Iterative reconstruction for MR imaging

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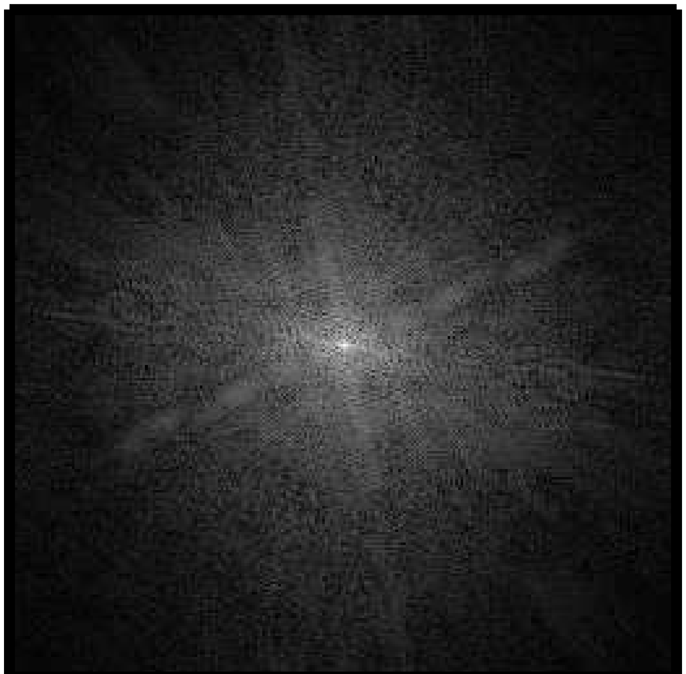
Outline

- MR image reconstruction introduction
  (k-space, FFT, gridding, density compensation)
- Model-based reconstruction overview
- Iterations and computation (NUFFT etc.)
- Regularization
- Field inhomogeneity correction
- Parallel (sensitivity encoded) imaging
- Iterative methods for RF pulse design
- Examples

Introduction

Standard MR Image Reconstruction

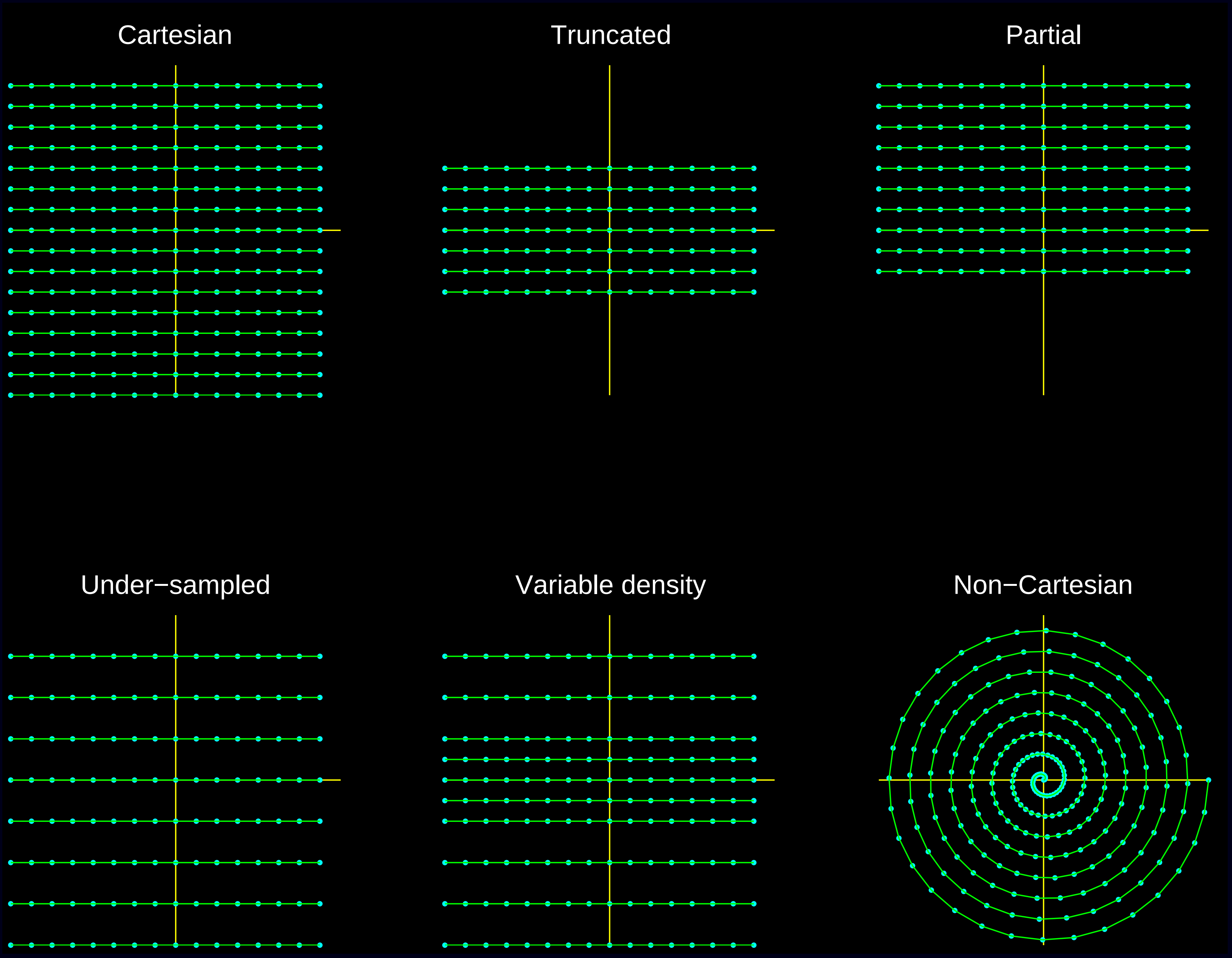
MR k−space data Reconstructed Image



Cartesian sampling in k-space. An inverse FFT. End of story.

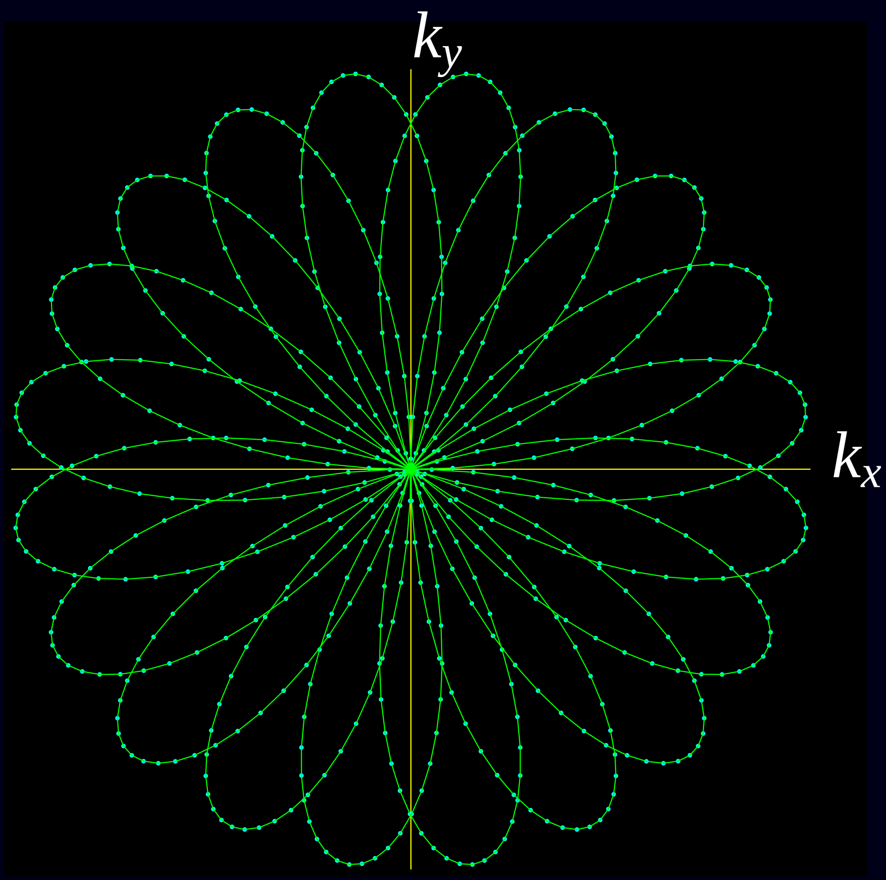
Commercial MR system quotes 400 FFTs (256^2) per second.

K-space Sampling Strategies



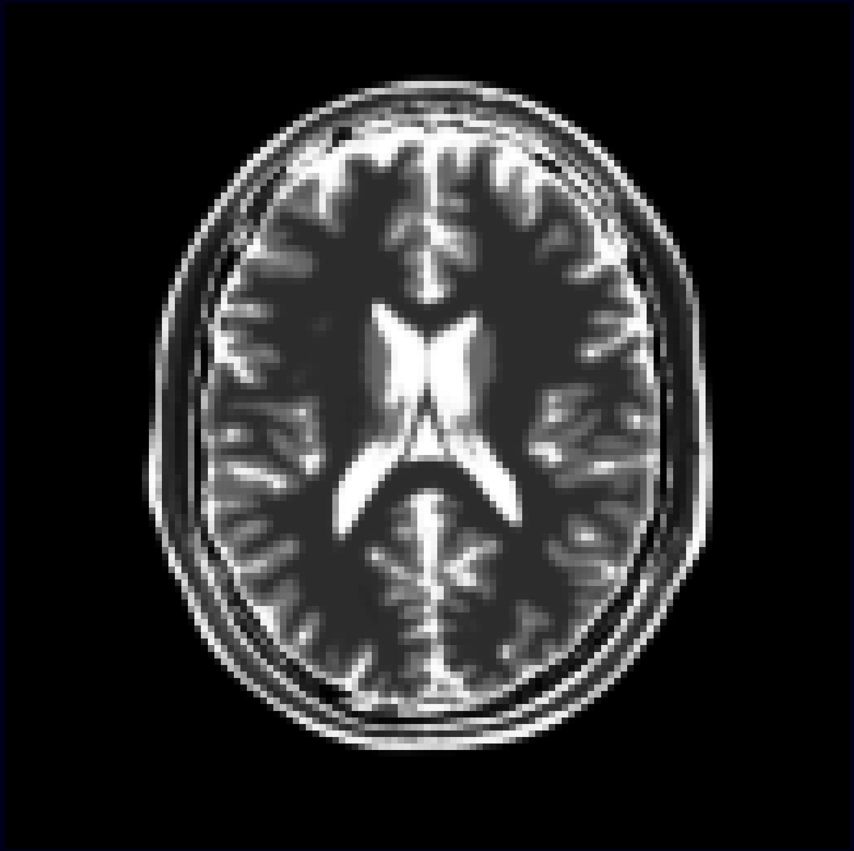
Non-Cartesian MR Image Reconstruction

"k-space"

