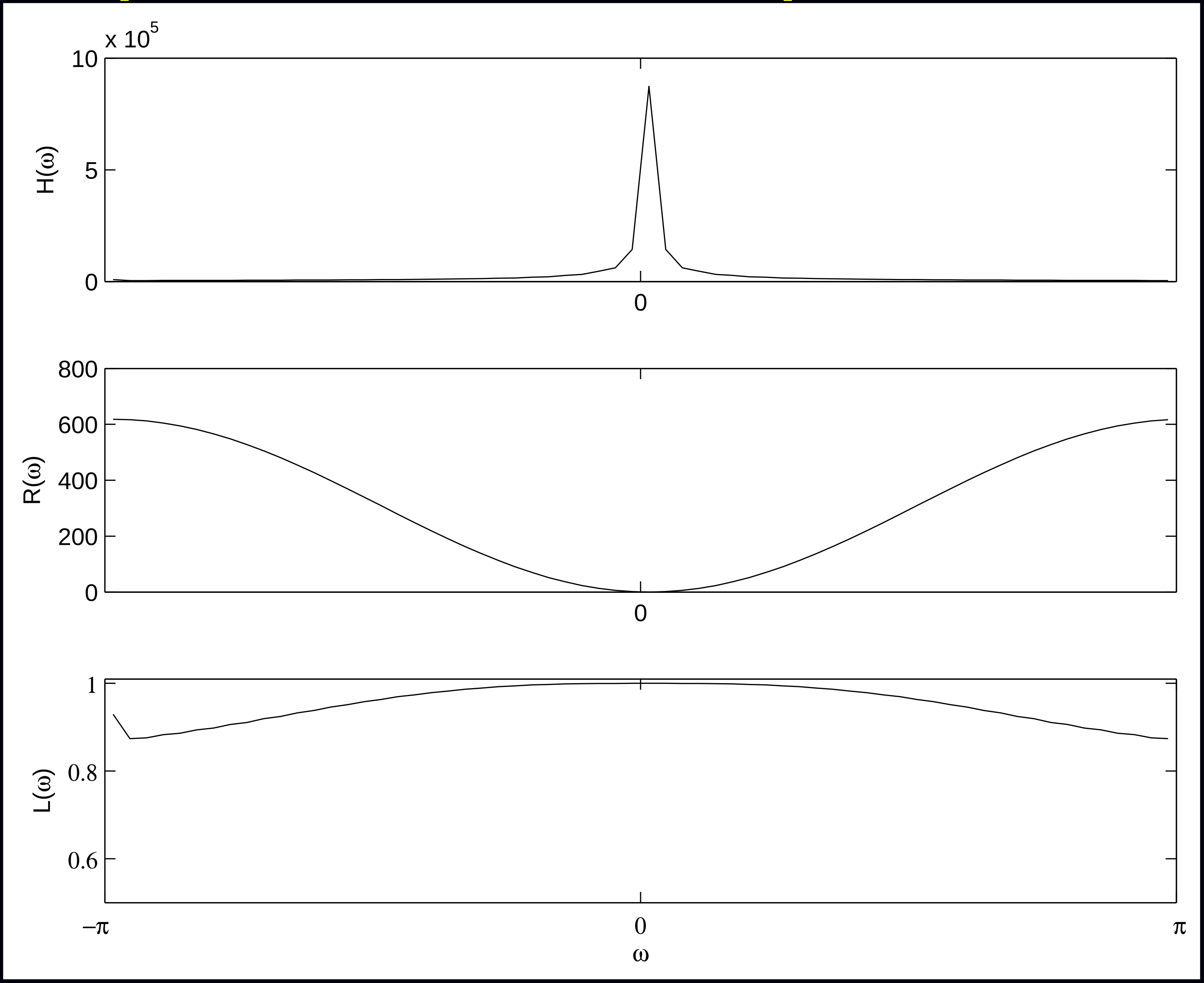
Adjust α to achieve desired spatial resolution.

