# Advanced methods for image reconstruction in fMRI

Jeffrey A. Fessler

EECS Department
The University of Michigan

Regional Symposium on MRI
Sep. 28, 2007

Acknowledgements: Doug Noll, Brad Sutton,

# Outline

- MR image reconstruction problem description
- Overview of image reconstruction methods
- Model-based image reconstruction
- Iterations and computation (NUFFT etc.)
- Regularization
- Myths about iterative reconstruction
- Field inhomogeneity correction
- Parallel (sensitivity encoded) imaging

Image reconstruction toolbox:
`http://www.eecs.umich.edu/~fessler`

# Why Iterative Image Reconstruction?

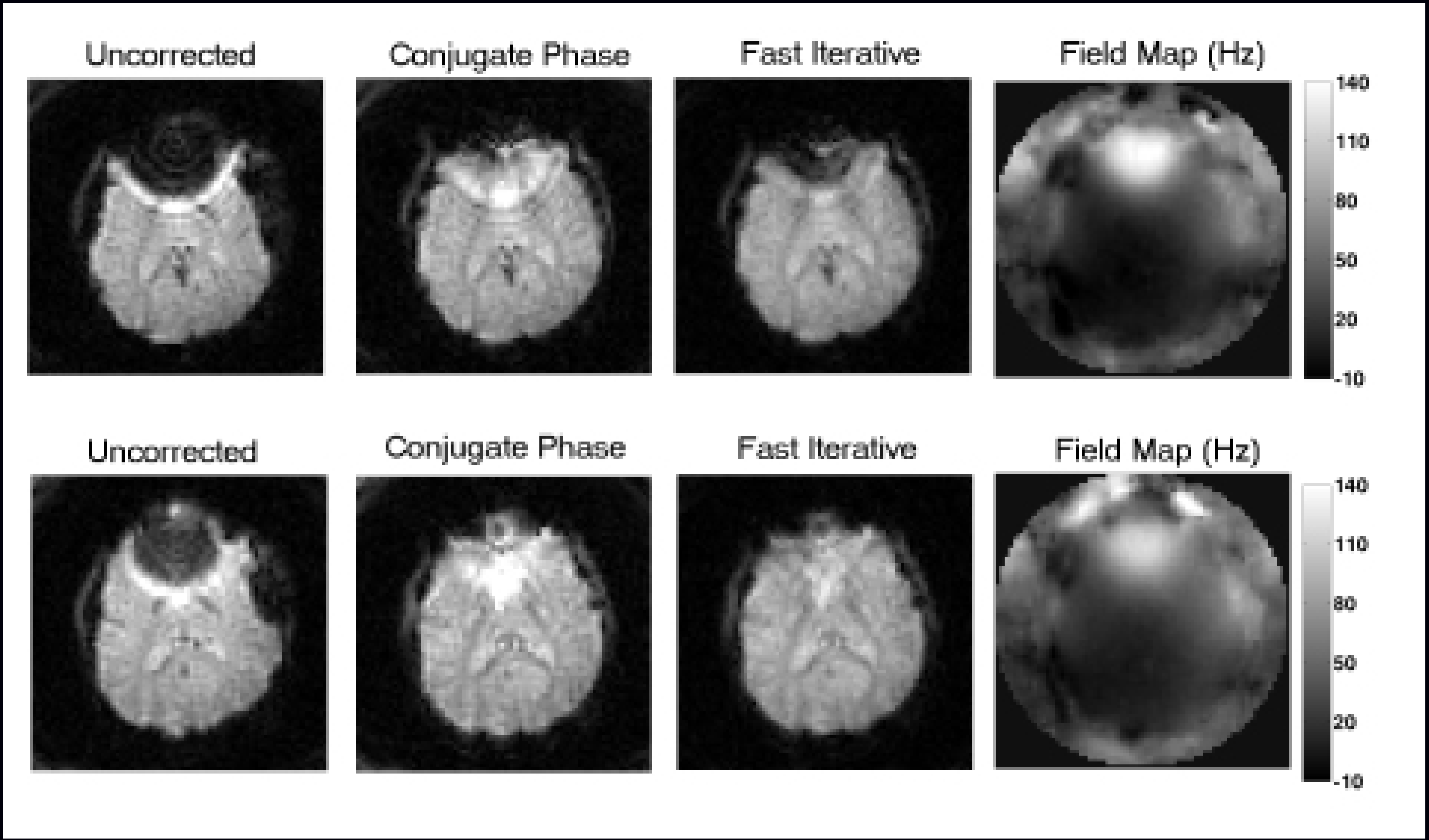
- Statistical modeling may reduce noise
- Incorporate prior information, *e.g.*:
  - support constraints
  - (piecewise) smoothness
  - phase constraints
- No density compensation needed
- “Non-Fourier” physical effects such as field inhomogeneity
- Incorporation of coil sensitivity maps
- Improved results for under-sampled trajectories (?)
- ...

(“Avoiding k-space interpolation” is not a compelling reason!)

# Primary drawbacks of Iterative Methods

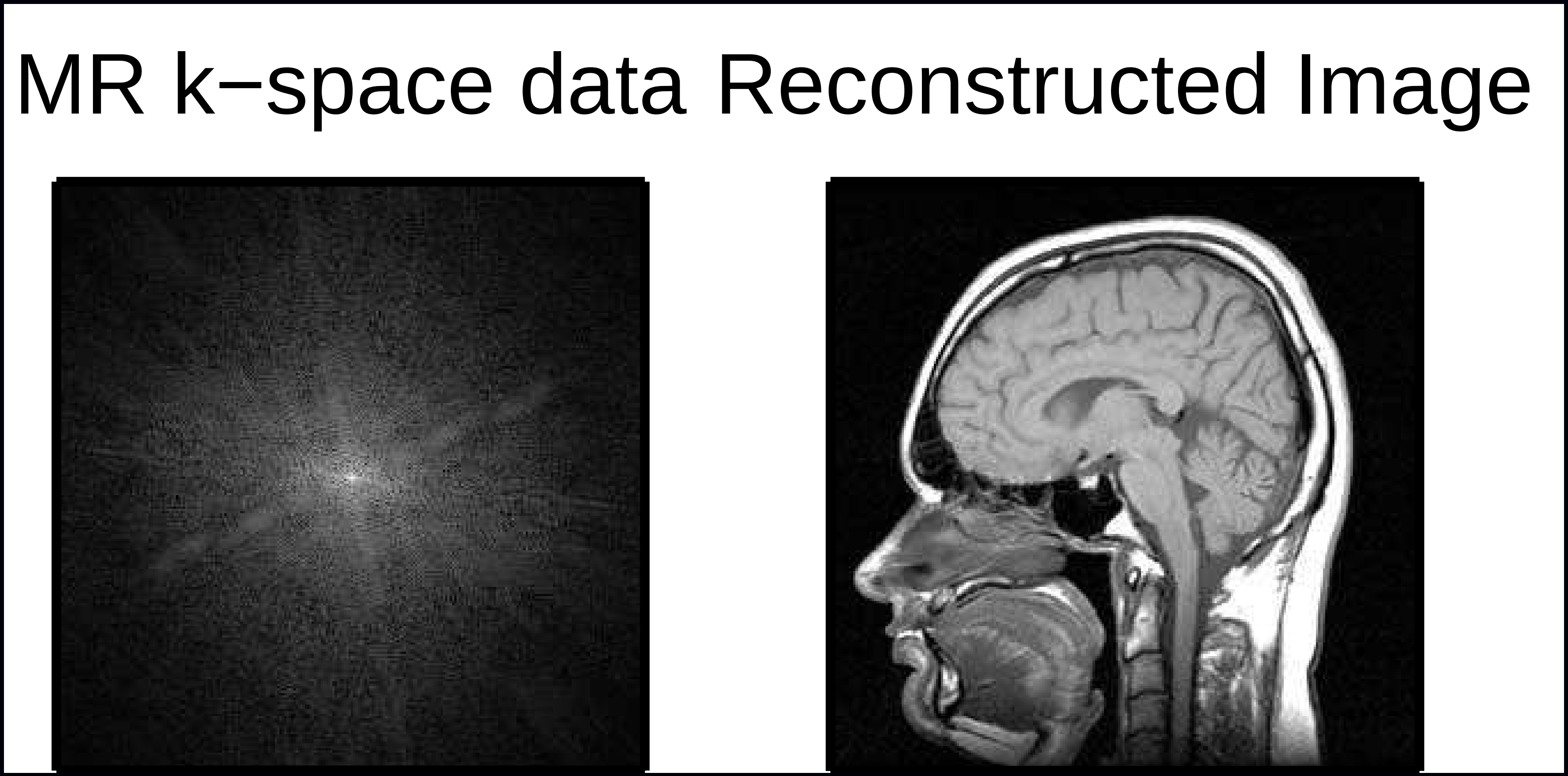
- Choosing regularization parameter(s)
- Algorithm speed

# Example: Iterative Reconstruction under $\Delta B_0$



# Introduction to Reconstruction

# Standard MR Image Reconstruction




Cartesian sampling in k-space. An inverse FFT. End of story.

Commercial MR system quotes 400 FFTs ($256^2$) per second.

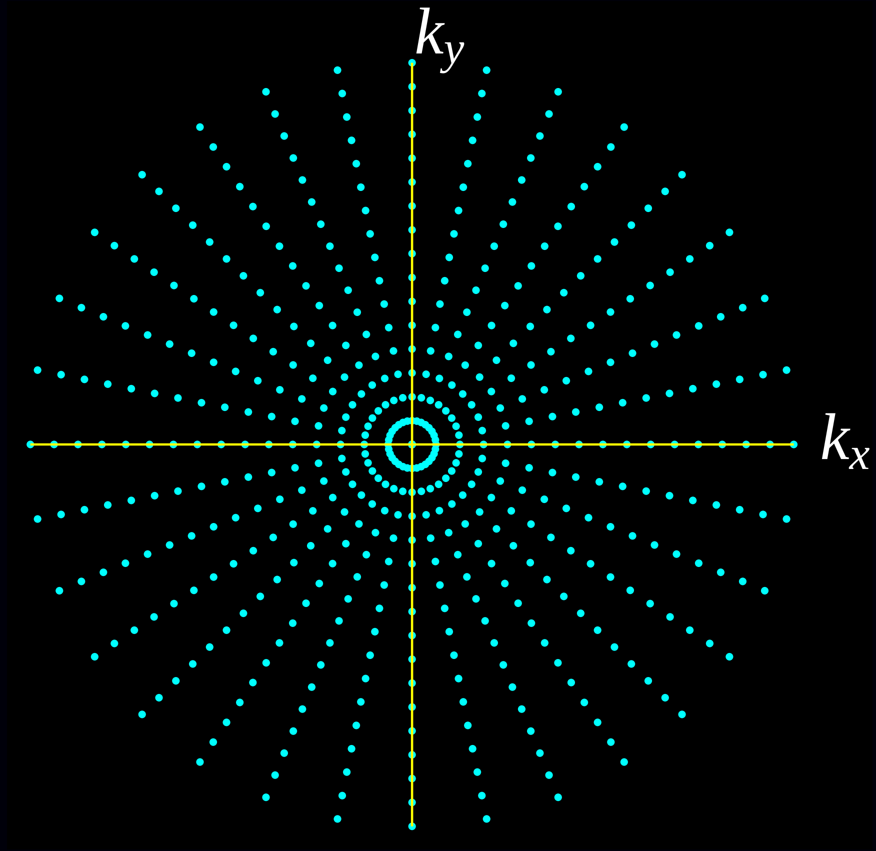
# Non-Cartesian MR Image Reconstruction

"k-space" data

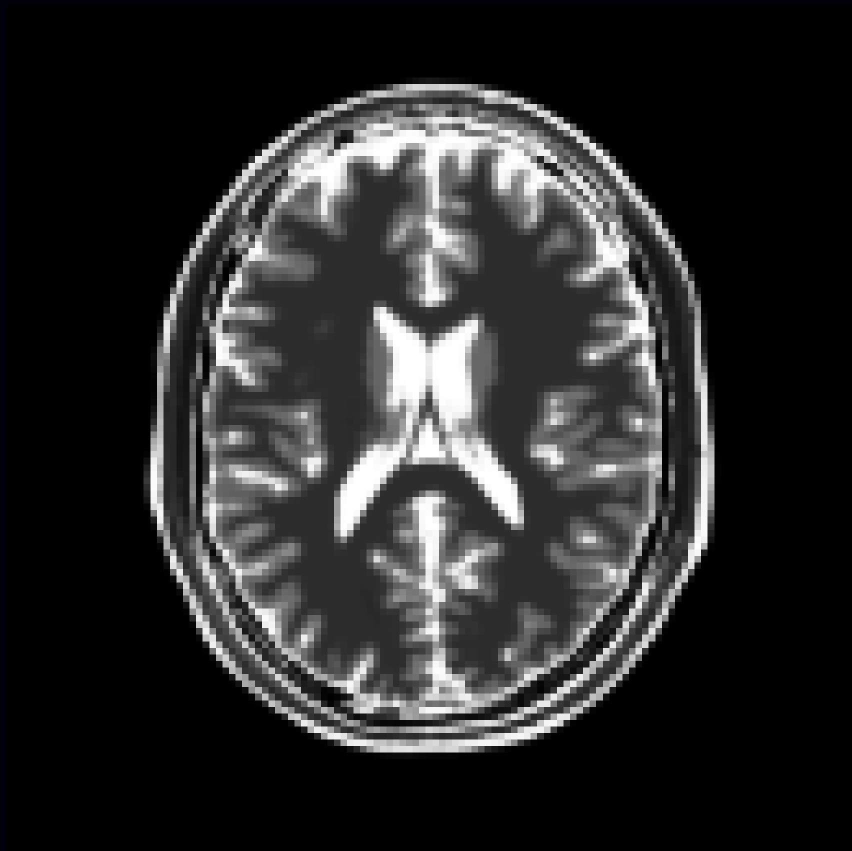
$\boldsymbol{y} = (y_1, \ldots, y_M)$

image

$f(\vec{r})$




$\Longrightarrow$



k-space trajectory:

$\vec{\kappa}(t) = (k_x(t), k_y(t))$

spatial coordinates:

$\vec{r} \in \mathbb{R}^{\bar{d}}$

# Textbook MRI Measurement Model

Ignoring *lots* of things, the standard measurement model is:

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, M$$

$$s(t) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r} = F(\vec{\kappa}(t)).$$

$\vec{r}$: spatial coordinates
$\vec{\kappa}(t)$: k-space trajectory of the MR pulse sequence
$f(\vec{r})$: object's unknown transverse magnetization
$F(\vec{\kappa})$: Fourier transform of $f(\vec{r})$. We get noisy samples of this!
$e^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}$ provides spatial information $\Longrightarrow$ Nobel Prize

Goal of image reconstruction: find $f(\vec{r})$ from measurements $\{y_i\}_{i=1}^{M}$.