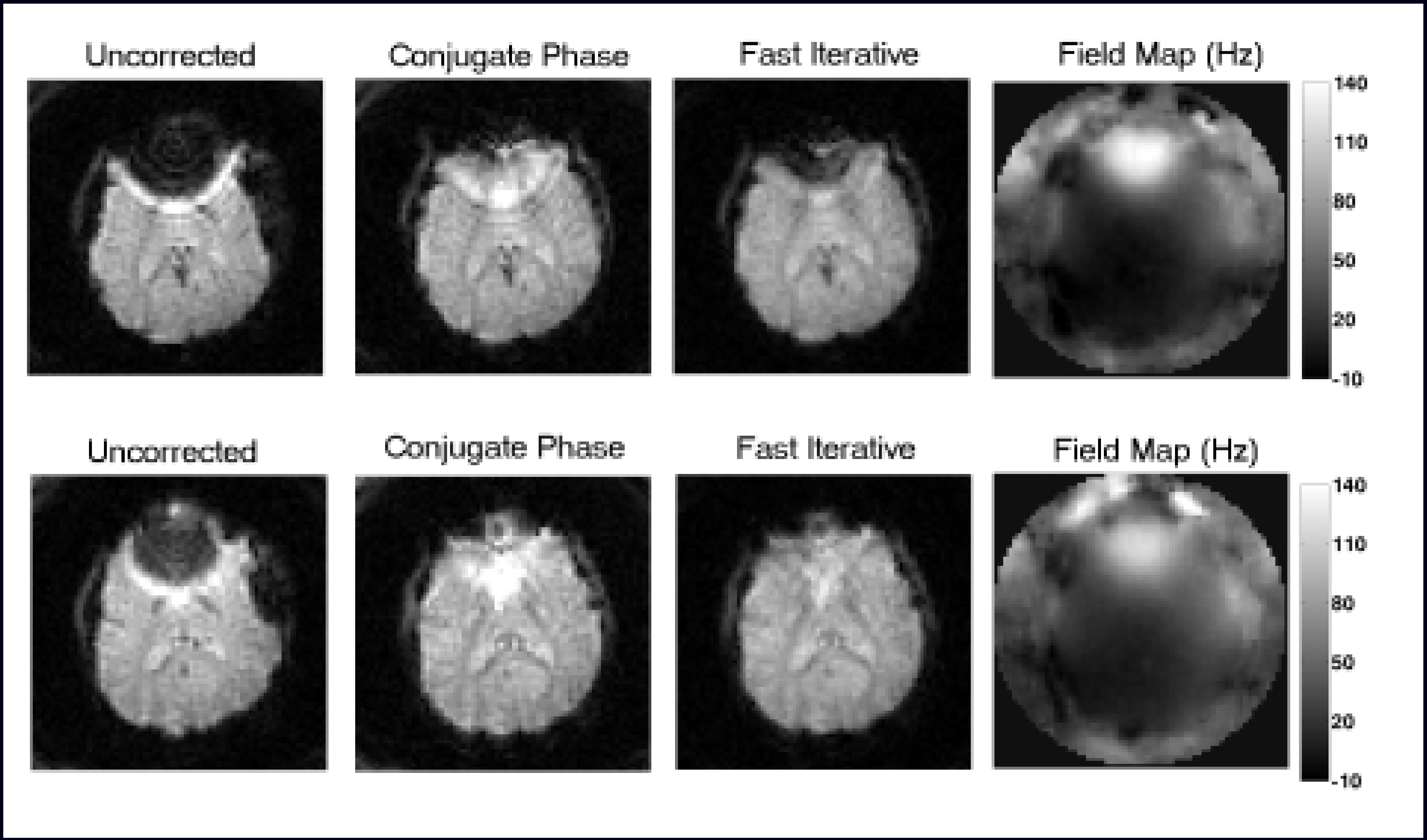


$\Longrightarrow$

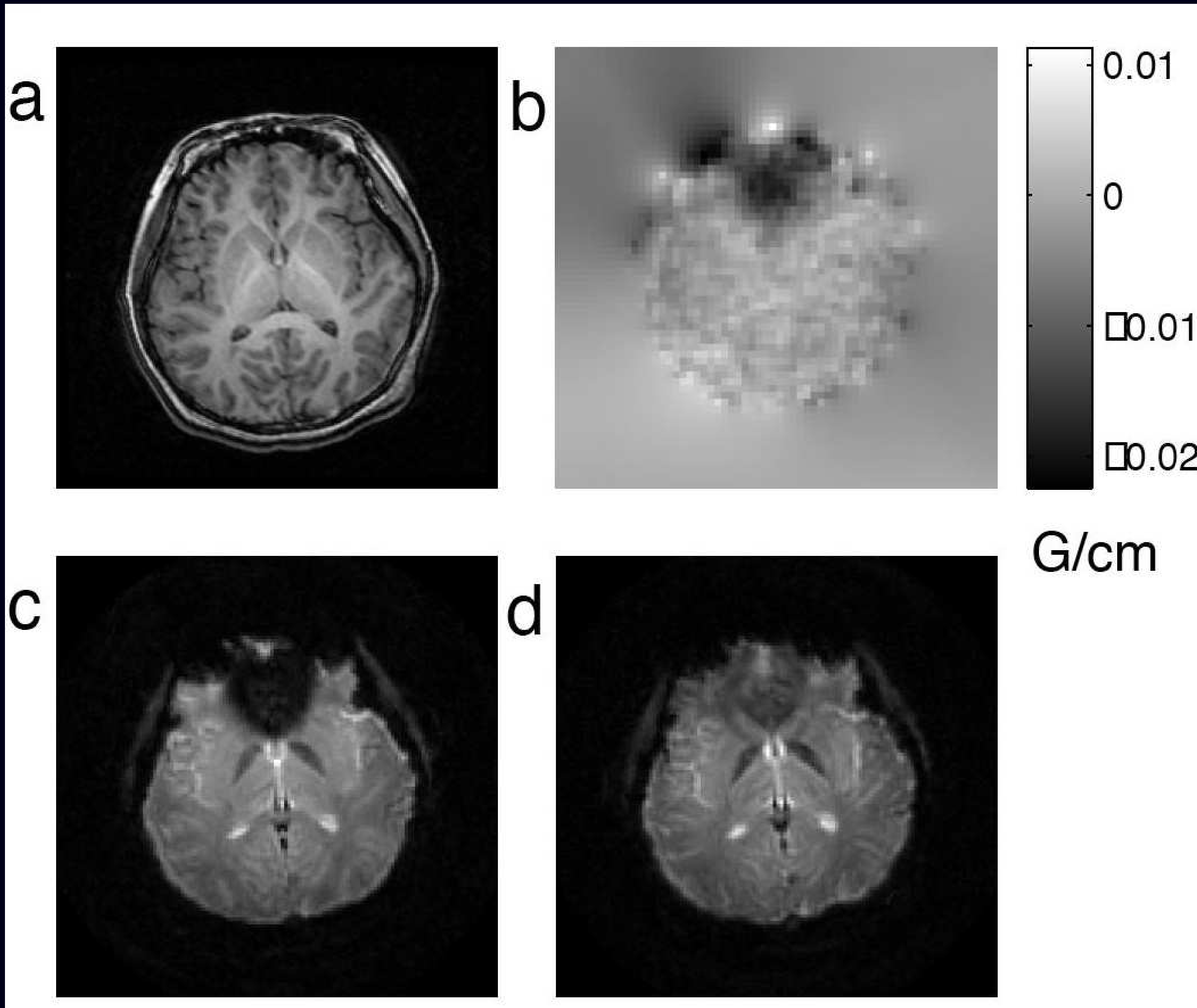
image



Example: Iterative Reconstruction under ΔB_0



Example: Iterative RF Pulse Design



Textbook MRI Measurement Model

Ignoring *lots* of things:

$$y_i = s(t_i) + \mathrm{noise}_i, \qquad i = 1, \ldots, N$$
$$s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r},$$

where $\vec{r}$ denotes spatial position, and
$\vec{\kappa}(t)$ denotes the “k-space trajectory” of the MR pulse sequence,
determined by user-controllable magnetic field gradients.

$\mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}$ provides spatial information $\Longrightarrow$ Nobel Prize

- MRI measurements are (roughly) samples of the Fourier transform $F(\vec{\kappa})$ of the object’s transverse magnetization $f(\vec{r})$.
- Basic image reconstruction problem:

  recover $f(\vec{r})$ from measurements $\{y_i\}_{i=1}^N$.

Inherently under-determined (ill posed) problem

$\Longrightarrow$ no canonical solution.

Image Reconstruction Strategies

The unknown object $f(\vec{r})$ is a continuous-space function,
but the recorded measurements $\boldsymbol{y} = (y_1, \ldots, y_N)$ are finite.

Options?

- Continuous-discrete formulation using many-to-one linear model:

$$\boldsymbol{y} = \mathcal{A} f + \boldsymbol{\varepsilon}.$$

Minimum norm solution (*cf.* "natural pixels"):

$$\min_{\hat{f}} \left\| \hat{f} \right\| \text{ subject to } \boldsymbol{y} = \mathcal{A} \hat{f}$$

$$\hat{f} = \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\boldsymbol{y} = \sum_{i=1}^N c_i \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}, \text{ where } \mathcal{A}\mathcal{A}^* \boldsymbol{c} = \boldsymbol{y}.$$

- Discrete-discrete formulation
Assume parametric model for object:

$$f(\vec{r}) = \sum_{j=1}^{M} f_j \, p_j(\vec{r}) \, .$$

- Continuous-continuous formulation

  Pretend that a continuum of measurements are available:

$$F(\vec{\kappa}) = \int f(\vec{r})\, e^{-\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, d\vec{r},$$

  vs samples $y_i = F(\vec{\kappa}_i) + \varepsilon_i$, where $\vec{\kappa}_i \triangleq \vec{\kappa}(t_i)$.

  The "solution" is an inverse Fourier transform:

$$f(\vec{r}) = \int F(\vec{\kappa})\, e^{\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, d\vec{\kappa}.$$

  Now discretize the integral solution (two approximations!):

$$\hat{f}(\vec{r}) = \sum_{i=1}^{N} F(\vec{\kappa}_i)\, e^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}}\, w_i \approx \sum_{i=1}^{N} y_i w_i\, e^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}},$$

  where w_i values are "sampling density compensation factors."