Spatial Resolution Example

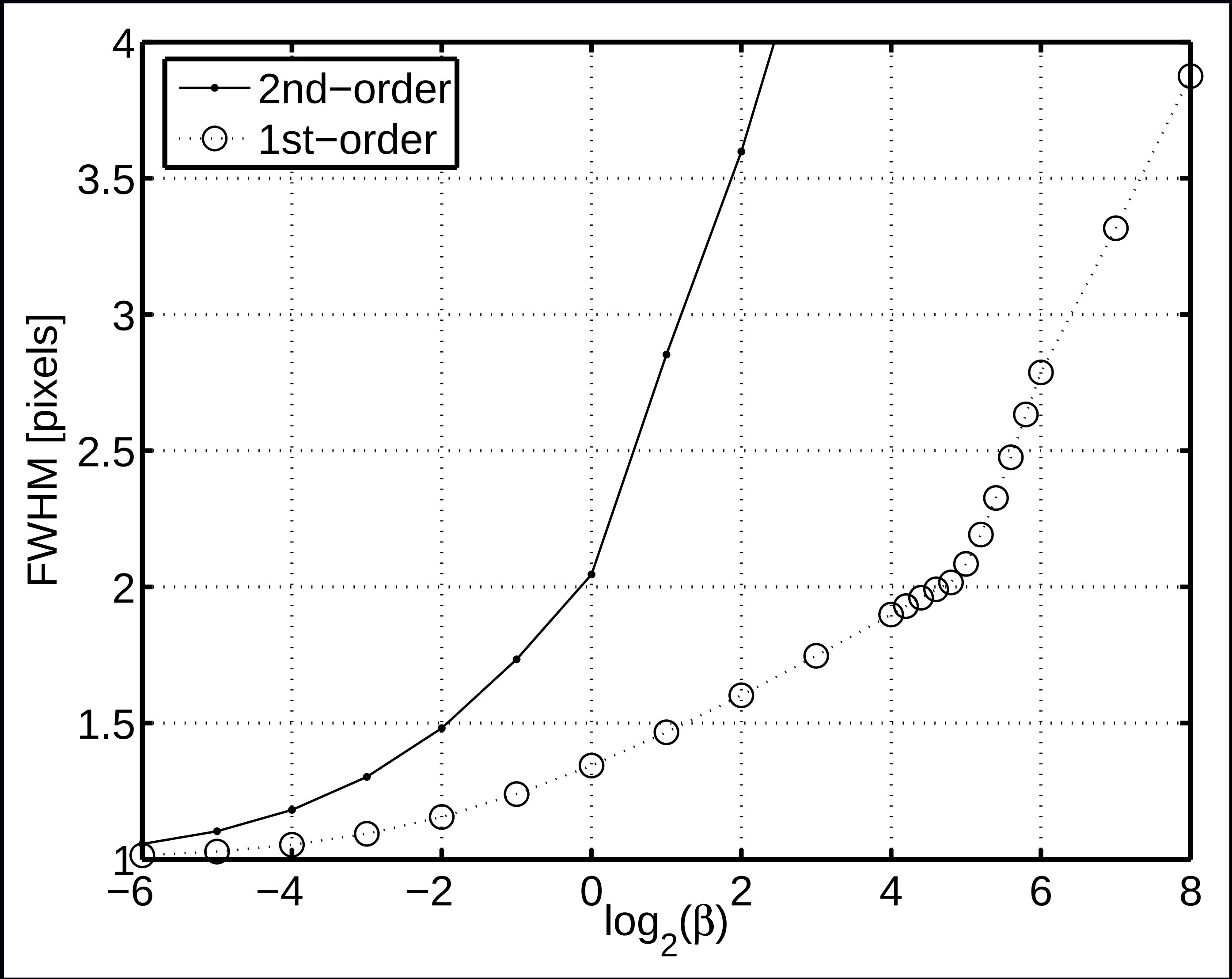


Radial k-space trajectory, FWHM of PSF is 1.2 pixels

Spatial Resolution Example: Profiles



Tabulating Spatial Resolution vs Regularization



Trajectory specific, but easily computed using a few FFTs
Works only for quadratic regularization