The unknown object $f(\vec{r})$ is a continuous-space function,
but the recorded measurements $\boldsymbol{y} = (y_1, \ldots, y_M)$ are finite.

Under-determined (ill posed) problem $\Longrightarrow$ no canonical solution.

*All MR scans provide only "partial" k-space data.*

# Image Reconstruction Strategies

- Continuous-continuous formulation

  Pretend that a continuum of measurements are available:

$$F(\vec{\kappa}) = \int f(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, \mathrm{d}\vec{r}.$$

  The “solution” is an inverse Fourier transform:

$$f(\vec{r}) = \int F(\vec{\kappa})\, \mathrm{e}^{\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, \mathrm{d}\vec{\kappa}.$$

  Now discretize the integral solution:

$$\hat{f}(\vec{r}) = \sum_{i=1}^{M} F(\vec{\kappa}_i)\, \mathrm{e}^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}}\, w_i \approx \sum_{i=1}^{M} y_i w_i\, \mathrm{e}^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}},$$

  where $w_i$ values are “sampling density compensation factors.”
  Numerous methods for choosing $w_i$ values in the literature.

  For Cartesian sampling, using $w_i = 1/N$ suffices,
  and the summation is an inverse FFT.
  For non-Cartesian sampling, replace summation with gridding.

- Continuous-discrete formulation

  Use many-to-one linear model:

$$\boldsymbol{y} = \mathcal{A} f + \boldsymbol{\varepsilon}, \text{ where } \mathcal{A} : \mathcal{L}_2(\mathbb{R}^{\bar{d}}) \to \mathbb{C}^M.$$

  Minimum norm solution (*cf.* “natural pixels”):

$$\min_{\hat{f}} \left\| \hat{f} \right\| \text{ subject to } \boldsymbol{y} = \mathcal{A} \hat{f}$$

$$\hat{f} = \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\boldsymbol{y} = \sum_{i=1}^M c_i \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}, \text{ where } \mathcal{A}\mathcal{A}^*\boldsymbol{c} = \boldsymbol{y}.$$

- Discrete-discrete formulation

  Assume parametric model for object:

$$f(\vec{r}) = \sum_{j=1}^N f_j \, p_j(\vec{r}) \,.$$

  Estimate parameter vector $\boldsymbol{f} = (f_1, \ldots, f_N)$ from data vector $\boldsymbol{y}$.

# Model-Based Image Reconstruction: Overview

# Model-Based Image Reconstruction

MR signal equation with more complete physics:

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\,\omega(\vec{r})\,t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, M$$

- $s^{\text{coil}}(\vec{r})$ Receive-coil sensitivity pattern(s) (for SENSE)
- $\omega(\vec{r})$ Off-resonance frequency map
  (due to field inhomogeneity / magnetic susceptibility)
- $R_2^*(\vec{r})$ Relaxation map

Other physical factors (?)
- Eddy current effects; in $\vec{\kappa}(t)$
- Concomitant gradient terms
- Chemical shift
- Motion

Goal?

(it depends)

# Field Inhomogeneity-Corrected Reconstruction

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r})\, t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given field map $\omega(\vec{r})$.
(Assume all other terms are known or unimportant.)

## Motivation

Essential for functional MRI of brain regions near sinus cavities!

(Sutton *et al.*, ISMRM 2001; T-MI 2003)

# Sensitivity-Encoded (SENSE) Reconstruction

$$s(t) = \int f(\vec{r}) s^{\text{coil}}(\vec{r}) \, \mathrm{e}^{-\imath\, \omega(\vec{r})\, t} \, \mathrm{e}^{-R_2^*(\vec{r})\, t} \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}} \, \mathrm{d}\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given sensitivity maps $s^{\text{coil}}(\vec{r})$.
(Assume all other terms are known or unimportant.)

Can combine SENSE with field inhomogeneity correction "easily."

(Sutton *et al.*, ISMRM 2001, Olafsson *et al.*, ISBI 2006)

# Joint Estimation of Image and Field-Map

$$s(t) = \int f(\vec{r})\, s^{\mathrm{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, d\vec{r}$$

Goal: estimate *both* the image $f(\vec{r})$ and the field map $\omega(\vec{r})$
(Assume all other terms are known or unimportant.)

Analogy:
joint estimation of emission image and attenuation map in PET.

(Sutton *et al.*, ISMRM Workshop, 2001; ISBI 2002; ISMRM 2002; ISMRM 2003; MRM 2004)

# The Kitchen Sink

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate image $f(\vec{r})$, field map $\omega(\vec{r})$, *and* relaxation map $R_2^*(\vec{r})$

Requires "suitable" k-space trajectory.

(Sutton *et al.*, ISMRM 2002; Twieg, MRM, 2003)

# Estimation of Dynamic Maps

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate dynamic field map $\omega(\vec{r})$ and “BOLD effect” $R_2^*(\vec{r})$ given baseline image $f(\vec{r})$ in fMRI.

Motion...

# Model-Based Image Reconstruction: Details

# Basic Signal Model

$$y_i = s(t_i) + \varepsilon_i, \qquad \mathsf{E}[y_i] = s(t_i) = \int f(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Goal: reconstruct $f(\vec{r})$ from $\boldsymbol{y} = (y_1, \ldots, y_M)$.

Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{N} f_j\, p(\vec{r} - \vec{r}_j) \longleftarrow \text{usually 2D rect functions.}$$

Substituting into signal model yields

$$\mathsf{E}[y_i] = \int \left[ \sum_{j=1}^{N} f_j\, p(\vec{r} - \vec{r}_j) \right] \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, \mathrm{d}\vec{r} = \sum_{j=1}^{N} \left[ \int p(\vec{r} - \vec{r}_j)\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, \mathrm{d}\vec{r} \right] f_j$$

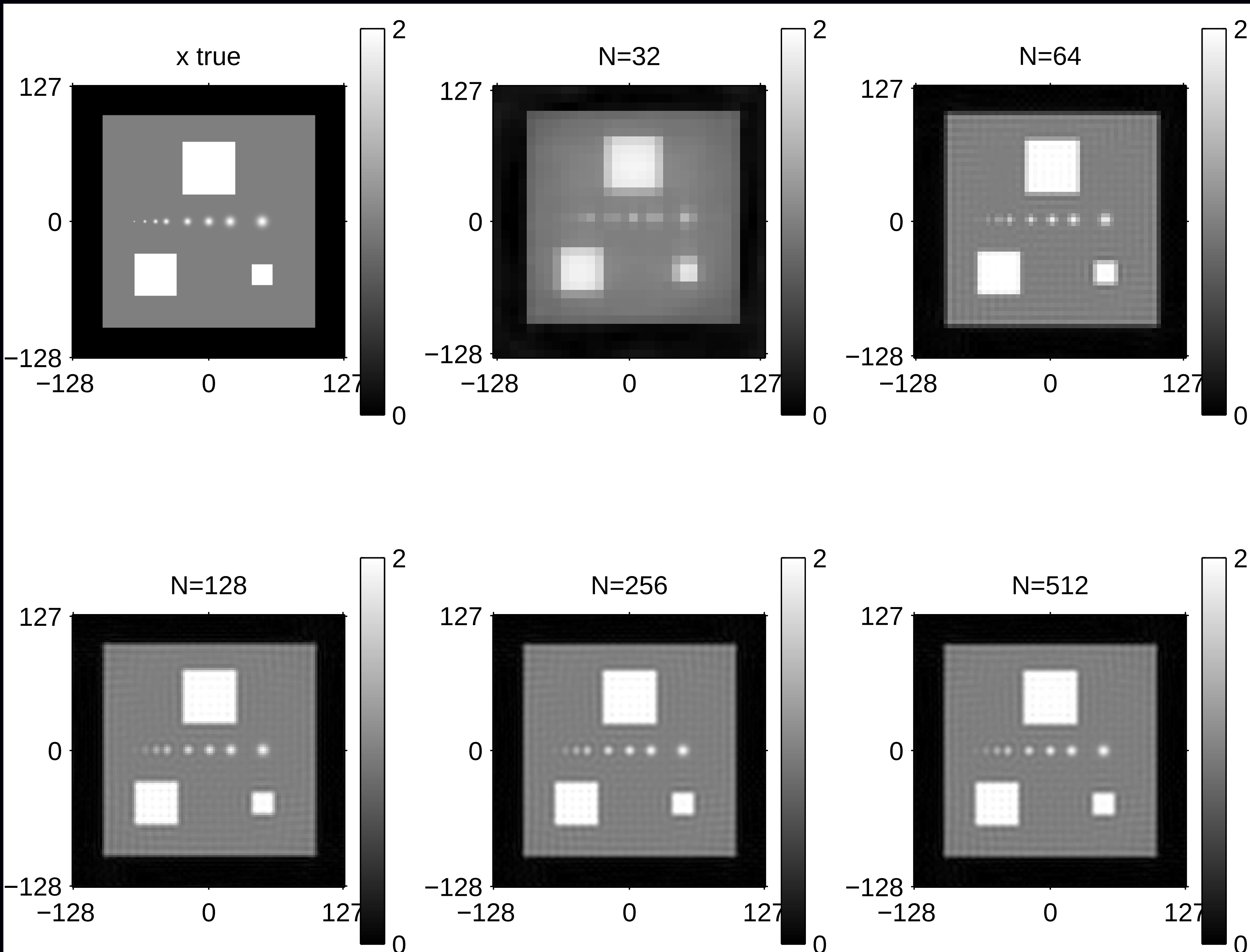
$$= \sum_{j=1}^{N} a_{ij} f_j, \qquad a_{ij} = P(\vec{\kappa}_i)\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}, \qquad p(\vec{r}) \stackrel{\text{FT}}{\Longleftrightarrow} P(\vec{\kappa}).$$

Discrete-discrete measurement model with system matrix $\boldsymbol{A} = \{a_{ij}\}$:

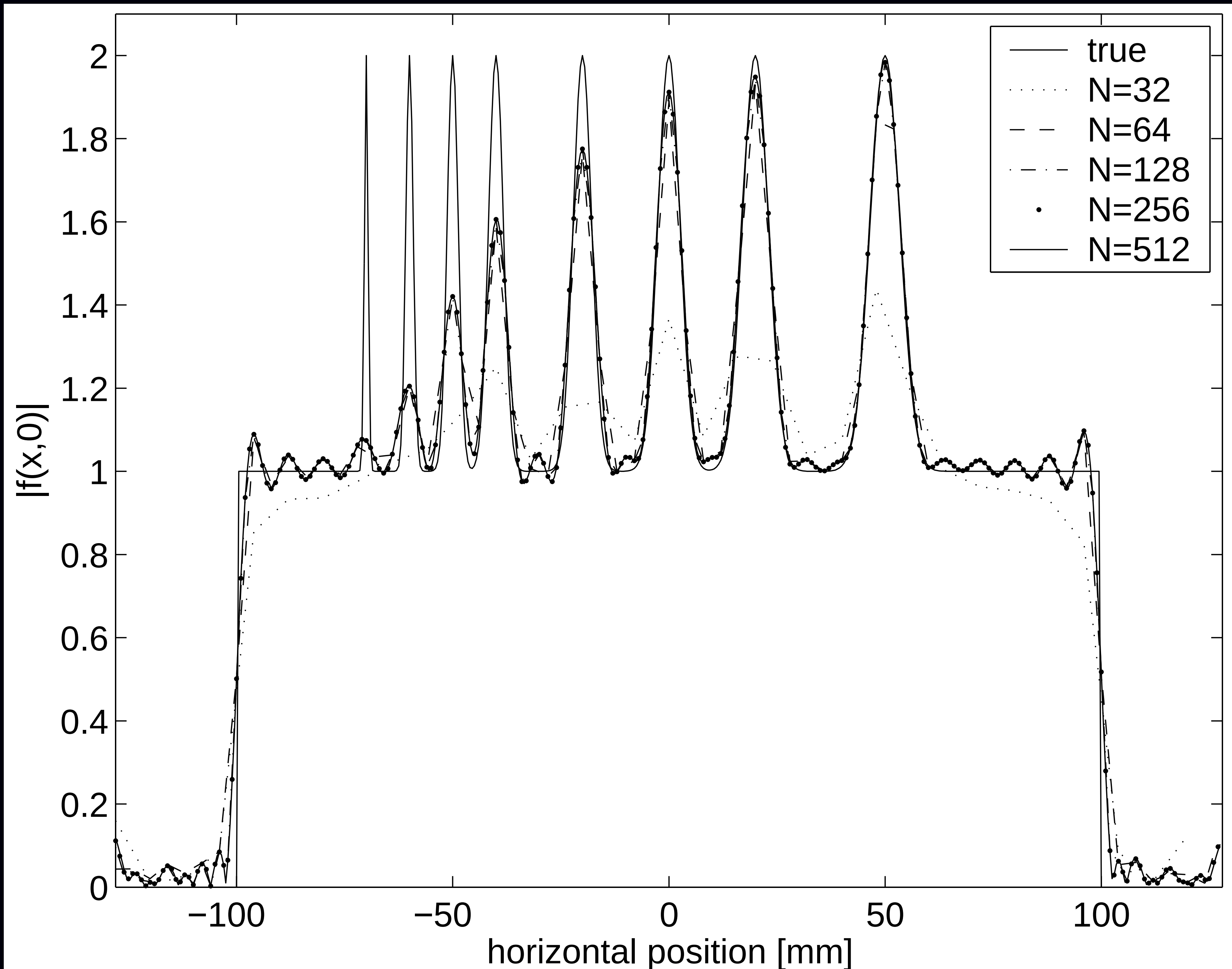
$$\boxed{\boldsymbol{y} = \boldsymbol{A}\boldsymbol{f} + \boldsymbol{\varepsilon}.}$$

Goal: estimate coefficients (pixel values) $\boldsymbol{f} = (f_1, \ldots, f_N)$ from $\boldsymbol{y}$.

# Small Pixel Size Need Not Matter



# Profiles



true
N=32
N=64
N=128
N=256
N=512
|f(x,0)|
horizontal position [mm]
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
−100
−50
0
50
100

# Regularized Least-Squares Estimation

Estimate object by minimizing a cost function:

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f} \in \mathbb{C}^N} \Psi(\boldsymbol{f}), \qquad \Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha \mathsf{R}(\boldsymbol{f})$$

- data fit term $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$
  corresponds to negative log-likelihood of Gaussian distribution
- regularizing term $\mathsf{R}(\boldsymbol{f})$ controls noise by penalizing roughness,

$$\textit{e.g.}: \quad \mathsf{R}(\boldsymbol{f}) \approx \int \|\nabla f\|^2 \, \mathrm{d}\vec{r}$$

- regularization parameter $\alpha > 0$
  controls tradeoff between spatial resolution and noise
- Equivalent to Bayesian MAP estimation with prior $\propto e^{-\alpha \mathsf{R}(\boldsymbol{f})}$

Issues:

- choosing $\mathsf{R}(f)$
- choosing $\alpha$
- computing minimizer rapidly.