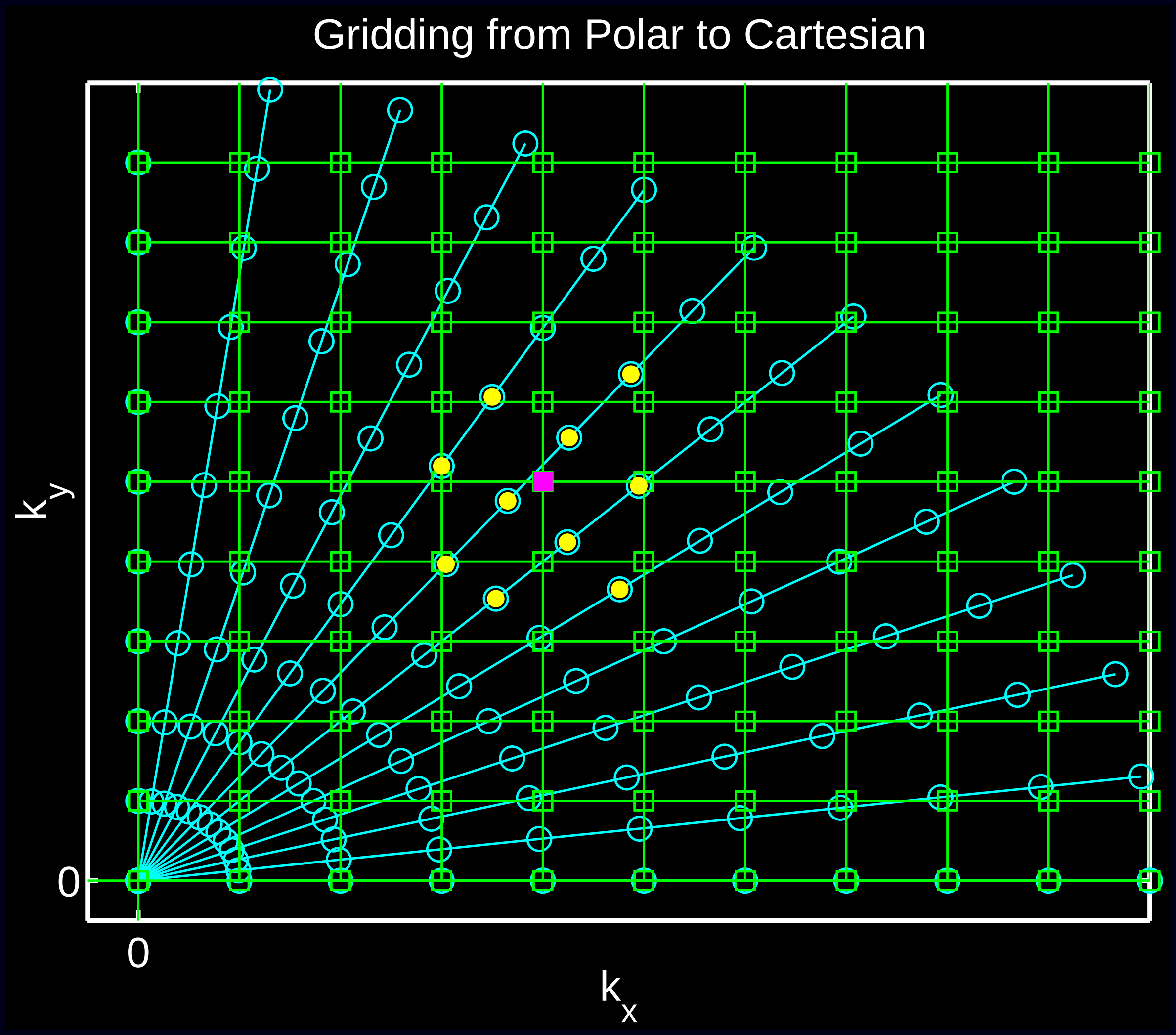
  Numerous methods for choosing w_i value in the literature.

  For Cartesian sampling, using $w_i = 1/N$ suffices,

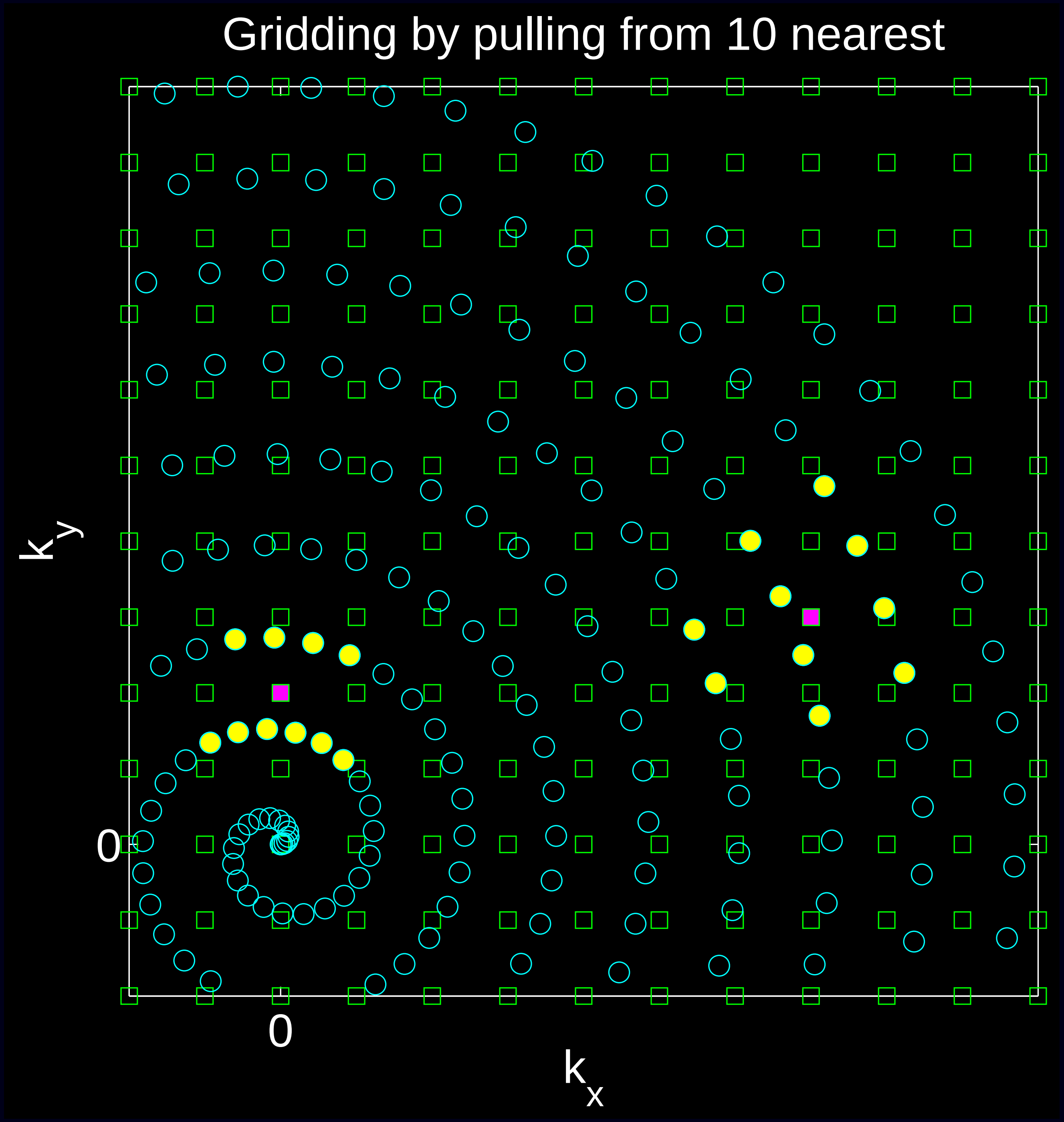
  and the summation is an inverse FFT.

Conventional MR Image Reconstruction

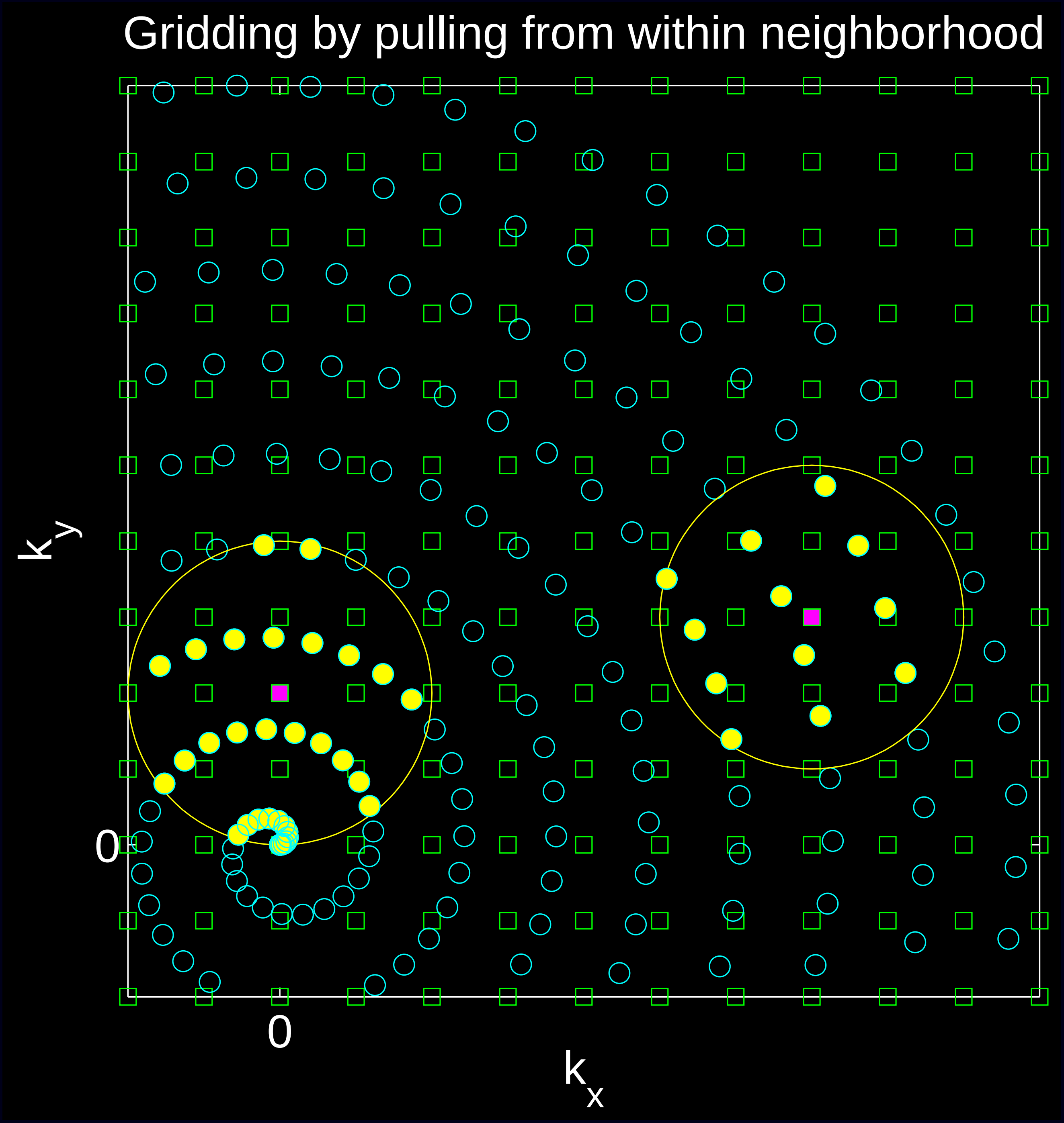
1. Interpolate measurements onto rectilinear grid ("gridding")
2. Apply inverse FFT to estimate samples of $f(\vec{r})$