Resolution/noise tradeoffs

Noise analysis:

$$\mathsf{Cov}\left\{\hat{\boldsymbol{f}}\right\} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}' \,\mathsf{Cov}\{\boldsymbol{y}\}\, \boldsymbol{A} \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1}$$

Using circulant approximations to $\boldsymbol{A}'\boldsymbol{A}$ and $\boldsymbol{C}'\boldsymbol{C}$ yields:

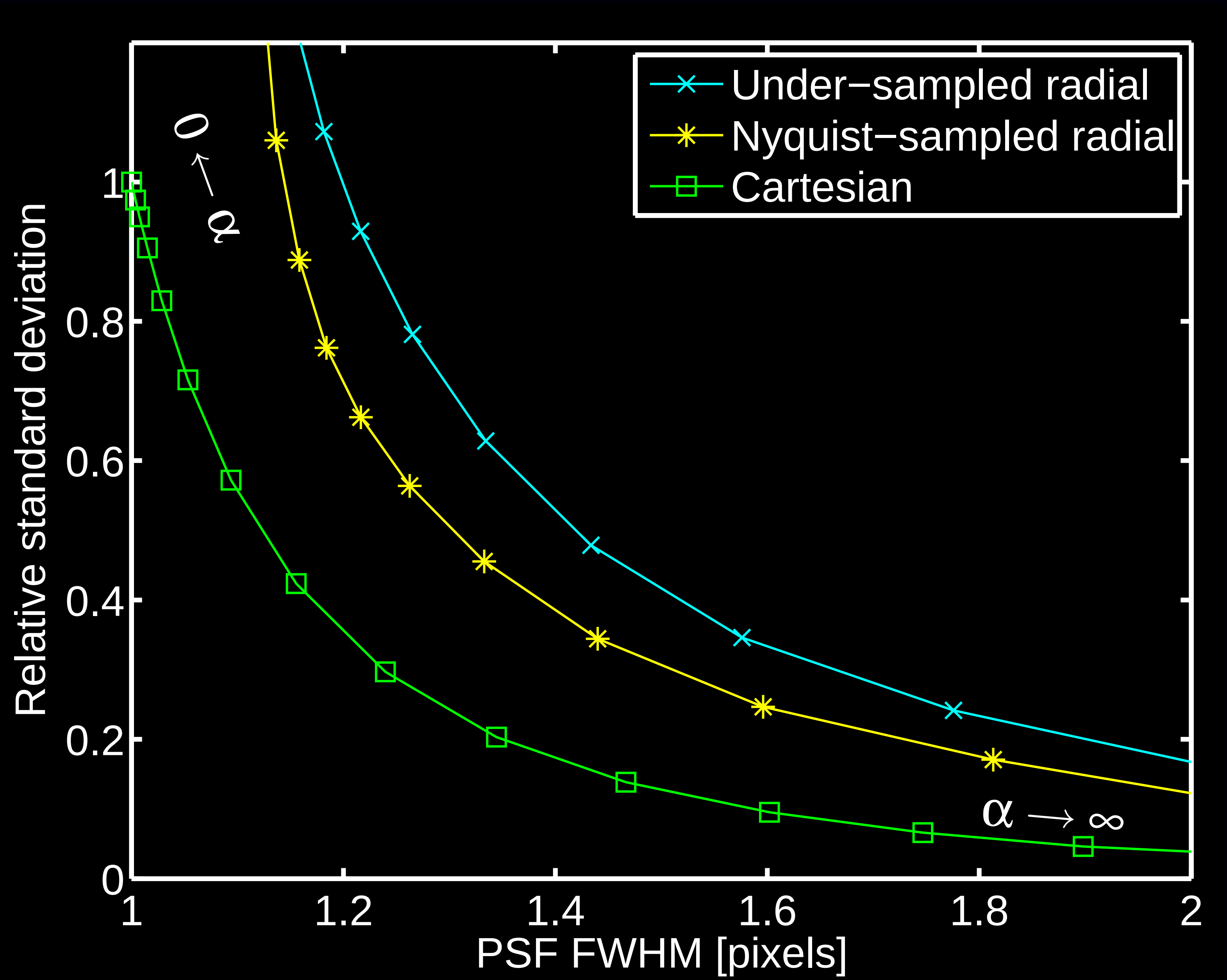
$$\mathsf{Var}\left\{\hat{f}_j\right\} \approx \sigma_\varepsilon^2 \sum_k \frac{H(\omega_k)}{(H(\omega_k) + \alpha R(\omega_k))^2}$$