# Quadratic regularization

1D example: squared differences between neighboring pixel values:

$$\mathsf{R}(f) = \sum_{j=2}^{N} \frac{1}{2} |f_j - f_{j-1}|^2.$$

In matrix-vector notation, $\mathsf{R}(\boldsymbol{f}) = \frac{1}{2} \|\boldsymbol{C}\boldsymbol{f}\|^2$ where

$$\boldsymbol{C} = \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ & & & \ddots & \ddots & \\ 0 & \dots & 0 & 0 & -1 & 1 \end{bmatrix}, \text{ so } \boldsymbol{C}\boldsymbol{f} = \begin{bmatrix} f_2 - f_1 \\ \vdots \\ f_N - f_{N-1} \end{bmatrix}.$$

For 2D and higher-order differences, modify differencing matrix $\boldsymbol{C}$.

Leads to closed-form solution:

$$\begin{aligned} \hat{\boldsymbol{f}} &= \arg\min_{\boldsymbol{f}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha \|\boldsymbol{C}\boldsymbol{f}\|^2 \\ &= \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}. \end{aligned}$$

(a formula of limited practical use for computing $\hat{\boldsymbol{f}}$)

# Choosing the Regularization Parameter

Spatial resolution analysis (Fessler & Rogers, IEEE T-IP, 1996):

$$\hat{\boldsymbol{f}} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}$$

$$\mathsf{E}\left[\hat{\boldsymbol{f}}\right] = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}' \,\mathsf{E}[\boldsymbol{y}]$$

$$\mathsf{E}\left[\hat{\boldsymbol{f}}\right] = \underbrace{\left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{A}}_{\text{blur}} \,\boldsymbol{f}$$

$\boldsymbol{A}'\boldsymbol{A}$ and $\boldsymbol{C}'\boldsymbol{C}$ are Toeplitz $\Longrightarrow$ blur is approximately shift-invariant.

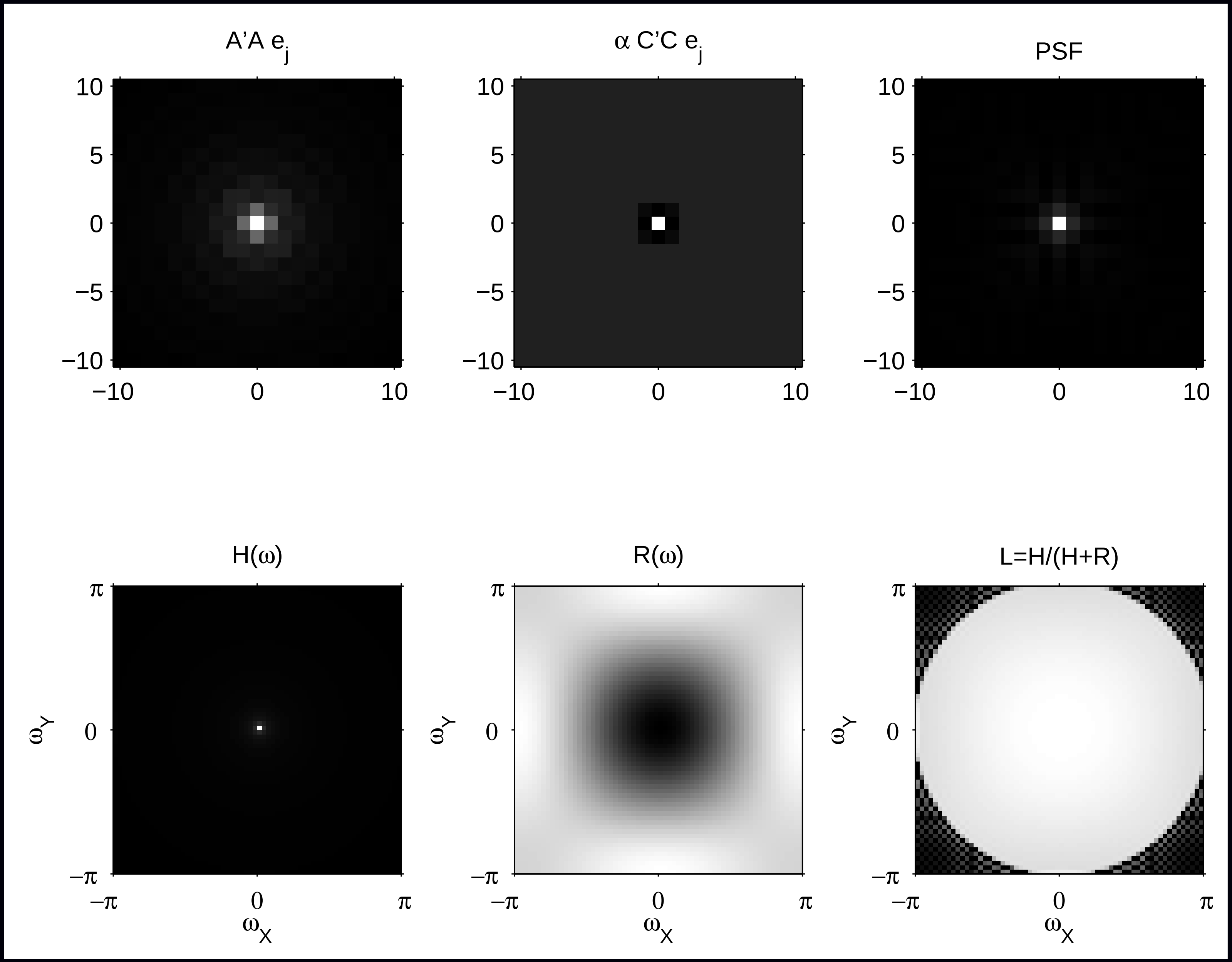
Frequency response of blur:

$$L(\omega) = \frac{H(\omega)}{H(\omega) + \alpha R(\omega)}$$

- $H(\omega_k) = \mathrm{FFT}(\boldsymbol{A}'\boldsymbol{A}\, e_j)$ (lowpass)
- $R(\omega_k) = \mathrm{FFT}(\boldsymbol{C}'\boldsymbol{C}\, e_j)$ (highpass)

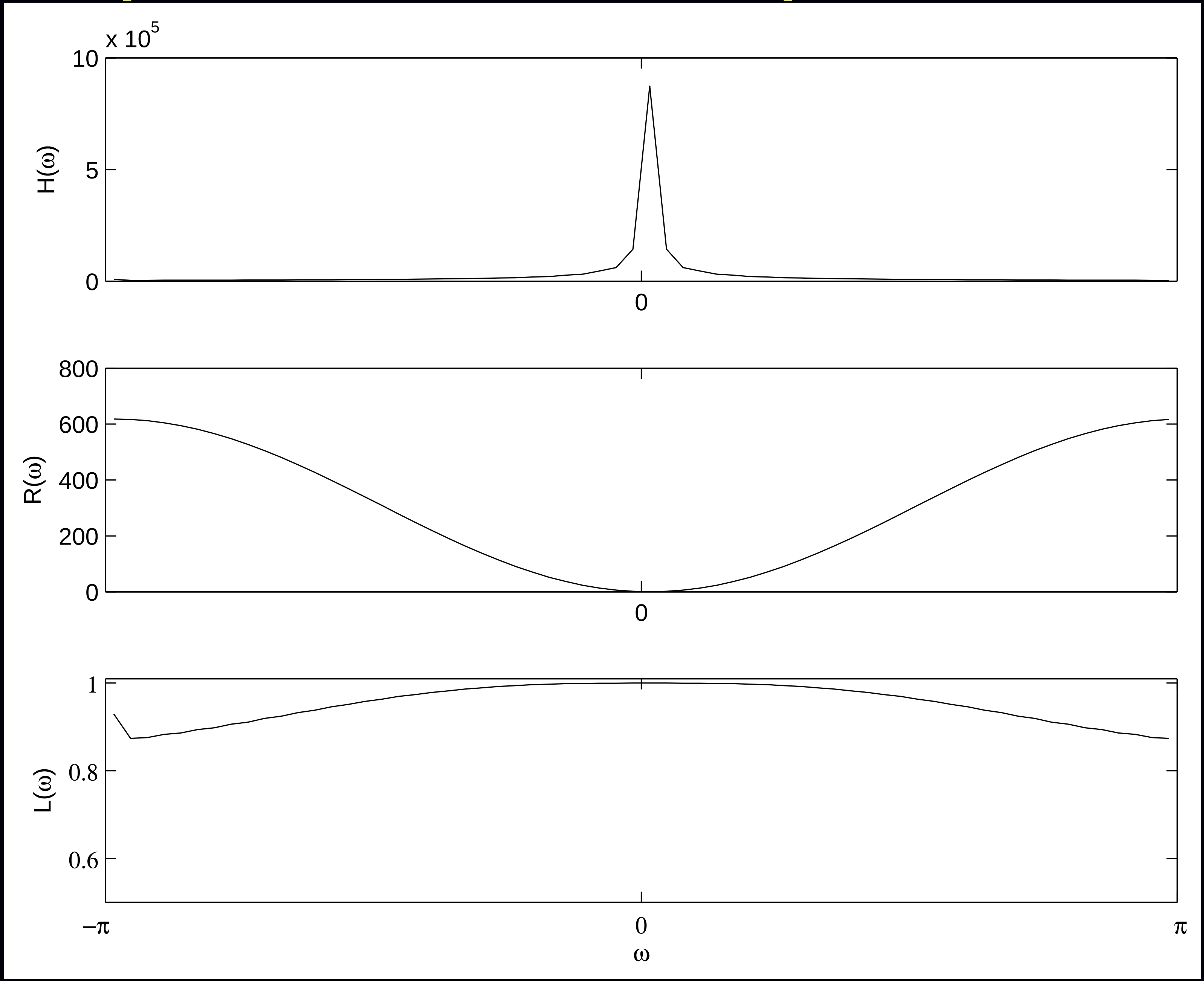
Adjust $\alpha$ to achieve desired spatial resolution.

# Spatial Resolution Example




Radial k-space trajectory, FWHM of PSF is 1.2 pixels

# Spatial Resolution Example: Profiles



# Resolution/noise tradeoffs

Noise analysis:

$$\mathsf{Cov}\left\{\hat{\boldsymbol{f}}\right\} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}' \,\mathsf{Cov}\{\boldsymbol{y}\}\, \boldsymbol{A} \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1}$$

Using circulant approximations to $\boldsymbol{A}'\boldsymbol{A}$ and $\boldsymbol{C}'\boldsymbol{C}$ yields:

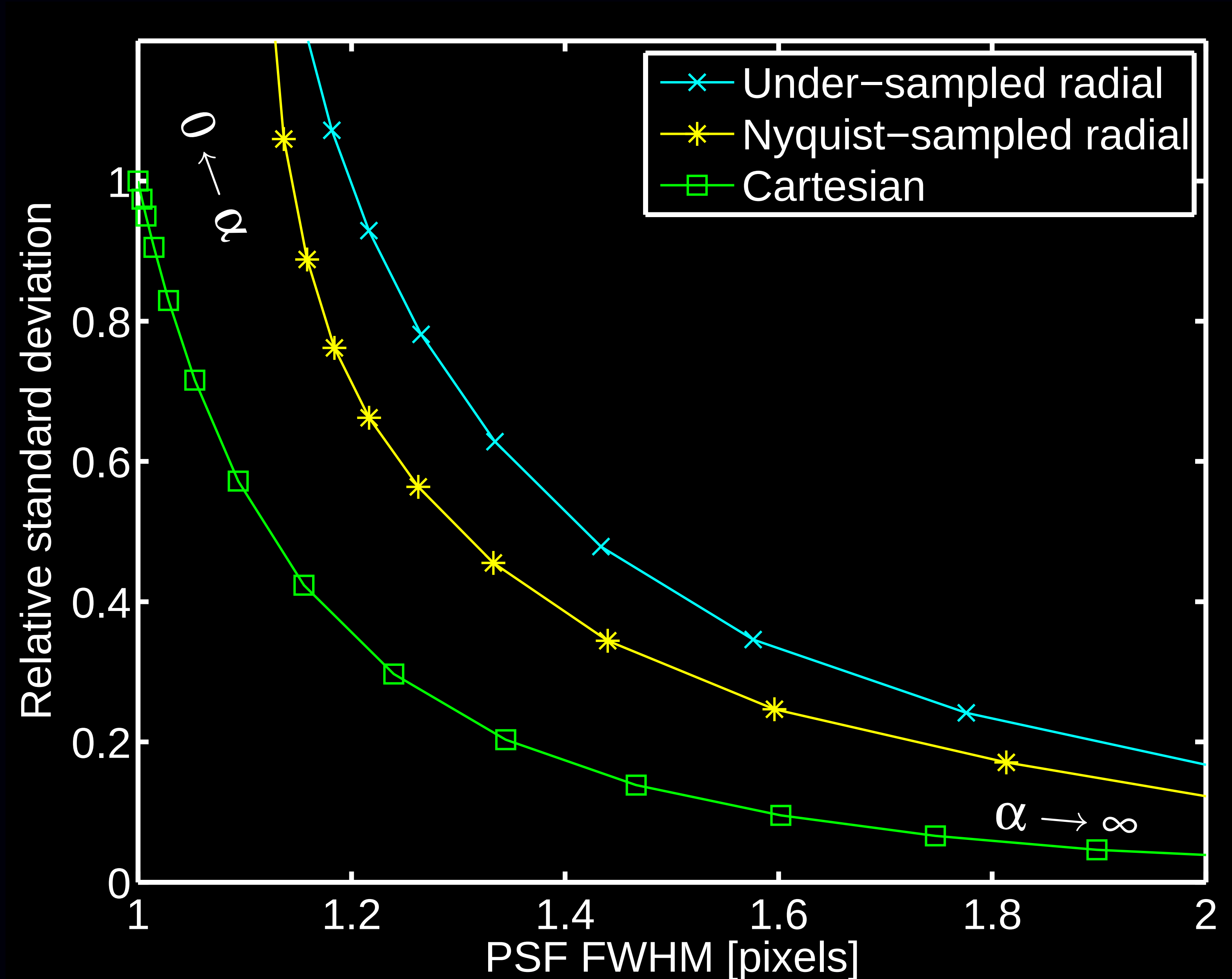
$$\mathsf{Var}\left\{\hat{f}_j\right\} \approx \sigma_\varepsilon^2 \sum_k \frac{H(\omega_k)}{(H(\omega_k) + \alpha R(\omega_k))^2}$$

- $H(\omega_k) = \mathrm{FFT}(\boldsymbol{A}'\boldsymbol{A}\, e_j)$ (lowpass)
- $R(\omega_k) = \mathrm{FFT}(\boldsymbol{C}'\boldsymbol{C}\, e_j)$ (highpass)

$\Longrightarrow$ Predicting reconstructed image noise requires just 2 FFTs.
(*cf.* gridding approach?)