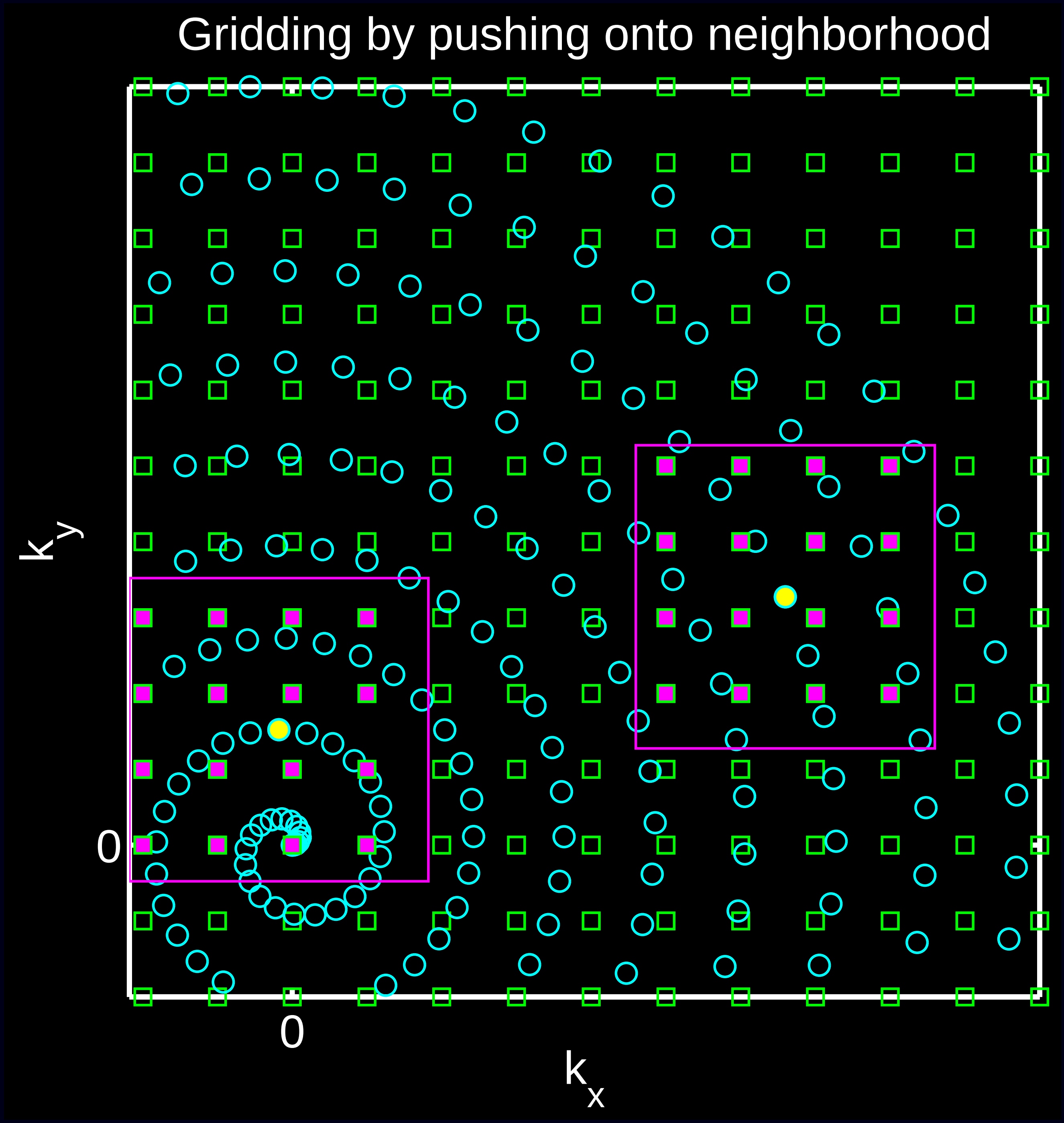
Gridding Approach 1: Pull from K nearest



Gridding Approach 2: Pull from neighborhood



Gridding Approach 3: Push to neighborhood



Gridding Approaches

Ignore noise: $y_i = F(\vec{\kappa}_i)$

Pull:
for each Cartesian grid point, use weighted average of nonuniform k-space samples within some neighborhood

- Does not require density compensation
- Requires cumbersome search/indexing to find neighbors

Push:
each nonuniform k-space sample onto a Cartesian neighborhood

$$\hat{F}(\vec{\kappa}) = \sum_{i=1}^{N} y_i w_i C(\vec{\kappa} - \vec{\kappa}_i)$$

- $C(\vec{\kappa})$ denotes the *gridding kernel*, typically separable Kaiser-Bessel
  Jackson *et al.*, IEEE T-MI, 1991
- "$*$" denotes convolution.
- $\delta(\cdot)$ denotes the Dirac impulse
- density compensation factors w_i essential

Post-iFFT Gridding Correction

Gridding as convolution in k-space:

$$\hat{F}(\vec{\kappa}) = \sum_{i=1}^{N} y_i w_i C(\vec{\kappa} - \vec{\kappa}_i) = C(\vec{\kappa}) * \sum_{i=1}^{N} y_i w_i \delta(\vec{\kappa} - \vec{\kappa}_i).$$

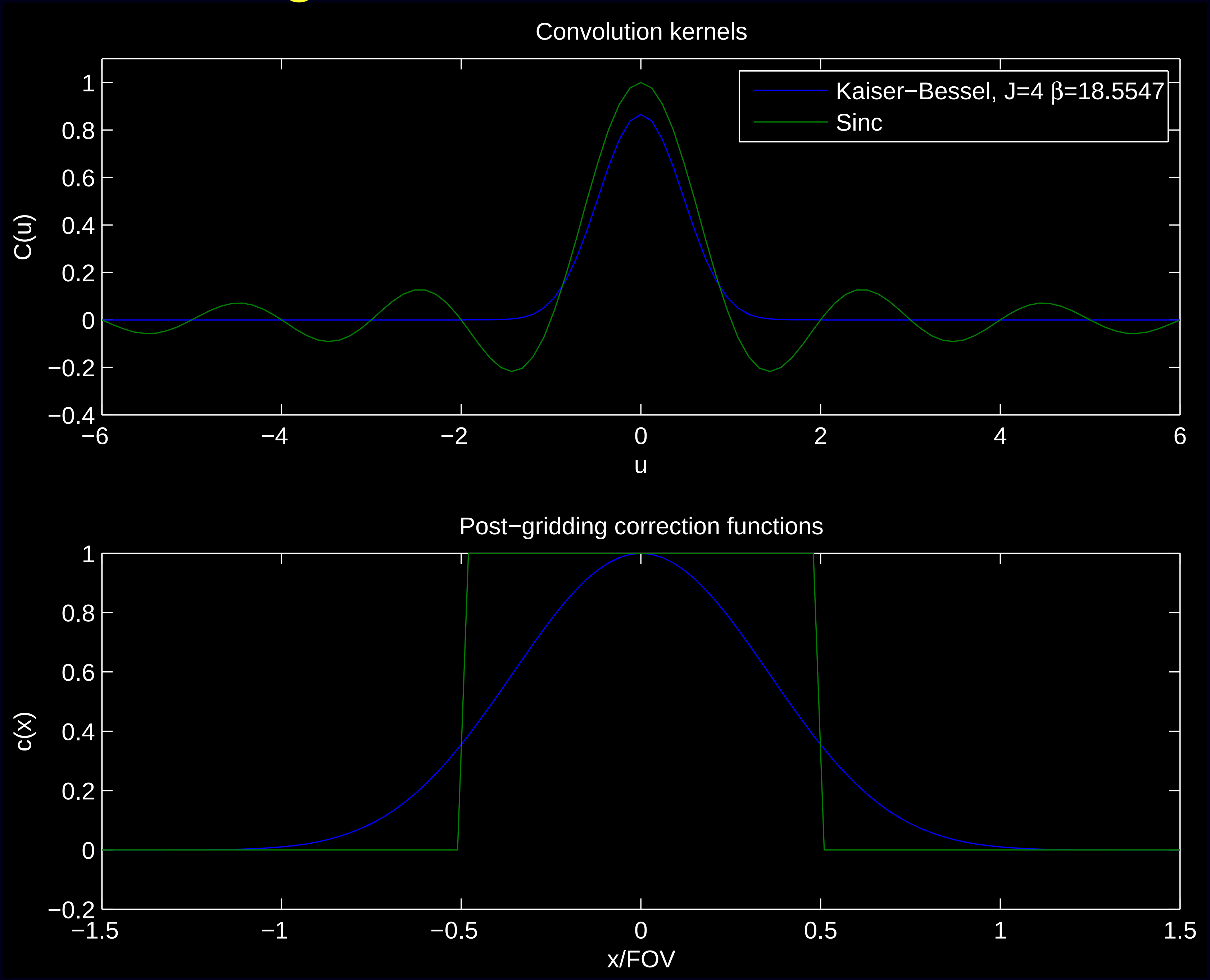
Inverse FT reconstruction:

$$\hat{f}_{\text{initial}}(\vec{r}) = \mathcal{F}^{-1}\left\{\hat{F}(\vec{\kappa})\right\} = c(\vec{r}) \sum_{i=1}^{N} y_i w_i \, e^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}.$$

Post-correction:

$$\hat{f}_{\text{final}}(\vec{r}) = \frac{\hat{f}_{\text{initial}}(\vec{r})}{c(\vec{r})}.$$

Gridding Kernels and Post-corrections



Density Compensation

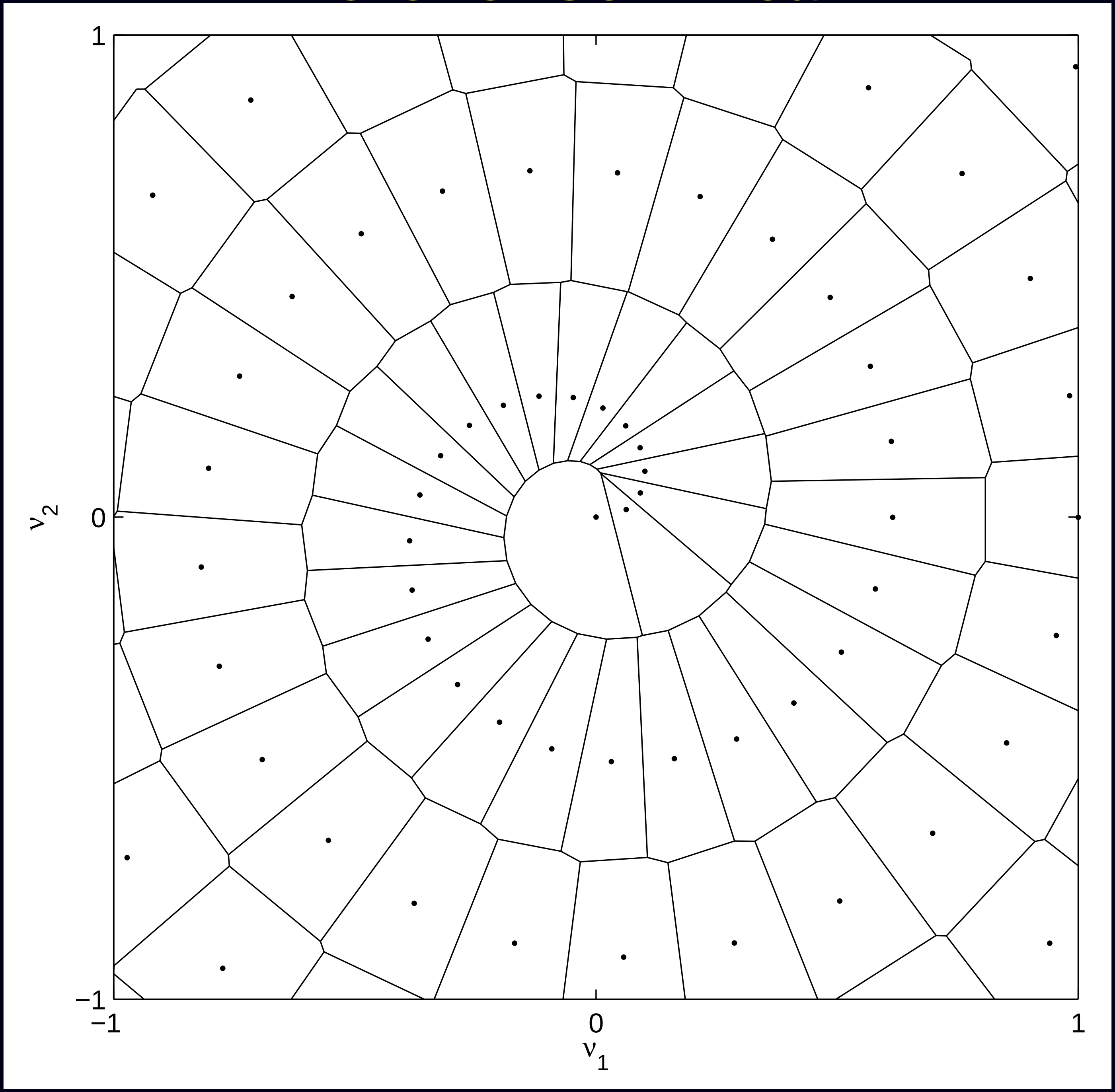
$$f(\vec{r}) = \int F(\vec{\kappa})\, e^{\imath 2\pi \vec{\kappa}\cdot\vec{r}}\, d\vec{\kappa} \approx \sum_{i=1}^{N} y_i\, e^{\imath 2\pi \vec{\kappa}_i\cdot\vec{r}}\, w_i.$$