- $H(\omega_k) = \mathrm{FFT}(\boldsymbol{A}'\boldsymbol{A}\, e_j)$ (lowpass)
- $R(\omega_k) = \mathrm{FFT}(\boldsymbol{C}'\boldsymbol{C}\, e_j)$ (highpass)

$\Longrightarrow$ Predicting reconstructed image noise requires just 2 FFTs.
(*cf.* gridding approach?)

Adjust α to achieve desired spatial resolution / noise tradeoff.

Resolution/Noise Tradeoff Example



In short: one can choose α rapidly and predictably for quadratic regularization.

NUFFT with Field Inhomogeneity?

Combine NUFFT with min-max temporal interpolator
(Sutton *et al.*, IEEE T-MI, 2003)
(forward version of “time segmentation”, Noll, T-MI, 1991)

Recall signal model including field inhomogeneity:

$$s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}\,.$$

Temporal interpolation approximation (aka “time segmentation”):

$$\mathrm{e}^{-\imath \omega(\vec{r}) t} \approx \sum_{l=1}^{L} a_l(t)\, \mathrm{e}^{-\imath \omega(\vec{r}) \tau_l}$$

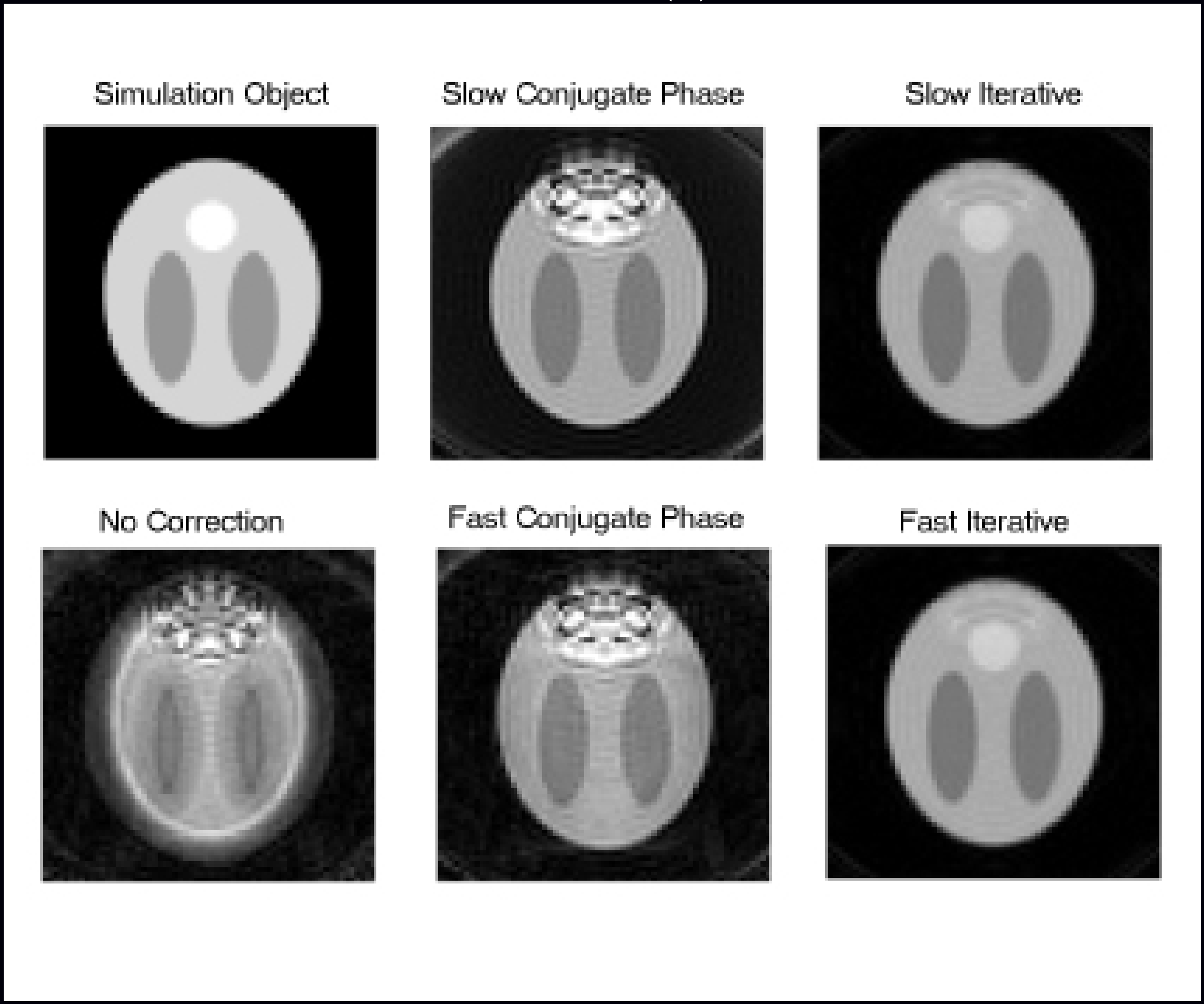
for min-max optimized temporal interpolation functions $\{a_l(\cdot)\}_{l=1}^{L}$.

$$s(t) \approx \sum_{l=1}^{L} a_l(t) \int \left[f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) \tau_l} \right] \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Linear combination of L NUFFT calls.

Field Corrected Reconstruction Example

Simulation using known field map $\omega(\vec{r})$.