Adjust $\alpha$ to achieve desired spatial resolution / noise tradeoff.

# Resolution/Noise Tradeoff Example




In short: one can choose $\alpha$ rapidly and predictably for quadratic regularization.

# Iterative Minimization by Conjugate Gradients

Choose initial guess $\boldsymbol{f}^{(0)}$ (*e.g.*, fast conjugate phase / gridding).
Iteration (unregularized):

$$
\begin{aligned}
\boldsymbol{g}^{(n)} &= \nabla\Psi\left(\boldsymbol{f}^{(n)}\right) = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) && \text{gradient} \\
\boldsymbol{p}^{(n)} &= \boldsymbol{P}\boldsymbol{g}^{(n)} && \text{precondition} \\
\gamma_n &= \begin{cases} 0, & n = 0 \\ \dfrac{\langle \boldsymbol{g}^{(n)}, \boldsymbol{p}^{(n)} \rangle}{\langle \boldsymbol{g}^{(n-1)}, \boldsymbol{p}^{(n-1)} \rangle}, & n > 0 \end{cases} \\
\boldsymbol{d}^{(n)} &= -\boldsymbol{p}^{(n)} + \gamma_n \boldsymbol{d}^{(n-1)} && \text{search direction} \\
\boldsymbol{v}^{(n)} &= \boldsymbol{A}\boldsymbol{d}^{(n)} \\
\alpha_n &= \langle \boldsymbol{d}^{(n)}, -\boldsymbol{g}^{(n)} \rangle / \left\|\boldsymbol{v}^{(n)}\right\|^2 && \text{step size} \\
\boldsymbol{f}^{(n+1)} &= \boldsymbol{f}^{(n)} + \alpha_n \boldsymbol{d}^{(n)} && \text{update}
\end{aligned}
$$

Bottlenecks: computing $\boldsymbol{A}\boldsymbol{f}^{(n)}$ and $\boldsymbol{A}'\boldsymbol{r}$.

- $\boldsymbol{A}$ is too large to store explicitly (not sparse)
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{f}$ is $O(MN)$ per iteration, whereas FFT is only $O(M \log M)$.

# Computing $\boldsymbol{A}\boldsymbol{f}$ Rapidly

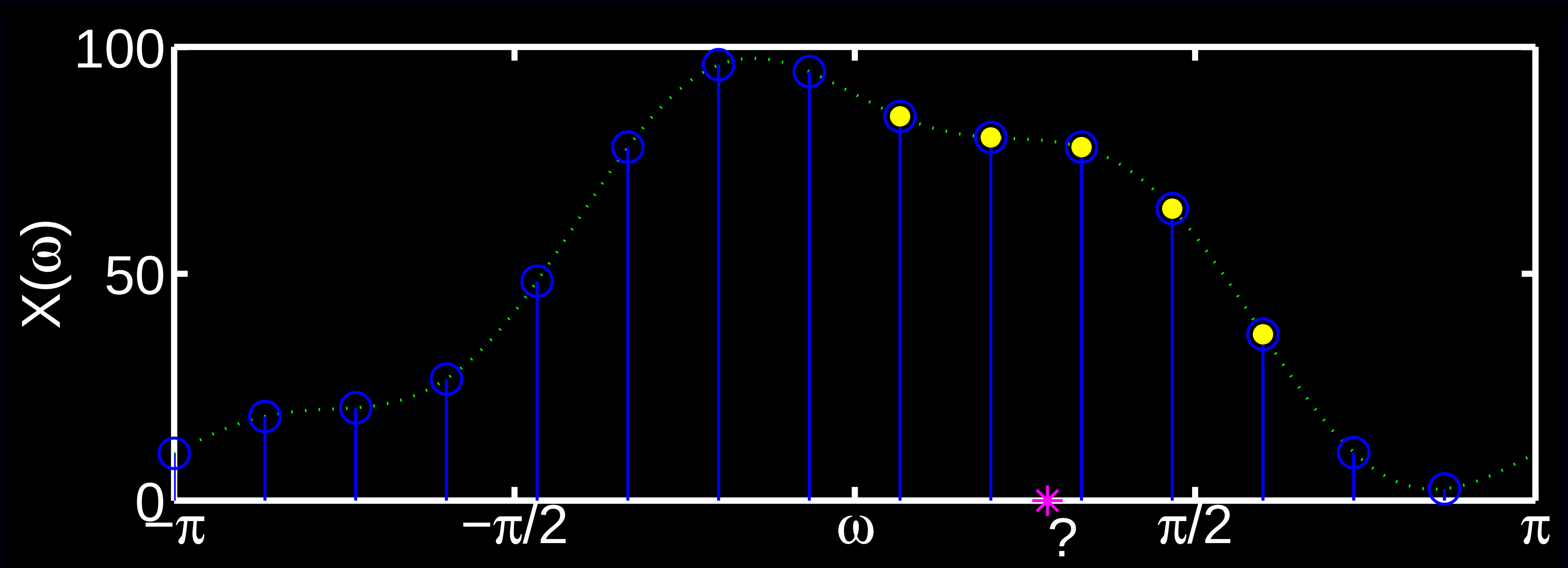
$$[\boldsymbol{A}\boldsymbol{f}]_i = \sum_{j=1}^{N} a_{ij} f_j = P(\vec{\kappa}_i) \sum_{j=1}^{N} \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j} f_j, \qquad i = 1, \ldots, M$$

- Pixel locations $\{\vec{r}_j\}$ are uniformly spaced
- k-space locations $\{\vec{\kappa}_i\}$ are unequally spaced

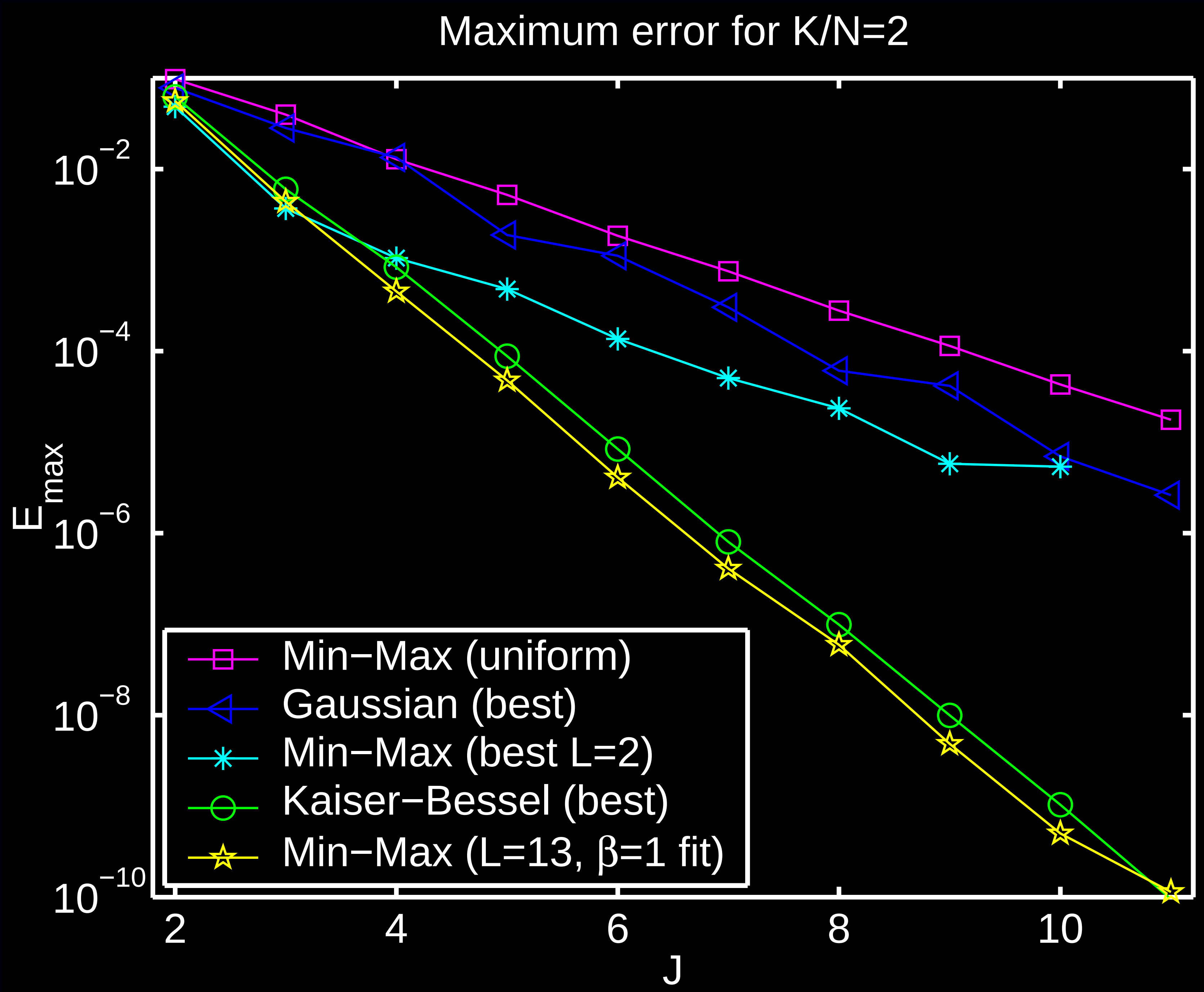
$\Longrightarrow$ needs nonuniform fast Fourier transform (NUFFT)

# NUFFT (Type 2)

- Compute over-sampled FFT of equally-spaced signal samples
- Interpolate onto desired unequally-spaced frequency locations
- Dutt & Rokhlin, SIAM JSC, 1993, Gaussian bell interpolator
- Fessler & Sutton, IEEE T-SP, 2003, min-max interpolator
  and min-max optimized Kaiser-Bessel interpolator.
  NUFFT toolbox: `http://www.eecs.umich.edu/~fessler/code`



# Worst-Case NUFFT Interpolation Error



# Further Acceleration using Toeplitz Matrices

Cost-function gradient:

$$\begin{aligned} \boldsymbol{g}^{(n)} &= \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \\ &= \boldsymbol{T}\boldsymbol{f}^{(n)} - \boldsymbol{b}, \end{aligned}$$

where

$$\boldsymbol{T} \triangleq \boldsymbol{A}'\boldsymbol{A}, \qquad \boldsymbol{b} \triangleq \boldsymbol{A}'\boldsymbol{y}.$$

In the absence of field inhomogeneity, the Gram matrix $\boldsymbol{T}$ is Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{M} \left|P(\vec{\kappa}_i)\right|^2 \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}.$$

Computing $\boldsymbol{T}\boldsymbol{f}^{(n)}$ requires an ordinary ($2\times$ over-sampled) FFT.

(Chan & Ng, SIAM Review, 1996)

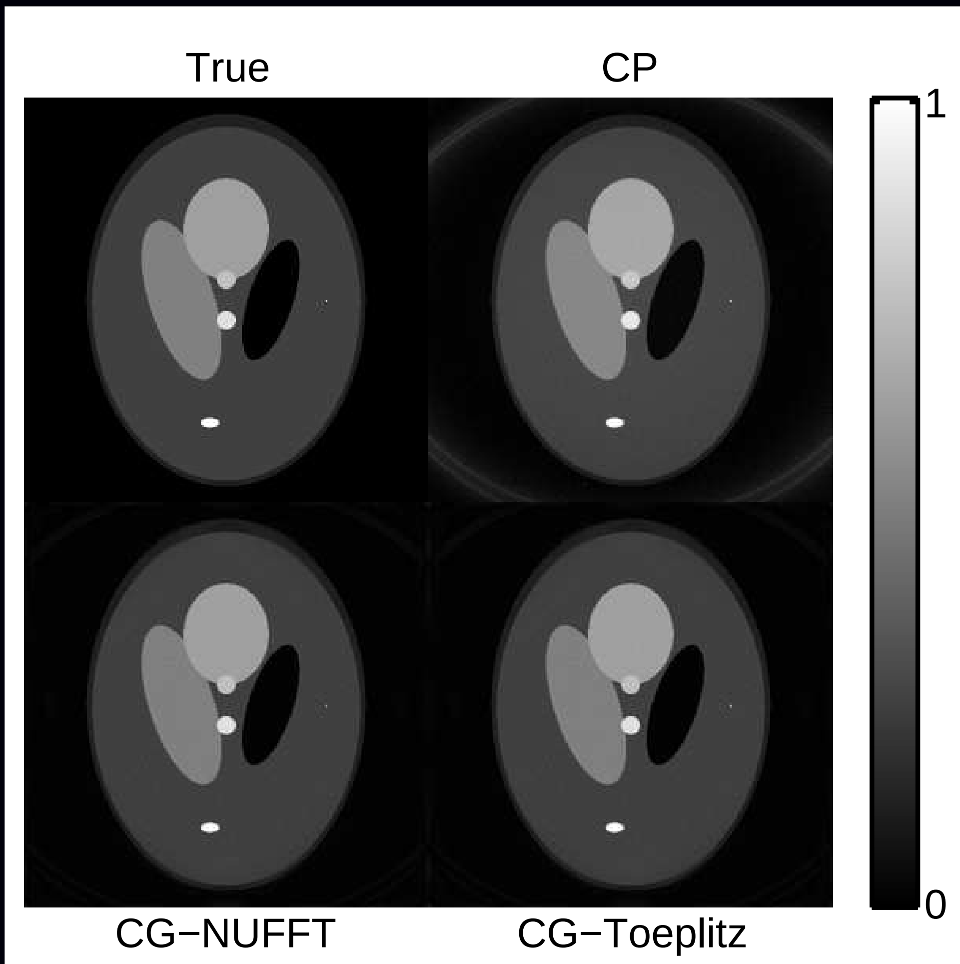
In 2D: block Toeplitz with Toeplitz blocks (BTTB).