- Voronoi cell area

  Bracewell, 1973, Astrophysical Journal; Rasche *et al.*, IEEE T-MI, 1999
- Jacobians — Norton, IEEE T-MI, 1987
- Jackson's area density — Jackson, IEEE T-MI, 1991
- Iterative methods — Pipe and Menon, MRM, 2000
- ...

Tradeoffs between simplicity and accuracy.

Voronoi Cell Area



Limitations of Gridding Reconstruction

1. Artifacts/inaccuracies due to interpolation
2. Contention about sample density "weighting"
3. Oversimplifications of Fourier transform signal model:
   - Magnetic field inhomogeneity
   - Magnetization decay (T_2)
   - Eddy currents
   - ...
4. Sensitivity encoding ?
5. ...

(But it is faster than iterative methods...)

Model-Based Image Reconstruction: Overview

Model-Based Image Reconstruction

MR signal equation with more complete physics:

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\,\omega(\vec{r})\,t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, N$$

- $s^{\text{coil}}(\vec{r})$ Receive-coil sensitivity pattern(s) (for SENSE)
- $\omega(\vec{r})$ Off-resonance frequency map
  (due to field inhomogeneity / magnetic susceptibility)
- $R_2^*(\vec{r})$ Relaxation map

Other physical factors (?)
- Eddy current effects; in $\vec{\kappa}(t)$
- Concomitant gradient terms
- Chemical shift
- Motion

Goal?

(it depends)

Field Inhomogeneity-Corrected Reconstruction

$$s(t) = \int f(\vec{r})\, s^{\mathrm{coil}}(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-R_2^*(\vec{r})\, t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given field map $\omega(\vec{r})$.
(Assume all other terms are known or unimportant.)

Motivation

Essential for functional MRI of brain regions near sinus cavities!

(Sutton *et al.*, ISMRM 2001; T-MI 2003)

Sensitivity-Encoded (SENSE) Reconstruction

$$s(t) = \int f(\vec{r}) s^{\text{coil}}(\vec{r}) \, \mathrm{e}^{-\imath \omega(\vec{r}) t} \, \mathrm{e}^{-R_2^*(\vec{r}) t} \, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}} \, d\vec{r}$$

Goal: reconstruct $f(\vec{r})$ given sensitivity maps $s^{\text{coil}}(\vec{r})$.
(Assume all other terms are known or unimportant.)

Can combine SENSE with field inhomogeneity correction “easily.”

(Sutton *et al.*, ISMRM 2001, Olafsson *et al.*, ISBI 2006)

Joint Estimation of Image and Field-Map

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate *both* the image $f(\vec{r})$ and the field map $\omega(\vec{r})$
(Assume all other terms are known or unimportant.)

Analogy:
joint estimation of emission image and attenuation map in PET.

(Sutton *et al.*, ISMRM Workshop, 2001; ISBI 2002; ISMRM 2002; ISMRM 2003; MRM 2004)

The Kitchen Sink

$$s(t) = \int f(\vec{r})\, s^{\mathrm{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})\,t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate image $f(\vec{r})$, field map $\omega(\vec{r})$, *and* relaxation map $R_2^*(\vec{r})$

Requires “suitable” k-space trajectory.

(Sutton *et al.*, ISMRM 2002; Twieg, MRM, 2003)

Estimation of Dynamic Maps

$$s(t) = \int f(\vec{r})\, s^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath\omega(\vec{r})t}\, \mathrm{e}^{-R_2^*(\vec{r})t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: estimate dynamic field map $\omega(\vec{r})$ and “BOLD effect” $R_2^*(\vec{r})$ given baseline image $f(\vec{r})$ in fMRI.

Motion...

Model-Based Image Reconstruction: Details

Back to Basic Signal Model

$$s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Goal: reconstruct $f(\vec{r})$ from $\boldsymbol{y} = (y_1, \ldots, y_N)$, where $y_i = s(t_i) + \varepsilon_i$.

Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{M} f_j\, p(\vec{r} - \vec{r}_j) \longleftarrow \text{usually 2D rect functions.}$$

$$y_i \approx \int \left[\sum_{j=1}^{M} f_j\, p(\vec{r} - \vec{r}_j) \right] \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, \mathrm{d}\vec{r} = \sum_{j=1}^{M} \left[\int p(\vec{r} - \vec{r}_j)\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}}\, \mathrm{d}\vec{r} \right] f_j$$

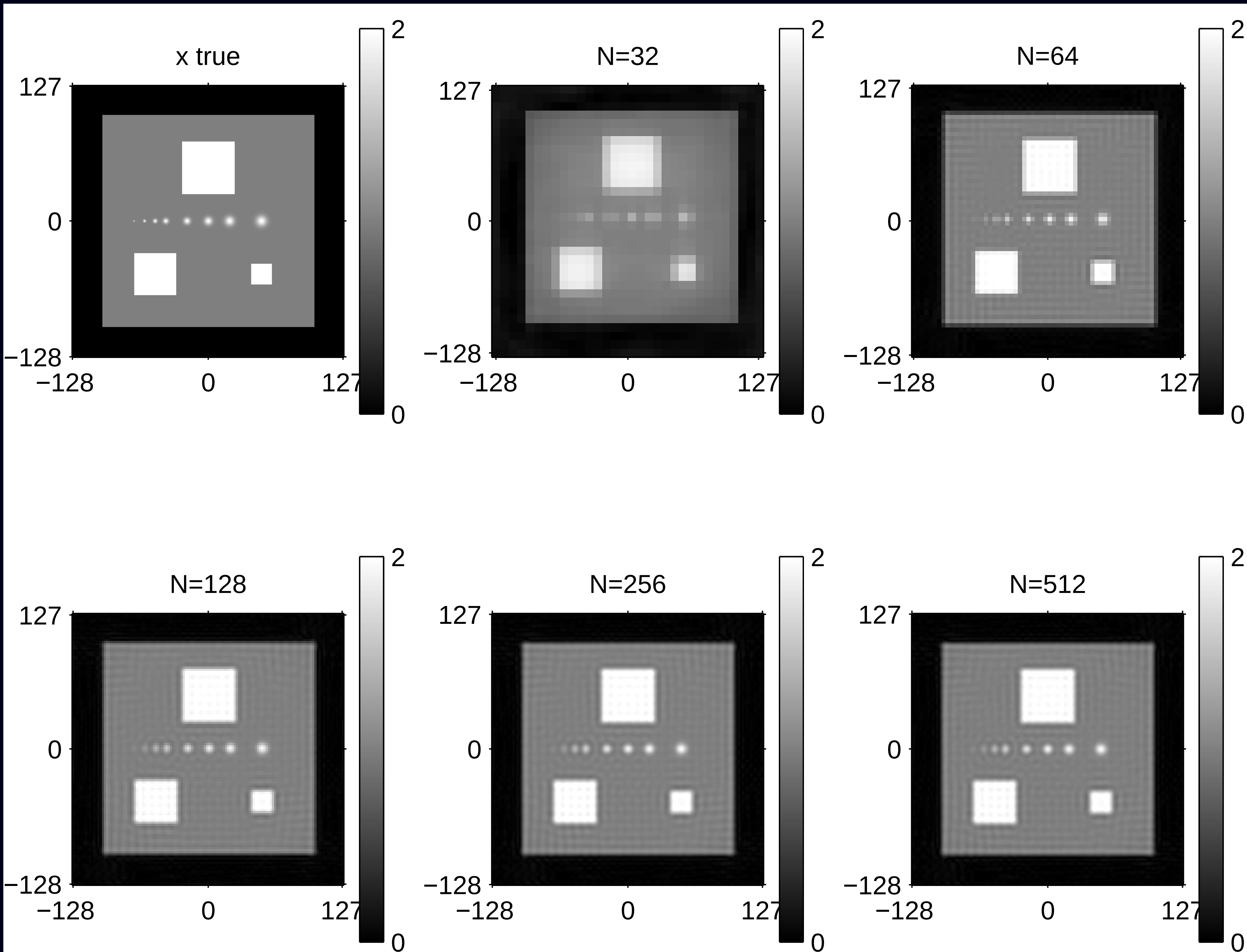
$$= \sum_{j=1}^{M} a_{ij} f_j, \qquad a_{ij} = P(\vec{\kappa}_i)\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}, \qquad p(\vec{r}) \overset{\text{FT}}{\Longleftrightarrow} P(\vec{\kappa}).$$

Discrete-discrete measurement model with system matrix $\boldsymbol{A} = \{a_{ij}\}$:

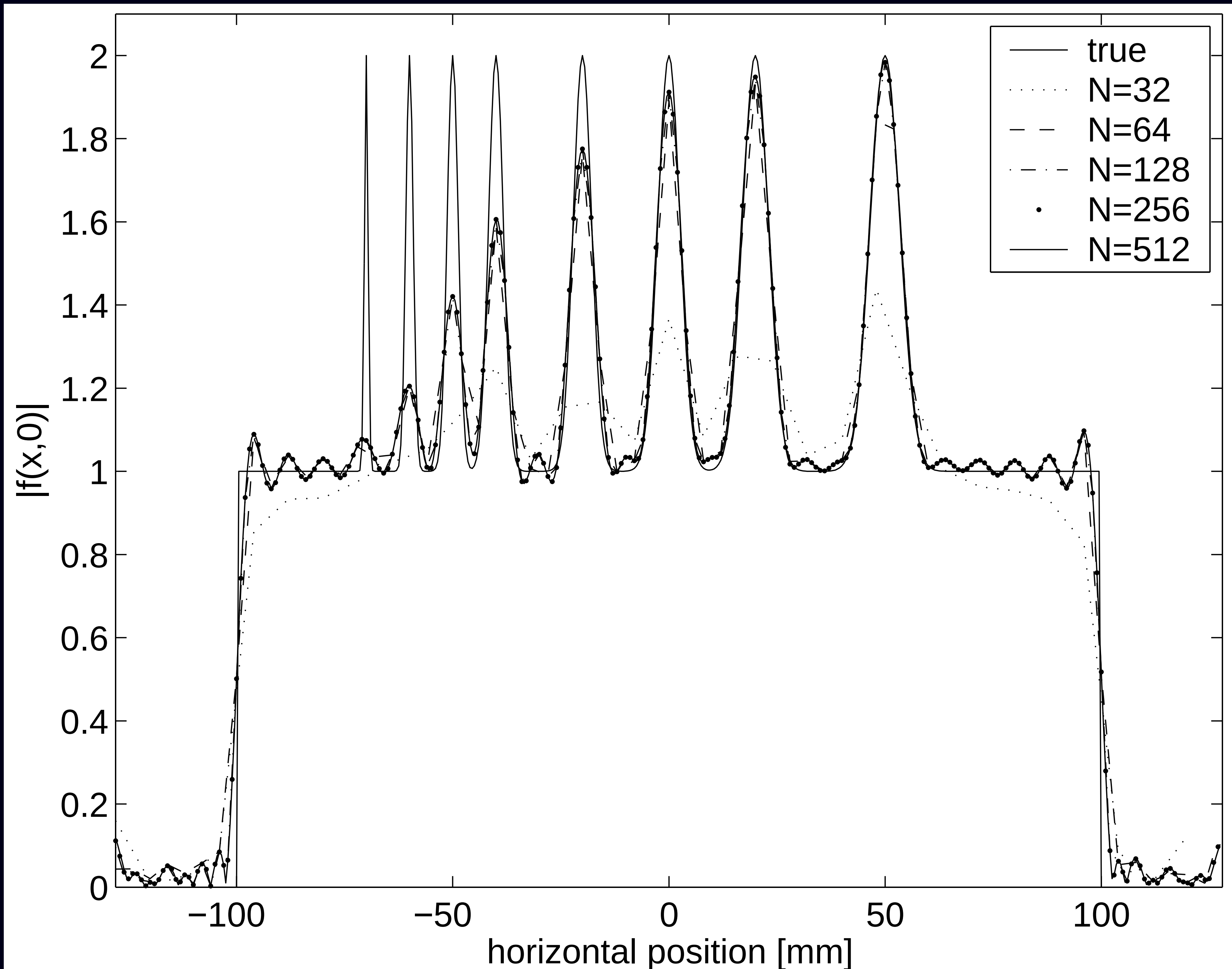
$$\boldsymbol{y} = \boldsymbol{A} \boldsymbol{f} + \boldsymbol{\varepsilon}.$$

Goal: estimate coefficients (pixel values) $\boldsymbol{f} = (f_1, \ldots, f_M)$ from $\boldsymbol{y}$.

Small Pixel Size Need Not Matter



Profiles



true
N=32
N=64
N=128
N=256
N=512
|f(x,0)|
horizontal position [mm]
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−100
−50
0
50
100

Regularized Least-Squares Estimation

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{f} \in \mathbb{C}^M}{\arg\min}\, \Psi(\boldsymbol{f}), \qquad \Psi(\boldsymbol{f}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2 + \alpha \mathsf{R}(\boldsymbol{f})$$

- data fit term $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}\|^2$
  corresponds to negative log-likelihood of Gaussian distribution
- regularizing roughness penalty term $\mathsf{R}(\boldsymbol{f})$ controls noise

$$\mathsf{R}(\boldsymbol{f}) \approx \int \|\nabla f\|^2 \,\mathrm{d}\vec{r}$$

- regularization parameter $\alpha > 0$
  controls tradeoff between spatial resolution and noise
  (Fessler & Rogers, IEEE T-IP, 1996)
- Equivalent to Bayesian MAP estimation with prior $\propto \mathrm{e}^{-\alpha \mathsf{R}(\boldsymbol{f})}$

Quadratic regularization $\mathsf{R}(\boldsymbol{f}) = \|\boldsymbol{C}\boldsymbol{f}\|^2$ leads to closed-form solution:

$$\hat{\boldsymbol{x}} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}$$

(a formula of limited practical use)

Choosing the Regularization Parameter

$$\hat{\boldsymbol{x}} = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{y}$$

$$\mathsf{E}[\hat{\boldsymbol{x}}] = \left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\,\mathsf{E}[\boldsymbol{y}]$$

$$\mathsf{E}[\hat{\boldsymbol{x}}] = \underbrace{\left[\boldsymbol{A}'\boldsymbol{A} + \alpha \boldsymbol{C}'\boldsymbol{C}\right]^{-1} \boldsymbol{A}'\boldsymbol{A}}_{\text{blur}}\,\boldsymbol{f}$$

$\boldsymbol{A}'\boldsymbol{A}$ and $\boldsymbol{C}'\boldsymbol{C}$ are Toeplitz $\Longrightarrow$ blur is approximately shift-invariant.

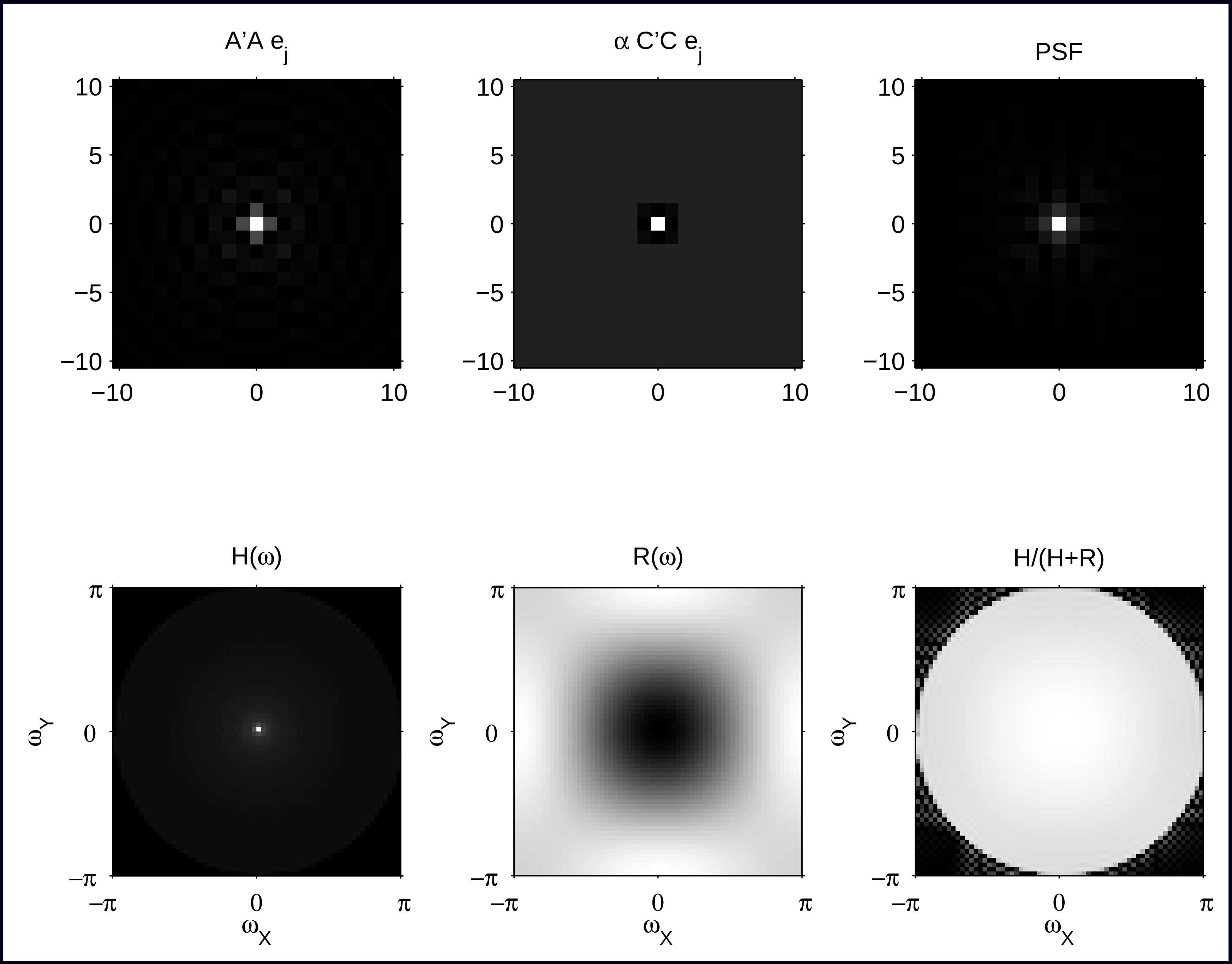
Frequency response of blur:

$$L(\omega) = \frac{H(\omega)}{H(\omega) + \alpha R(\omega)}$$

where $H = \mathrm{FFT}(\boldsymbol{A}'\boldsymbol{A}\,e_j)$ (lowpass) and $R = \mathrm{FFT}(\boldsymbol{C}'\boldsymbol{C}\,e_j)$ (highpass)