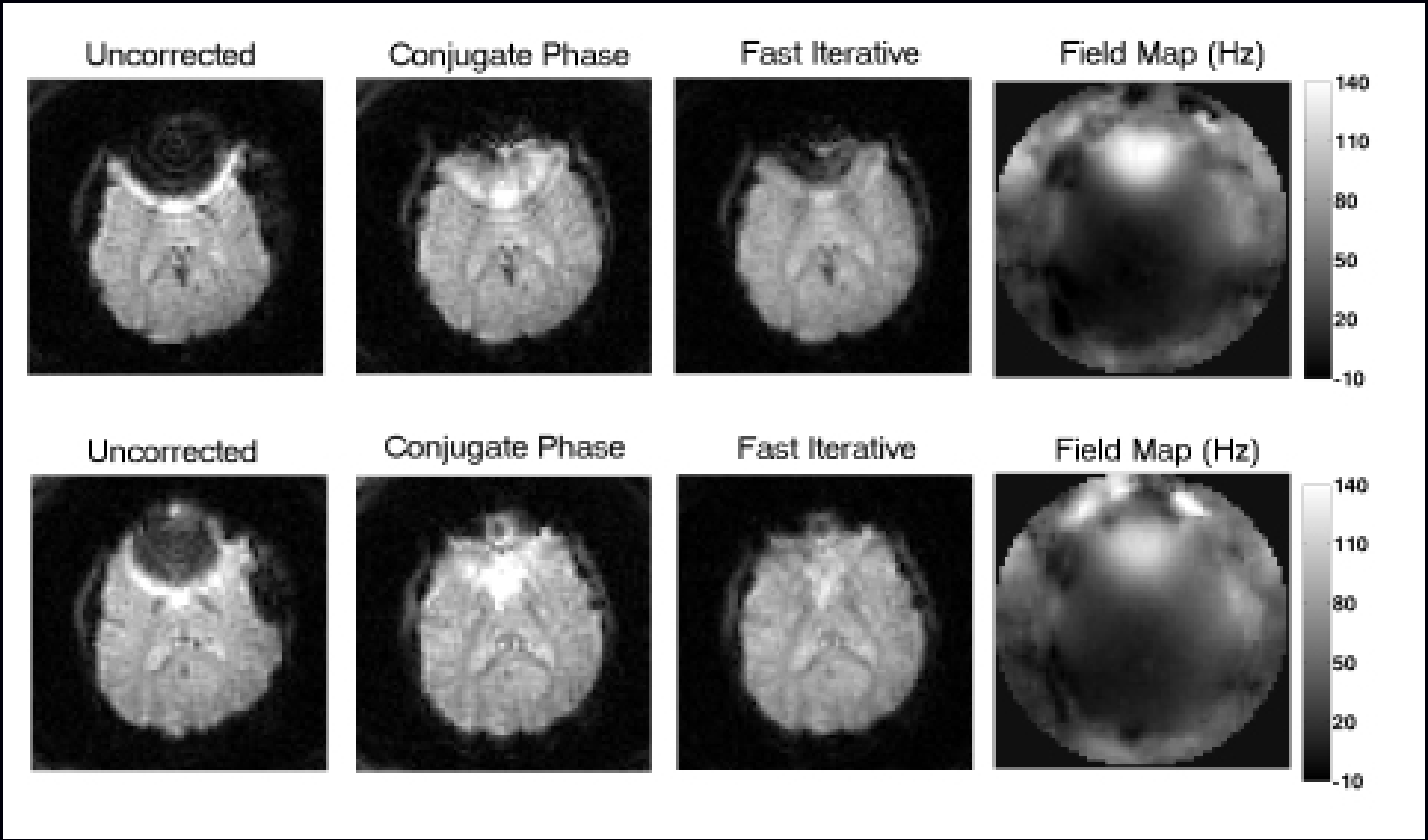


Simulation Quantitative Comparison

- Computation time?
- NRMSE between $\hat{f}$ and f^{true}?

Reconstruction Method	Time (s)	NRMSE complex	NRMSE magnitude
No Correction	0.06	1.35	0.22
Full Conjugate Phase	4.07	0.31	0.19
Fast Conjugate Phase	0.33	0.32	0.19
Fast Iterative (10 iters)	2.20	0.04	0.04
Exact Iterative (10 iters)	128.16	0.04	0.04

Human Data: Field Correction



Acceleration using Toeplitz Approximations

In the presence of field inhomogeneity, the system matrix is:

$$a_{ij} = P(\vec{\kappa}_i)\, e^{-\imath \omega(\vec{r}_j) t_i}\, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}$$

The Gram matrix $\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A}$ is *not* Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2\, e^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}\, e^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i}.$$

Approximation (“time segmentation”):

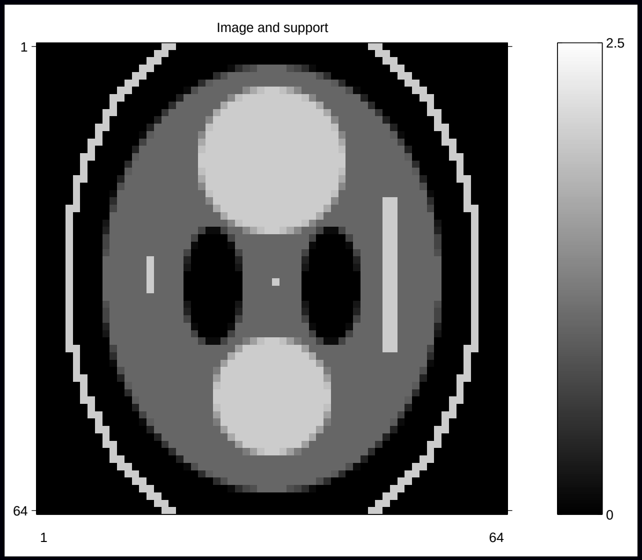
$$e^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i} \approx \sum_{l=1}^{L} b_{il}\, e^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) \tau_l}$$