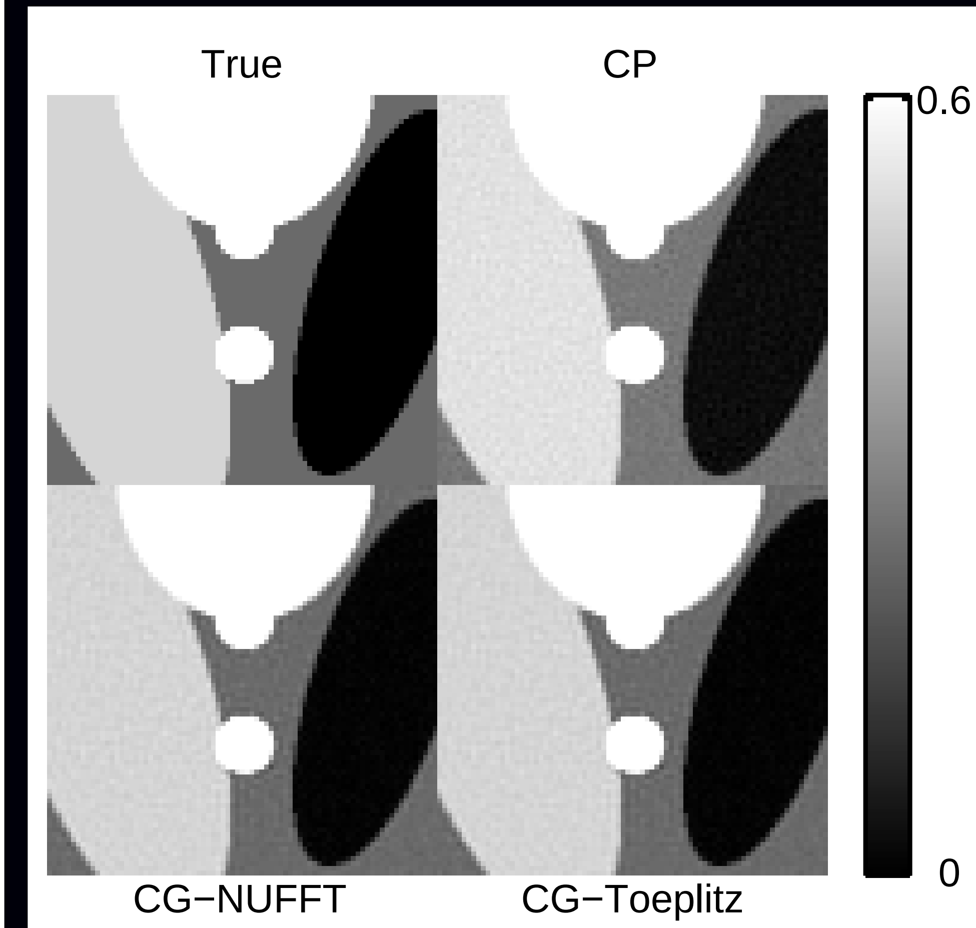
Precomputing the first column of $\boldsymbol{T}$ and $\boldsymbol{b}$ requires a couple NUFFTs.
(Wajer, ISMRM 2001, Eggers ISMRM 2002, Liu ISMRM 2005)

This formulation seems ideal for “hardware” FFT systems.

# Toeplitz Acceleration

Example: $256^2$ image. radial trajectory, $2\times$ angular under-sampling.




(Iterative provides reduced aliasing energy.)

# Toeplitz Acceleration

| Method | $\boldsymbol{A}'\boldsymbol{D}\boldsymbol{y}$ | $\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{y}$ | $\boldsymbol{T}$ | 20 iter | Total Time | NRMS (50dB) |
|---|---|---|---|---|---|---|
| Conj. Phase | 0.3 | | | | 0.3 | 7.8% |
| CG-NUFFT | | | | 12.5 | 12.5 | 4.1% |
| CG-Toeplitz | | 0.3 | 0.8 | 3.5 | 4.6 | 4.1% |

- Toeplitz aproach reduces CPU time by more than $2\times$ on conventional workstation (Xeon 3.4GHz)
- Eliminates k-space interpolations $\Longrightarrow$ ideal for FFT hardware
- No SNR compromise
- CG reduces NRMS error relative to CP, but $15\times$ slower... (More dramatic improvements seen in fMRI when correcting for field inhomogeneity.)

# NUFFT with Field Inhomogeneity?

Combine NUFFT with min-max temporal interpolator
(Sutton *et al.*, IEEE T-MI, 2003)
(forward version of “time segmentation”, Noll, T-MI, 1991)

Recall signal model including field inhomogeneity:

$$s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}.$$

Temporal interpolation approximation (aka “time segmentation”):

$$\mathrm{e}^{-\imath \omega(\vec{r}) t} \approx \sum_{l=1}^{L} a_l(t)\, \mathrm{e}^{-\imath \omega(\vec{r}) \tau_l}$$

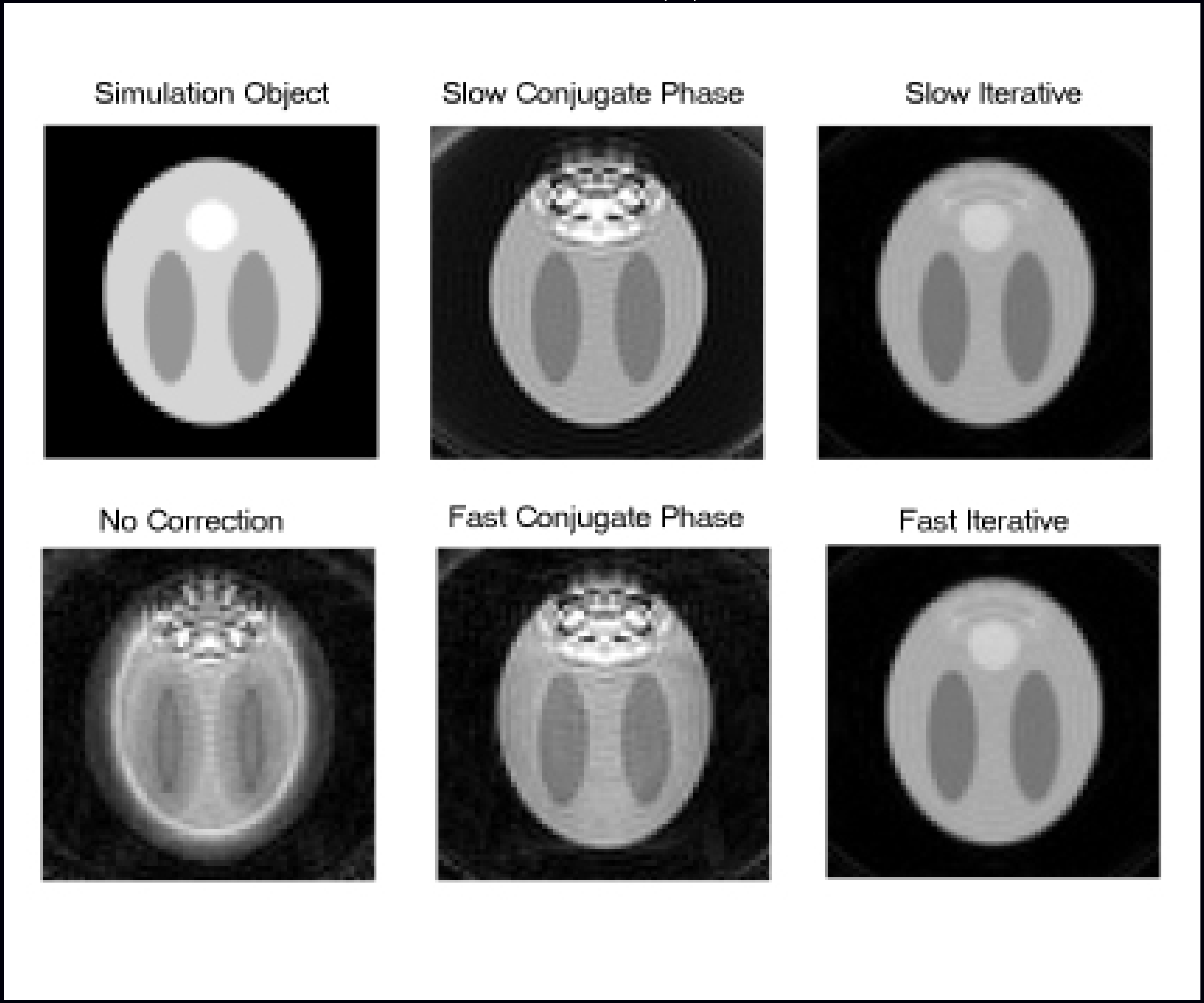
for min-max optimized temporal interpolation functions $\{a_l(\cdot)\}_{l=1}^{L}$.

$$s(t) \approx \sum_{l=1}^{L} a_l(t) \int \left[ f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) \tau_l} \right] \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Linear combination of $L$ NUFFT calls.

# Field Corrected Reconstruction Example

Simulation using known field map $\omega(\vec{r})$.

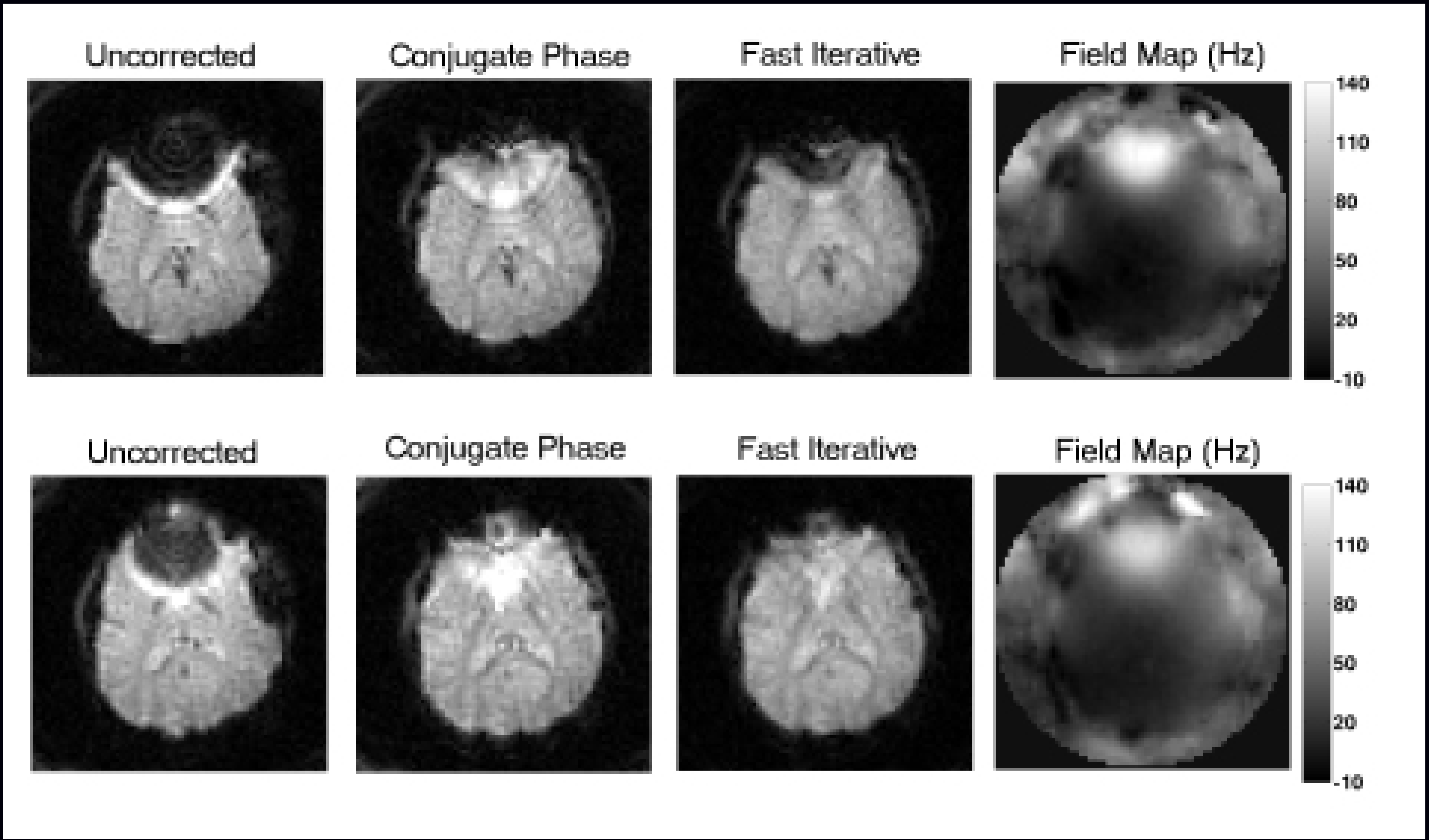


# Simulation Quantitative Comparison

- Computation time?
- NRMSE between $\hat{f}$ and $f^{\text{true}}$?

| Reconstruction Method | Time (s) | NRMSE complex | NRMSE magnitude |
|---|---|---|---|
| No Correction | 0.06 | 1.35 | 0.22 |
| Full Conjugate Phase | 4.07 | 0.31 | 0.19 |
| Fast Conjugate Phase | 0.33 | 0.32 | 0.19 |
| Fast Iterative (10 iters) | 2.20 | 0.04 | 0.04 |
| Exact Iterative (10 iters) | 128.16 | 0.04 | 0.04 |

# Human Data: Field Correction



# Acceleration using Toeplitz Approximations

In the presence of field inhomogeneity, the system matrix is:

$$a_{ij} = P(\vec{\kappa}_i)\, \mathrm{e}^{-\imath\omega(\vec{r}_j)t_i}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}$$

The Gram matrix $\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A}$ is *not* Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}\, \mathrm{e}^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i}.$$

Approximation ("time segmentation"):

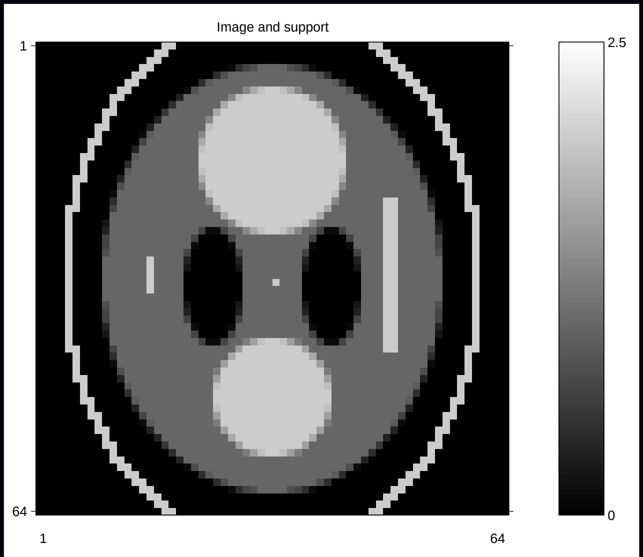
$$\mathrm{e}^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i} \approx \sum_{l=1}^{L} b_{il}\, \mathrm{e}^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) \tau_l}$$