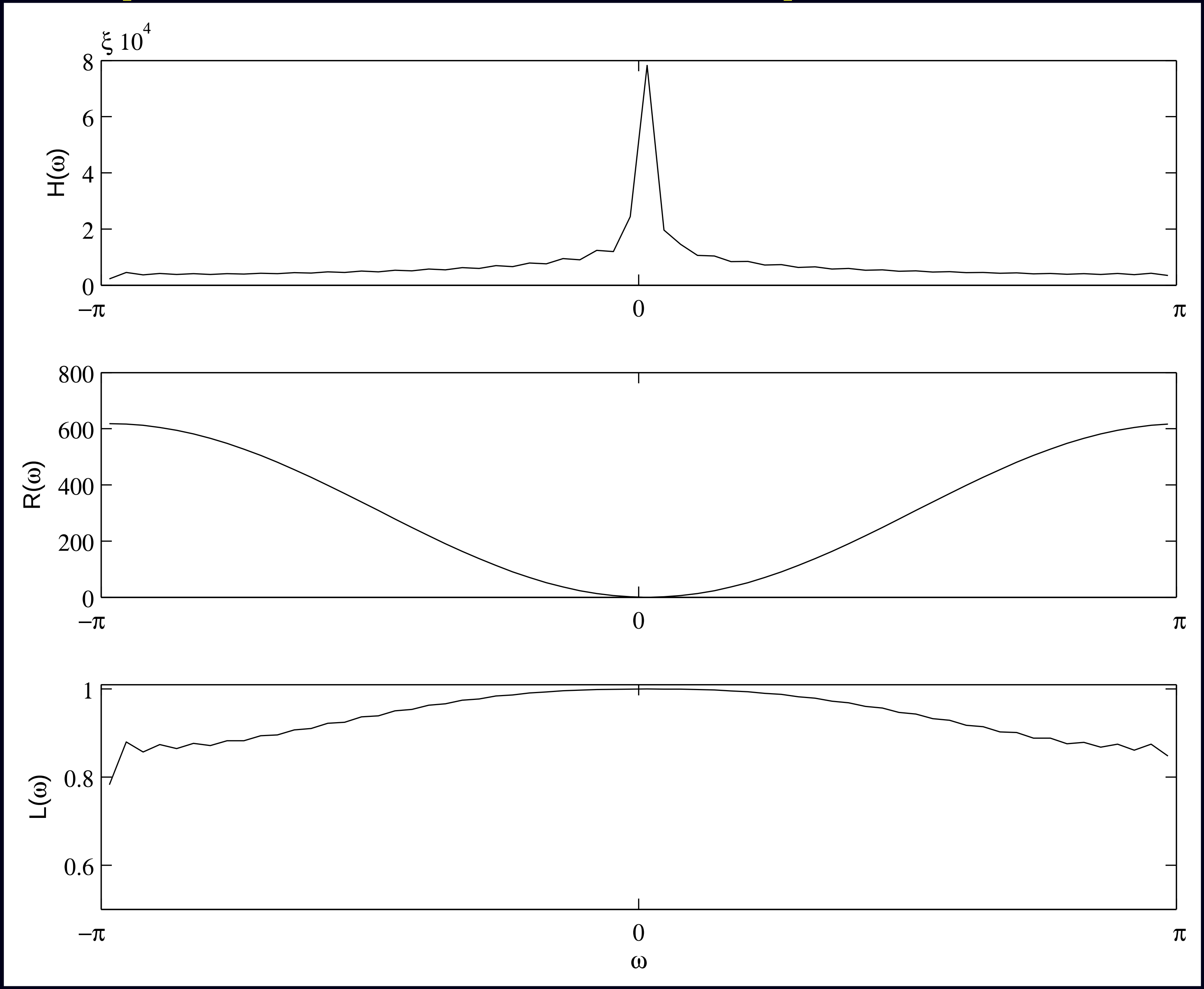
Adjust α to achieve desired spatial resolution.

Spatial Resolution Example



Spiral k-space trajectory, FWHM of PSF is 1.2 pixels

Spatial Resolution Example: Profiles



Iterative Minimization by Conjugate Gradients

Choose initial guess $\boldsymbol{f}^{(0)}$ (*e.g.*, fast conjugate phase / gridding).
Iteration (unregularized):

$$
\begin{aligned}
&\boldsymbol{g}^{(n)} = \nabla\Psi\big(\boldsymbol{f}^{(n)}\big) = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) && \text{gradient} \\
&\boldsymbol{p}^{(n)} = \boldsymbol{P}\boldsymbol{g}^{(n)} && \text{precondition} \\
&\gamma_n = \begin{cases} 0, & n = 0 \\ \dfrac{\langle \boldsymbol{g}^{(n)}, \boldsymbol{p}^{(n)} \rangle}{\langle \boldsymbol{g}^{(n-1)}, \boldsymbol{p}^{(n-1)} \rangle}, & n > 0 \end{cases} \\
&\boldsymbol{d}^{(n)} = -\boldsymbol{p}^{(n)} + \gamma_n \boldsymbol{d}^{(n-1)} && \text{search direction} \\
&\boldsymbol{v}^{(n)} = \boldsymbol{A}\boldsymbol{d}^{(n)} \\
&\alpha_n = \langle \boldsymbol{d}^{(n)}, -\boldsymbol{g}^{(n)} \rangle / \langle \boldsymbol{A}\boldsymbol{f}^{(n)}, \boldsymbol{A}\boldsymbol{f}^{(n)} \rangle && \text{step size} \\
&\boldsymbol{f}^{(n+1)} = \boldsymbol{f}^{(n)} + \alpha_n \boldsymbol{d}^{(n)} && \text{update}
\end{aligned}
$$

Bottlenecks: computing $\boldsymbol{A}\boldsymbol{f}$ and $\boldsymbol{A}'\boldsymbol{y}$.

- $\boldsymbol{A}$ is too large to store explicitly (not sparse)
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{f}$ is $O(NM)$ per iteration, whereas FFT is only $O(N \log N)$.

Computing $\boldsymbol{A f}$ Rapidly

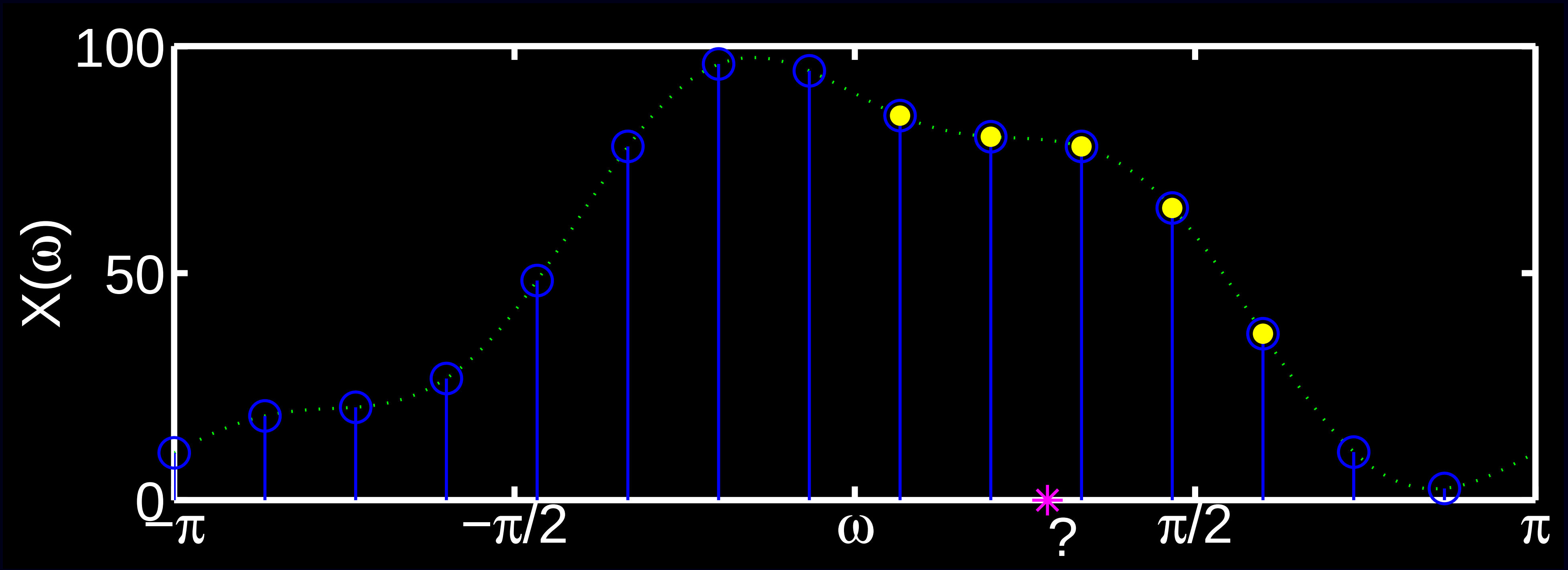
$$[\boldsymbol{A f}]_i = \sum_{j=1}^{M} a_{ij} f_j = P(\vec{\kappa}_i) \sum_{j=1}^{M} \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j} f_j, \qquad i = 1, \ldots, N$$

- Pixel locations $\{\vec{r}_j\}$ are uniformly spaced
- k-space locations $\{\vec{\kappa}_i\}$ are unequally spaced

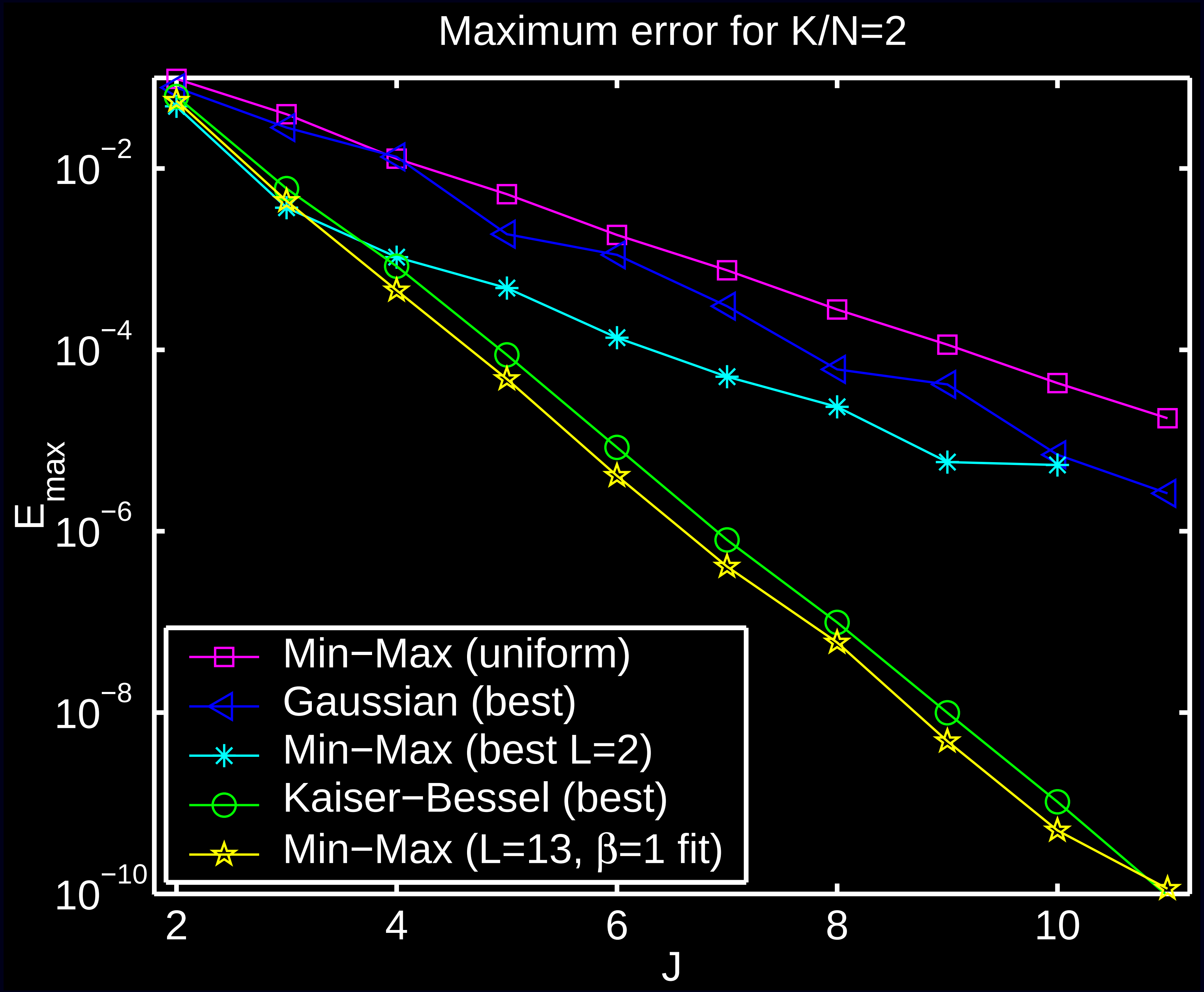
$\Longrightarrow$ needs nonuniform fast Fourier transform (NUFFT)