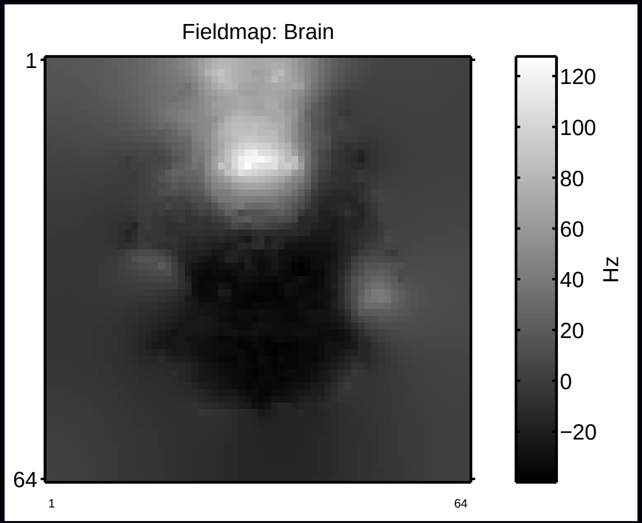
$$\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A} \approx \sum_{l=1}^{L} \boldsymbol{D}_l' \boldsymbol{T}_l \boldsymbol{D}_l, \qquad \begin{array}{l} \boldsymbol{D}_l \triangleq \operatorname{diag}\left\{e^{-\imath \omega(\vec{r}_j) \tau_l}\right\} \\ [\boldsymbol{T}_l]_{jk} \triangleq \sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2\, b_{il}\, e^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}. \end{array}$$