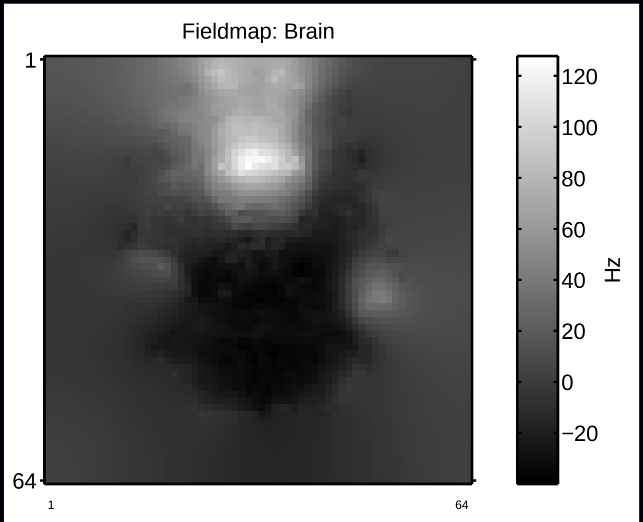
$$\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A} \approx \sum_{l=1}^{L} \boldsymbol{D}_l' \boldsymbol{T}_l \boldsymbol{D}_l, \qquad \begin{array}{l} \boldsymbol{D}_l \triangleq \mathrm{diag}\left\{\mathrm{e}^{-\imath\omega(\vec{r}_j)\tau_l}\right\} \\ [\boldsymbol{T}_l]_{jk} \triangleq \sum_{i=1}^{M} |P(\vec{\kappa}_i)|^2\, b_{il}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}. \end{array}$$

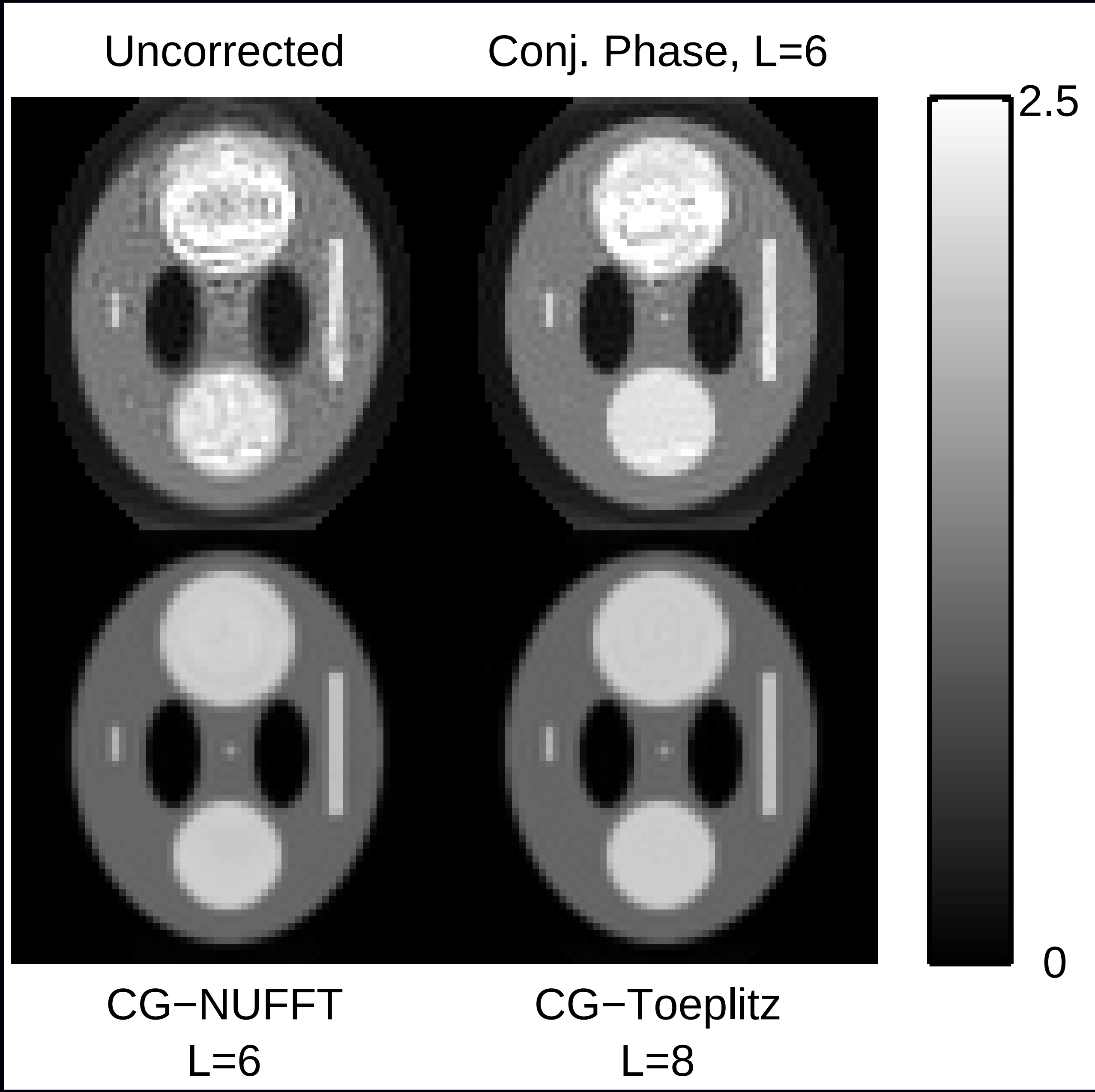
Each $\boldsymbol{T}_l$ is Toeplitz $\Longrightarrow$ $\boldsymbol{T}\boldsymbol{f}$ using $L$ pairs of FFTs.

(Fessler *et al.*, IEEE T-SP, Sep. 2005, brain imaging special issue)

# Toeplitz Results



# Toeplitz Acceleration

| | | Precomputation | | | | | | NRMS % vs SNR | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Method | $L$ | $\boldsymbol{B},\boldsymbol{C}$ | $\boldsymbol{A}'\boldsymbol{D}\boldsymbol{y}$ | $\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{y}$ | $\boldsymbol{T}_l$ | 15 iter | Total Time | $\infty$ | 50 dB | 40 dB | 30 dB | 20 dB |
| Conj. Phase | 6 | 0.4 | 0.2 | | | | 0.6 | 30.7 | 37.3 | 46.5 | 65.3 | 99.9 |
| CG-NUFFT | 6 | 0.4 | | | | 5.0 | 5.4 | 5.6 | 16.7 | 26.5 | 43.0 | 70.4 |
| CG-Toeplitz | 8 | 0.4 | | 0.2 | 0.6 | 1.3 | 2.5 | 5.5 | 16.7 | 26.4 | 42.9 | 70.4 |

- Reduces CPU time by $2\times$ on conventional workstation (Mac G5)
- No SNR compromise
- Eliminates k-space interpolations $\Longrightarrow$ ideal for FFT hardware

# Joint Field-Map / Image Reconstruction

Signal model:

$$y_i = s(t_i) + \varepsilon_i, \qquad s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}.$$

After discretization:

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{\omega})\, \boldsymbol{f} + \boldsymbol{\varepsilon}, \qquad a_{ij}(\boldsymbol{\omega}) = P(\vec{\kappa}_i)\, \mathrm{e}^{-\imath \omega_j t_i}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}.$$

Joint estimation via regularized (nonlinear) least-squares:

$$(\hat{\boldsymbol{f}}, \hat{\boldsymbol{\omega}}) = \operatorname*{arg\,min}_{\boldsymbol{f} \in \mathbb{C}^N, \boldsymbol{\omega} \in \mathbb{R}^N} \|\boldsymbol{y} - \boldsymbol{A}(\boldsymbol{\omega})\, \boldsymbol{f}\|^2 + \beta_1 R_1(\boldsymbol{f}) + \beta_2 R_2(\boldsymbol{\omega}).$$

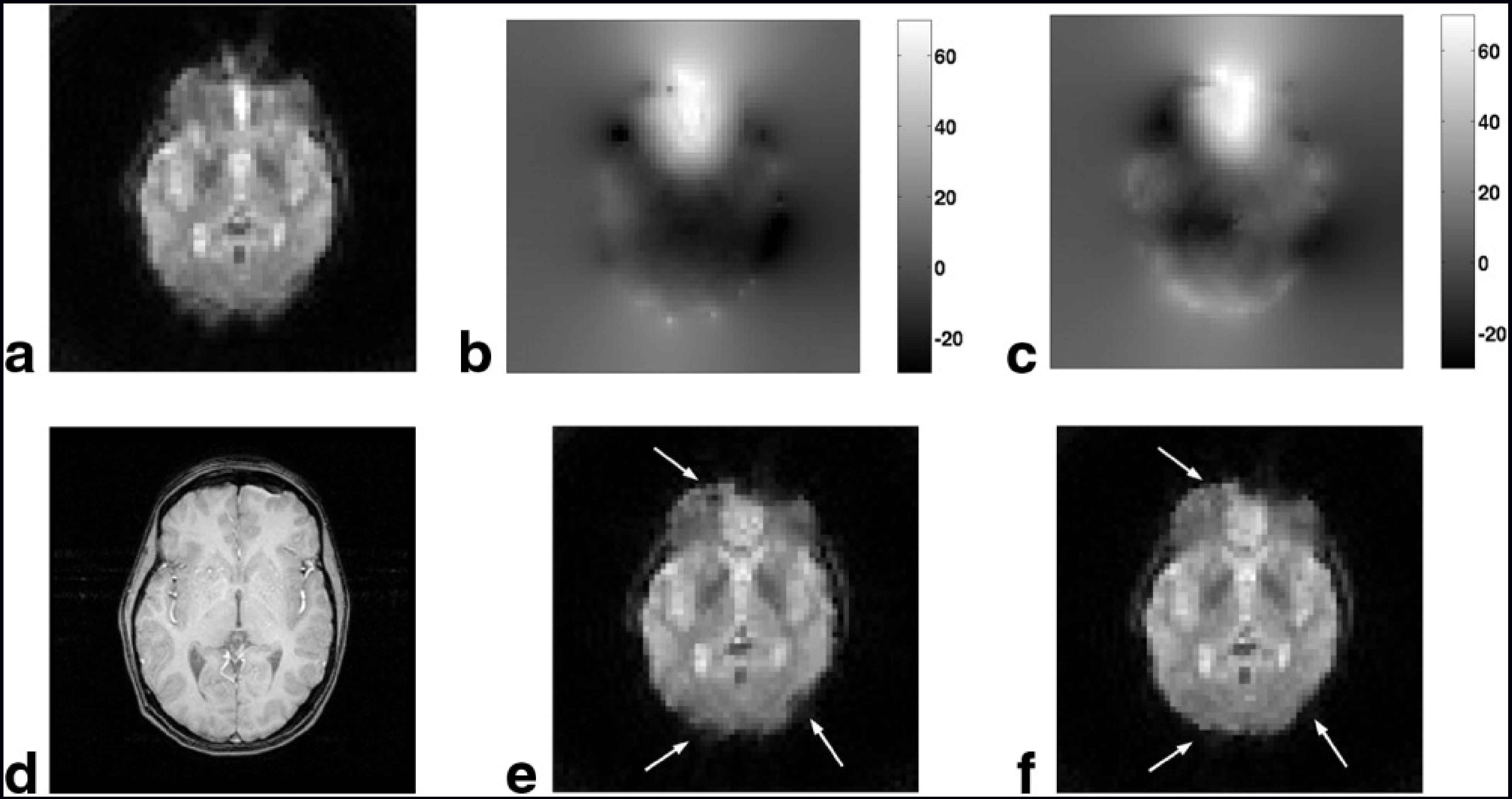
Alternating minimization:

- Using current estimate of fieldmap $\hat{\boldsymbol{\omega}}$, update $\hat{\boldsymbol{f}}$ using CG algorithm.
- Using current estimate $\hat{\boldsymbol{f}}$ of image, update fieldmap $\hat{\boldsymbol{\omega}}$ using gradient descent.

Use spiral-in / spiral-out sequence or “racetrack” EPI.

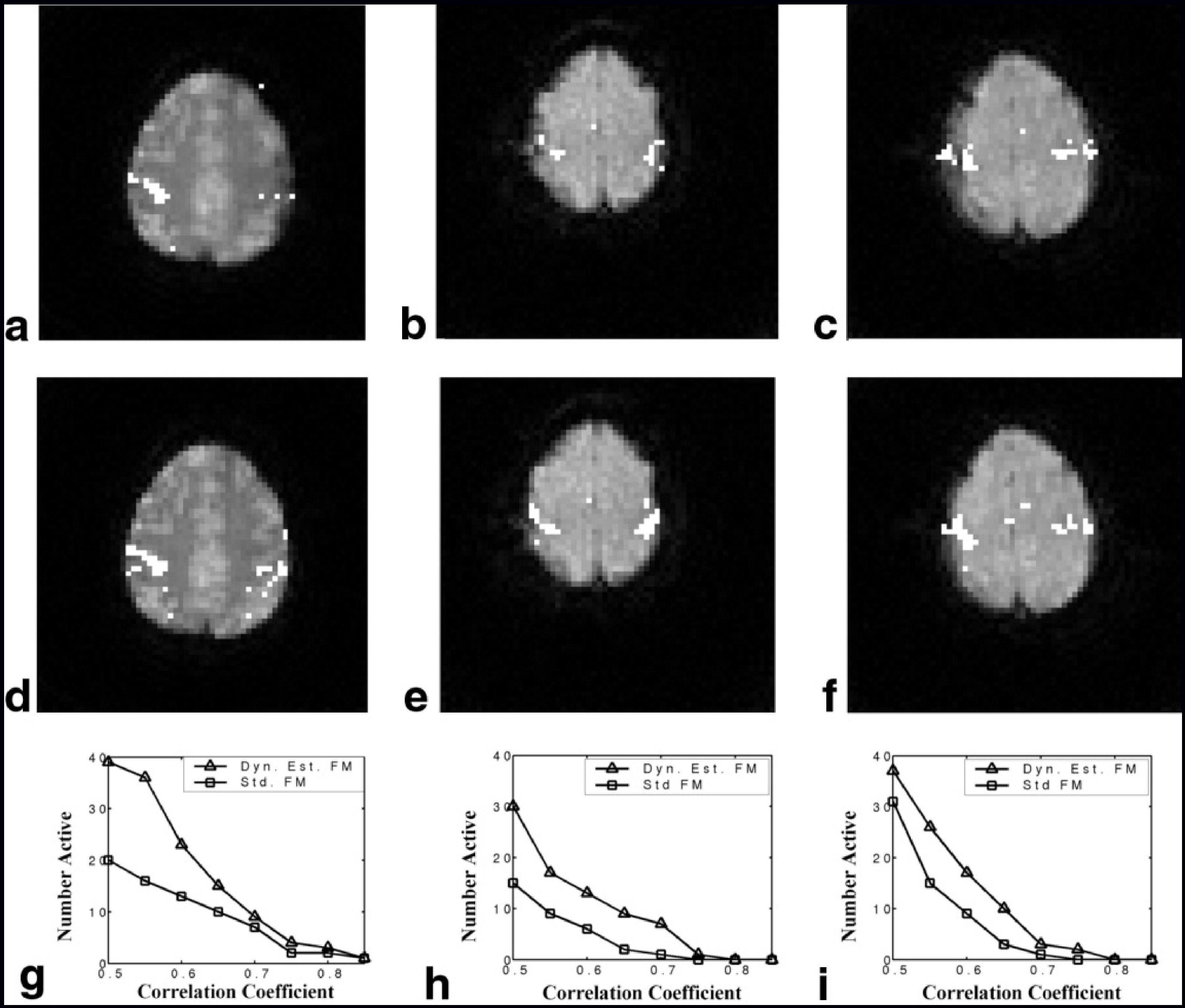
(Sutton *et al.*, MRM, 2004)

# Joint Estimation Example




(a) uncorr., (b) std. map, (c) joint map, (d) T1 ref, (e) using std, (f) using joint.

# Activation Results: Static vs Dynamic Field Maps



Functional results for the two reconstructions for 3 human subjects.

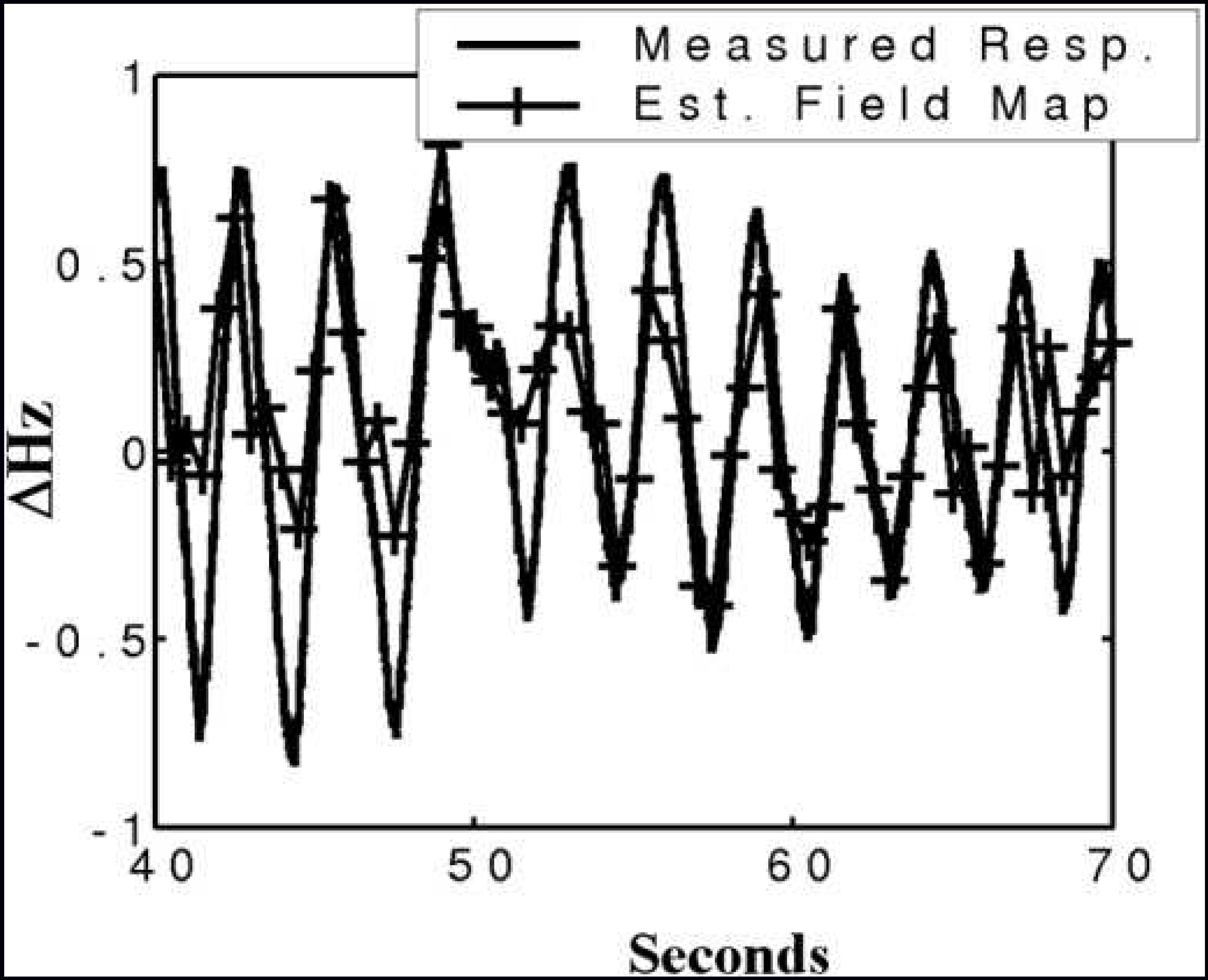
Reconstruction using the standard field map
for (a) subject 1, (b) subject 2, and (c) subject 3.

Reconstruction using the jointly estimated field map
for (d) subject 1, (e) subject 2, and (f) subject 3.

Number of pixels with correlation coefficients higher than thresholds
for (g) subject 1, (h) subject 2, and (i) subject 3.

Take home message: dynamic field mapping is possible, using iterative reconstruction as an essential tool.
(Standard field maps based on echo-time differences work poorly for spiral-in / spiral-out sequences due to phase discrepancies.)

# Tracking Respiration-Induced Field Changes



# Myths

- Choosing $\alpha$ is difficult
- Sample density weighting is desirable

# Sampling density weighted LS

Some researchers recommend using a weighted LS cost function:

$$\Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|_{\boldsymbol{W}}^2$$

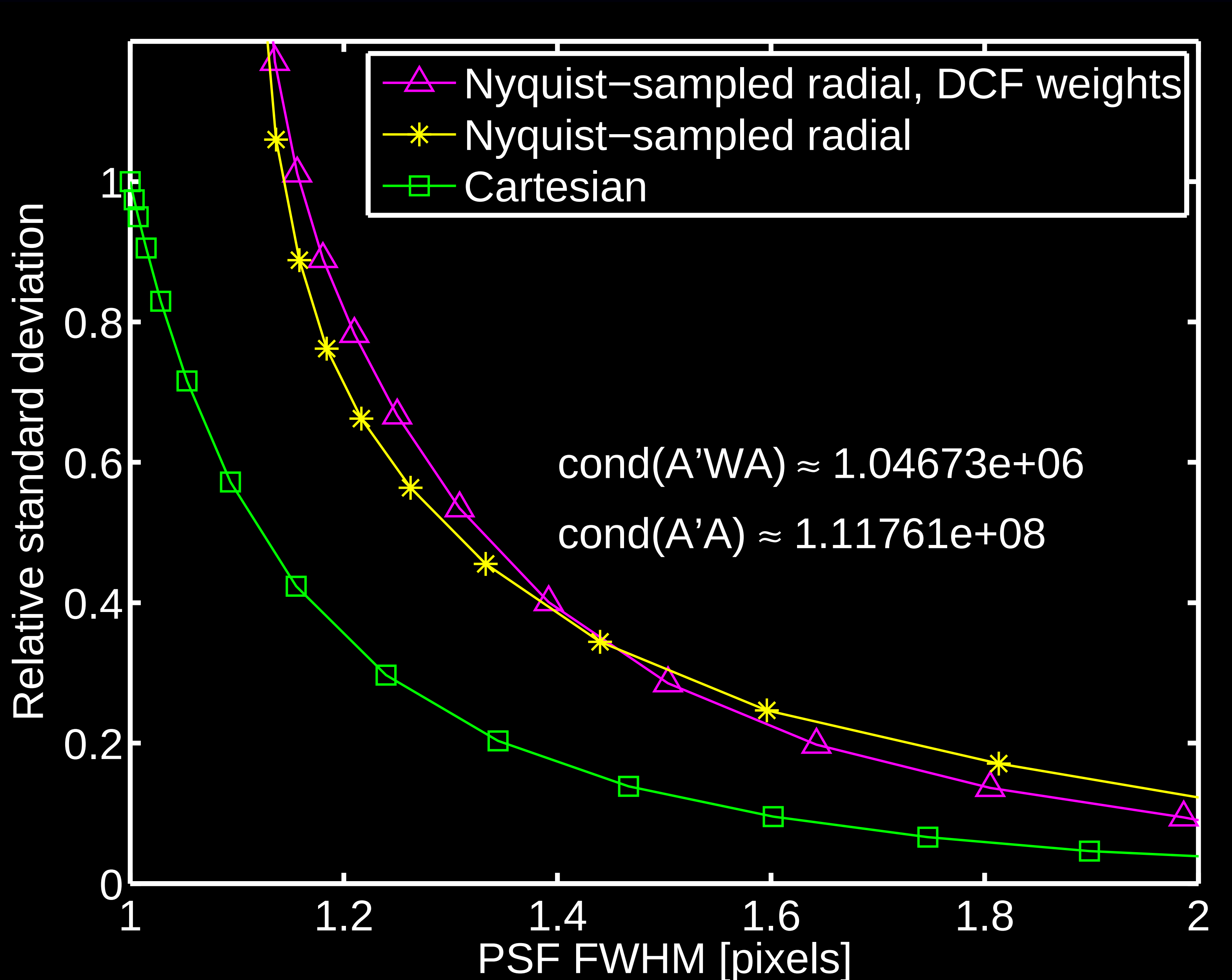
where the weighting matrix $\boldsymbol{W}$ is related to the k-space sample density compensation factors (DCF).

Purported benefits:

- Faster convergence
- Better conditioning

But, Gauss-Markov theorem from statistical estimation theory states that lowest estimator variance is realized by choosing $\boldsymbol{W} = \sigma_{\varepsilon}^{-2}\boldsymbol{I}$, the inverse of the data noise covariance.

# Resolution/Noise Tradeoff: Example with Weighting



# Don't just take it from me...

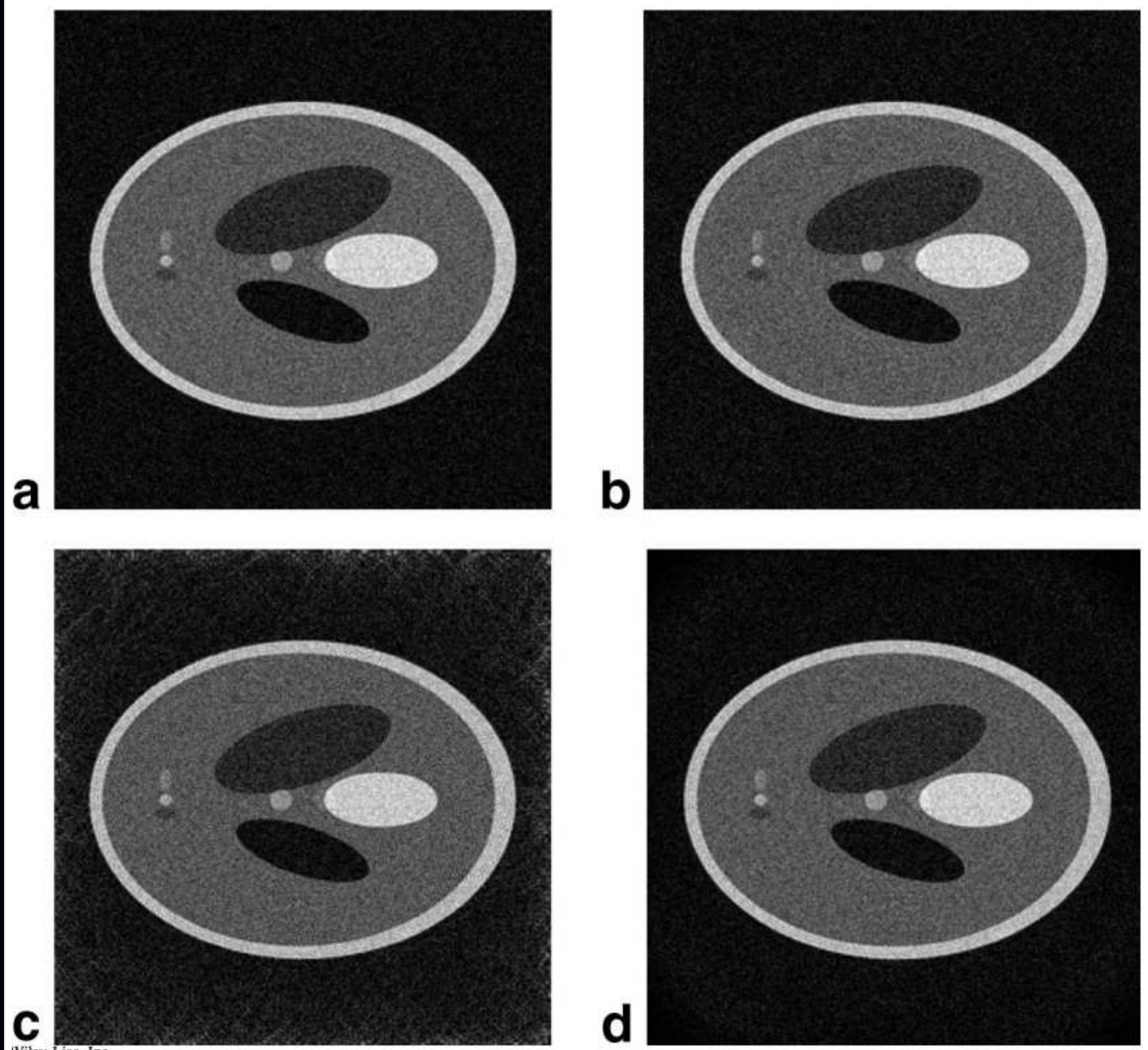





**Figure 5.** Reconstructed images for simulated spiral sampling of the noisy *k*-space of the SL phantom (60% noise) using conventional gridding (**a**), F-CG without (**b**) and with spiral DCF (**c**), and DING (**d**).