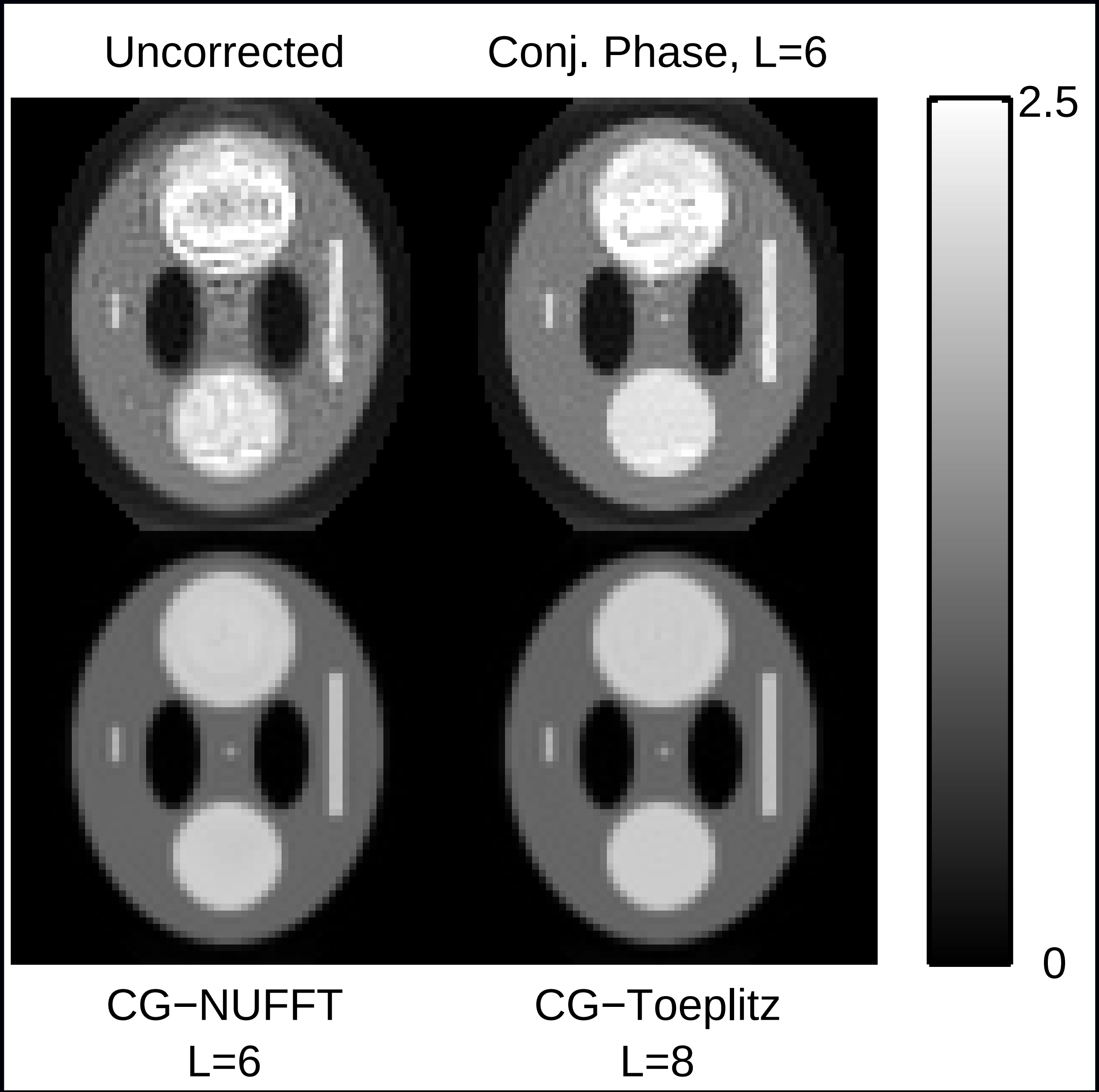
Each $\boldsymbol{T}_l$ is Toeplitz $\Longrightarrow \boldsymbol{T}\boldsymbol{f}$ using L pairs of FFTs.

(Fessler *et al.*, IEEE T-SP, Sep. 2005, brain imaging special issue)

Toeplitz Results







Toeplitz Acceleration

		Precomputation						NRMS % vs SNR				
Method	L	$\boldsymbol{B},\boldsymbol{C}$	$\boldsymbol{A}'\boldsymbol{D}\boldsymbol{y}$	$\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{y}$	$\boldsymbol{T}_l$	15 iter	Total Time	∞	50 dB	40 dB	30 dB	20 dB
Conj. Phase	6	0.4	0.2				0.6	30.7	37.3	46.5	65.3	99.9
CG-NUFFT	6	0.4				5.0	5.4	5.6	16.7	26.5	43.0	70.4
CG-Toeplitz	8	0.4		0.2	0.6	1.3	2.5	5.5	16.7	26.4	42.9	70.4

- Reduces CPU time by $2\times$ on conventional workstation (Mac G5)
- No SNR compromise
- Eliminates k-space interpolations $\Longrightarrow$ ideal for FFT hardware

Summary

- Iterative reconstruction: much potential in MRI
- Quadratic regularization parameter selection is tractable
- Computation: reduced by tools like NUFFT / Toeplitz
- But optimization algorithm design remains important
  (*cf.* Shepp and Vardi, 1982, PET)

Some current challenges

- Nonquadratic regularization: analysis / design
  Ahn and Leahy, IEEE T-MI, Mar. 2008
- Through-voxel field inhomogeneity gradients
- Motion / dynamics / partial k-space data
- Establishing diagnostic efficacy with clinical data...

Image reconstruction toolbox:
`http://www.eecs.umich.edu/~fessler`