Fig. 5 of Gabr *et al.*, MRM, Dec. 2006

# Acceleration via Weighting?

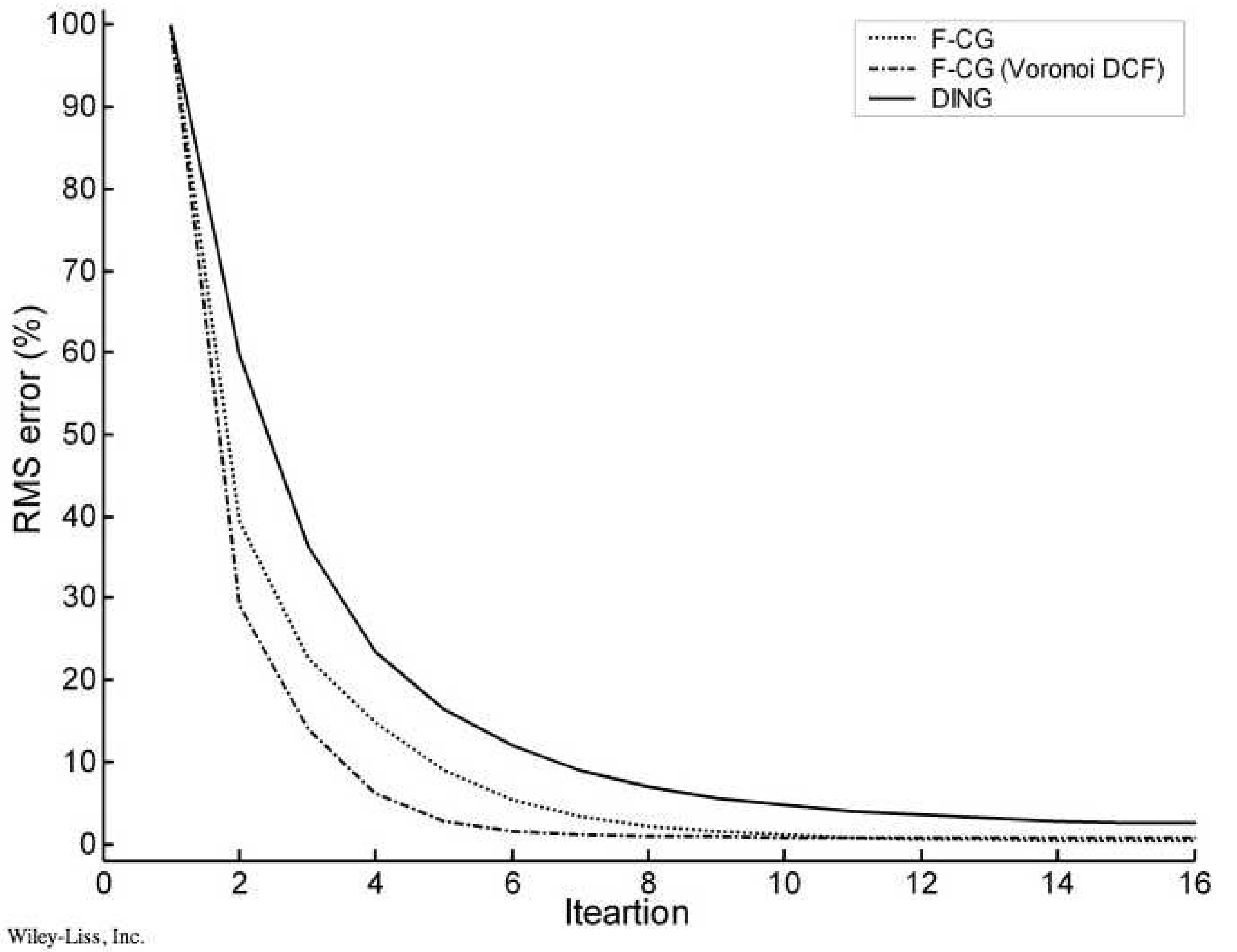
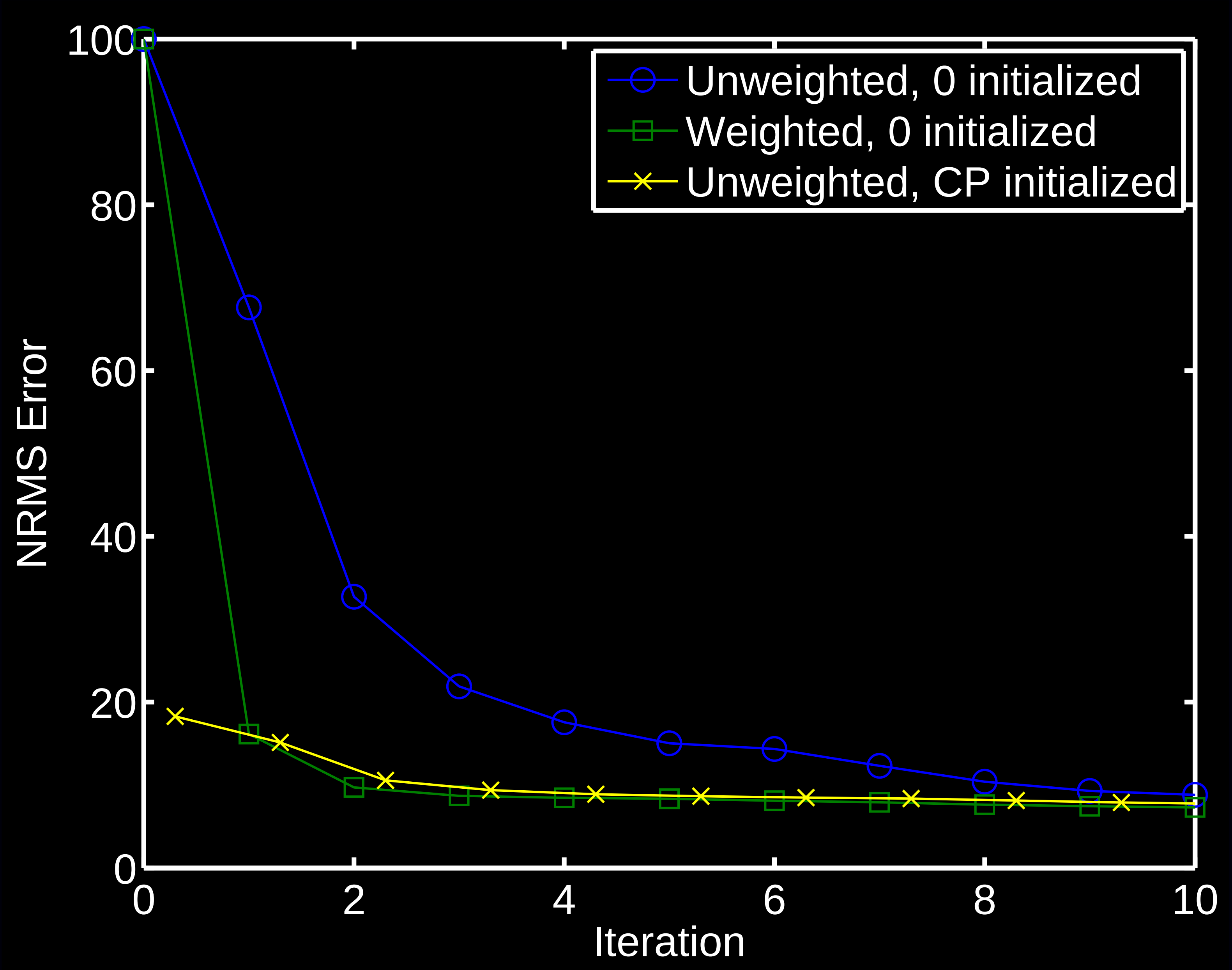



**Figure 8.** The RMS error of F-CG and DING vs. the number of CG iterations for the simulated noiseless SL the random trajectory. Errors stabilize after approximately 15 iterations.

Fig. 8 of Gabr *et al.*, MRM, Dec. 2006. Zero initialization!

# Acceleration via Initialization



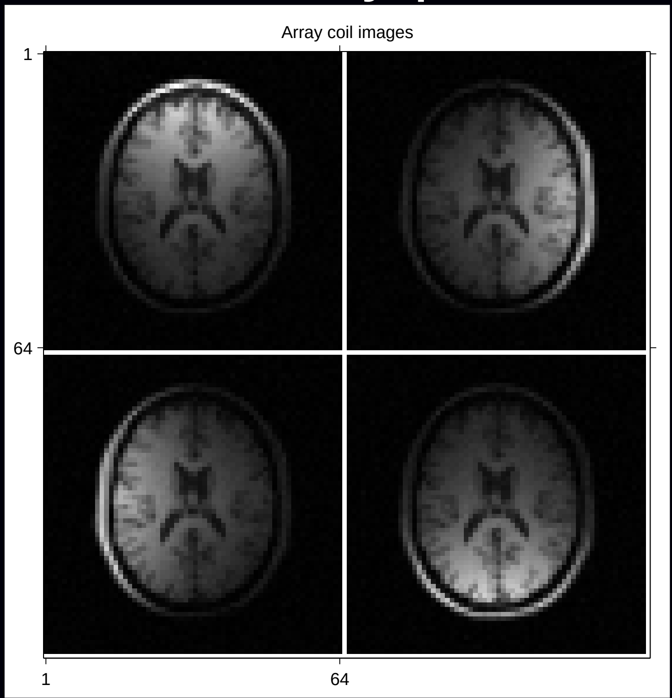
# Parallel Imaging

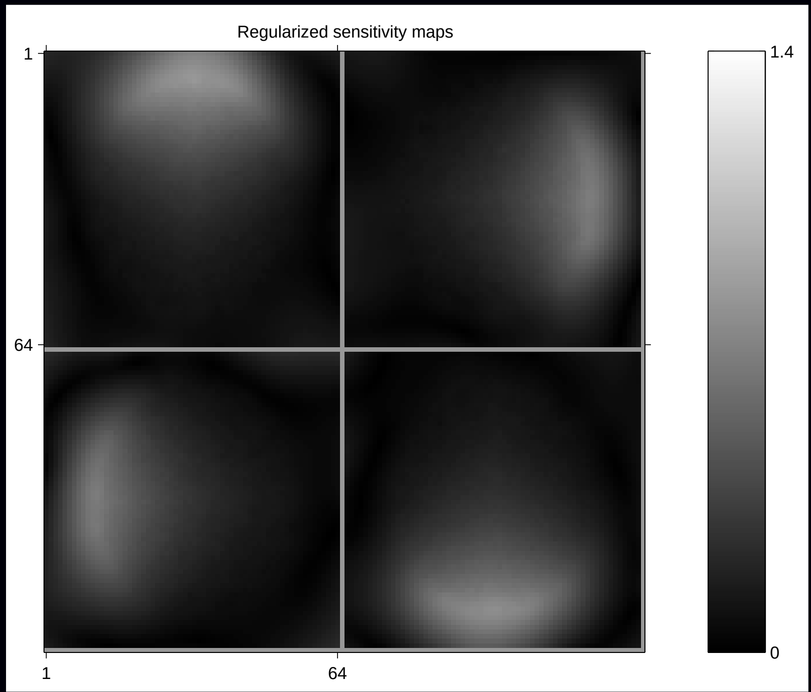
# Sensitivity encoded (SENSE) imaging

Use multiple receive coils (requires multiple RF channels).
Exploit spatial localization of sensitivity pattern of each coil.

Note: at 1.5T, RF is about 60MHz.
$\Longrightarrow$ RF wavelength is about $3 \cdot 10^8$m/s$/60 \cdot 10^6$Hz $= 5$ meters

RF coil sensitivity patterns




Pruessmann *et al.*, MRM, 1999

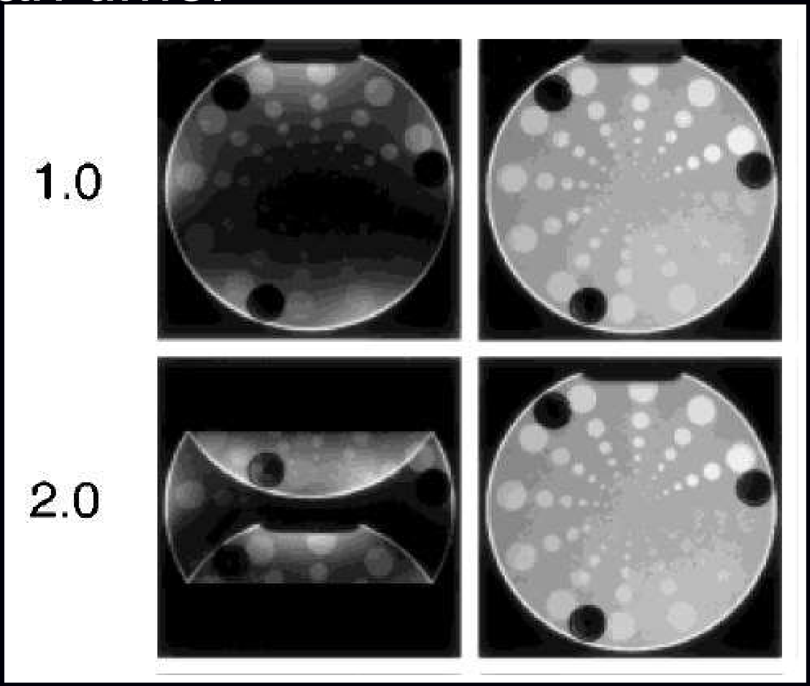
# SENSE Model

Multiple coil data:

$$y_{li} = s_l(t_i) + \varepsilon_{li}, \quad s_l(t) = \int f(\vec{r})\, s_l^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}, \quad l = 1,\ldots,L = N_{\text{coil}}$$

Goal: reconstruct $f(\vec{r})$ from coil data $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_L$
"given" sensitivity maps $\left\{s_l^{\text{coil}}(\vec{r})\right\}_{l=1}^{L}$.

Benefit: reduced scan time.




Left: sum of squares; right: SENSE.

# SENSE Reconstruction

Signal model:

$$s_l(t) = \int f(\vec{r})\, s_l^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}$$

Discretized form:

$$\boldsymbol{y}_l = \boldsymbol{A}\boldsymbol{D}_l \boldsymbol{f} + \boldsymbol{\varepsilon}_l, \quad l = 1, \ldots, L,$$

where $\boldsymbol{A}$ is the usual frequency/phase encoding matrix and
$\boldsymbol{D}_l$ contains the sensitivity pattern of the $l$th coil: $\boldsymbol{D}_l = \mathrm{diag}\left\{ s_l^{\text{coil}}(\vec{r}_j) \right\}$.

Regularized least-squares estimation:

$$\hat{\boldsymbol{f}} = \arg\min_{\boldsymbol{f}} \sum_{l=1}^{L} \|\boldsymbol{y}_l - \boldsymbol{A}\boldsymbol{D}_l \boldsymbol{f}\|^2 + \beta R(\boldsymbol{f}).$$

Can generalize to account for noise correlation due to coil coupling.
Easy to apply CG algorithm, including Toeplitz/NUFFT acceleration.

For Cartesian SENSE, iterations are not needed.
(Solve small system of linear equations for each voxel.)

# Summary

- Iterative reconstruction: much potential in MRI
- Quadratic regularization parameter selection is tractable
- Computation: reduced by tools like NUFFT / Toeplitz
- But optimization algorithm design remains important
  (*cf.* Shepp and Vardi, 1982, PET)

# Some current challenges

- Sensitivity pattern mapping for SENSE
- Through-voxel field inhomogeneity gradients
- Motion / dynamics / partial k-space data
- Establishing diagnostic efficacy with clinical data...

Image reconstruction toolbox:
`http://www.eecs.umich.edu/~fessler`