NUFFT (Type 2)

- Compute over-sampled FFT of equally-spaced signal samples
- Interpolate onto desired unequally-spaced frequency locations
- Dutt & Rokhlin, SIAM JSC, 1993, Gaussian bell interpolator
- Fessler & Sutton, IEEE T-SP, 2003, min-max interpolator
  and min-max optimized Kaiser-Bessel interpolator.
  NUFFT toolbox: `http://www.eecs.umich.edu/~fessler/code`



Worst-Case NUFFT Interpolation Error



Further Acceleration using Toeplitz Matrices

Cost-function gradient:

$$\begin{aligned} \boldsymbol{g}^{(n)} &= \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{f}^{(n)} - \boldsymbol{y}) \\ &= \boldsymbol{T}\boldsymbol{f}^{(n)} - \boldsymbol{b}, \end{aligned}$$

where

$$\boldsymbol{T} \triangleq \boldsymbol{A}'\boldsymbol{A}, \qquad \boldsymbol{b} \triangleq \boldsymbol{A}'\boldsymbol{y}.$$

In the absence of field inhomogeneity, the Gram matrix $\boldsymbol{T}$ is Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{N} \left|P(\vec{\kappa}_i)\right|^2 \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}.$$

Computing $\boldsymbol{T}\boldsymbol{f}^{(n)}$ requires an ordinary ($2\times$ over-sampled) FFT.

(Chan & Ng, SIAM Review, 1996)

In 2D: block Toeplitz with Toeplitz blocks (BTTB).

Precomputing the first column of $\boldsymbol{T}$ and $\boldsymbol{b}$ requires a couple NUFFTs.
(Wajer, ISMRM 2001, Eggers ISMRM 2002, Liu ISMRM 2005)

This formulation seems ideal for “hardware” FFT systems.

NUFFT with Field Inhomogeneity?

Combine NUFFT with min-max temporal interpolator
(Sutton *et al.*, IEEE T-MI, 2003)
(forward version of “time segmentation”, Noll, T-MI, 1991)

Recall signal model including field inhomogeneity:

$$s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r}.$$

Temporal interpolation approximation (aka “time segmentation”):

$$\mathrm{e}^{-\imath \omega(\vec{r}) t} \approx \sum_{l=1}^{L} a_l(t)\, \mathrm{e}^{-\imath \omega(\vec{r}) \tau_l}$$

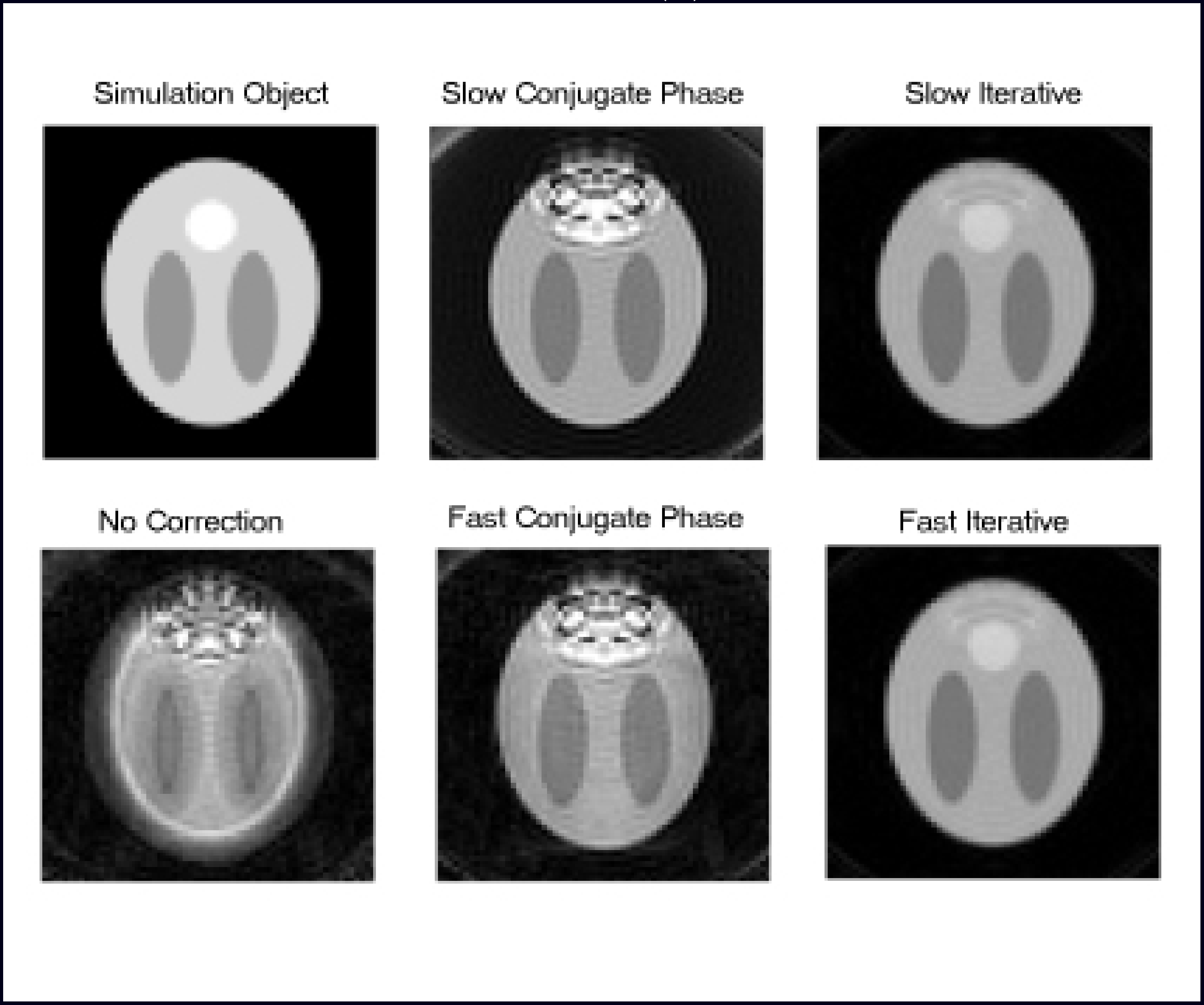
for min-max optimized temporal interpolation functions $\{a_l(\cdot)\}_{l=1}^{L}$.

$$s(t) \approx \sum_{l=1}^{L} a_l(t) \int \left[f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) \tau_l} \right] \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, d\vec{r}$$

Linear combination of L NUFFT calls.

Field Corrected Reconstruction Example

Simulation using known field map $\omega(\vec{r})$.

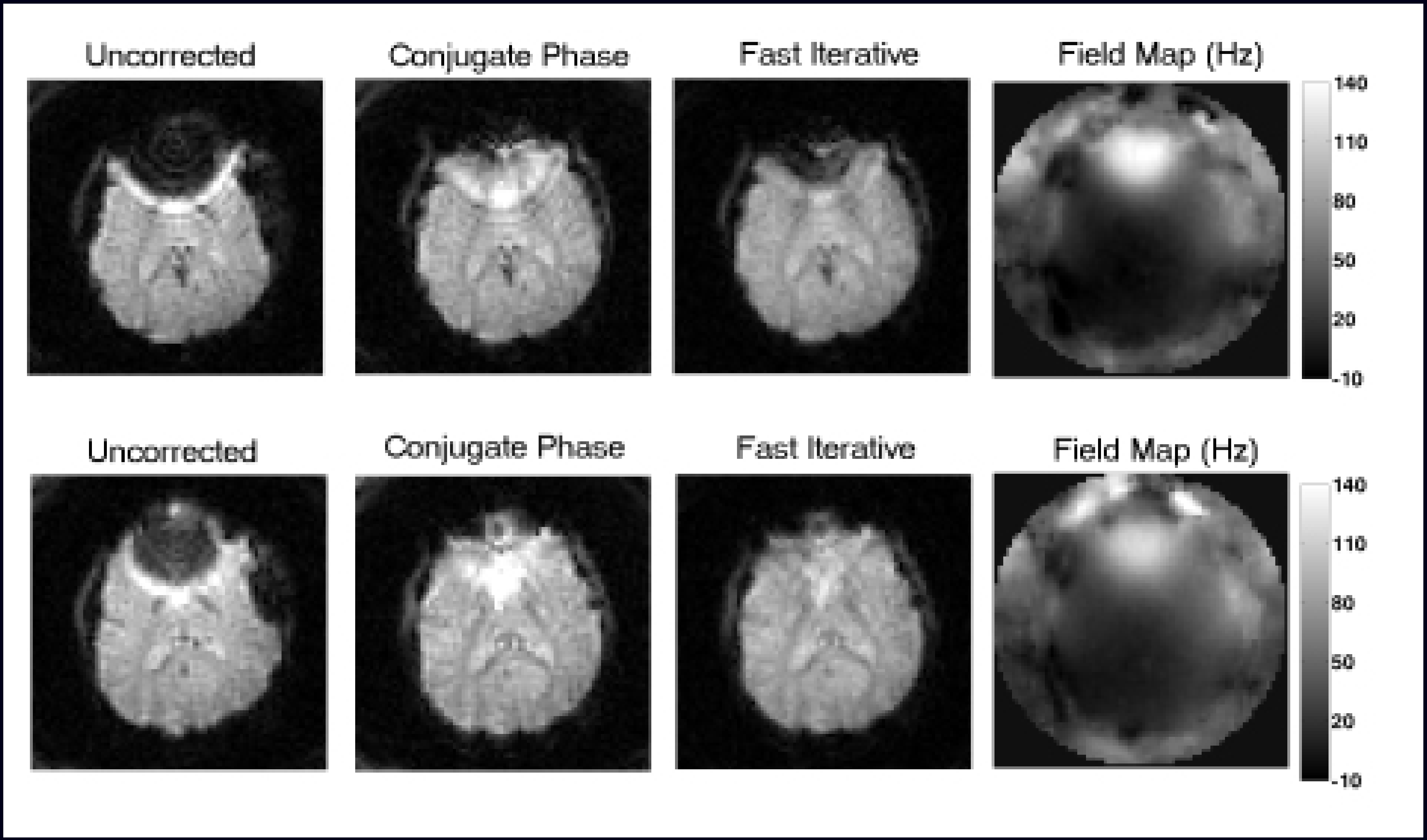


Simulation Quantitative Comparison

- Computation time?
- NRMSE between $\hat{\boldsymbol{x}}$ and $\boldsymbol{f}^{\mathrm{true}}$?

Reconstruction Method	Time (s)	NRMSE complex	NRMSE magnitude
No Correction	0.06	1.35	0.22
Full Conjugate Phase	4.07	0.31	0.19
Fast Conjugate Phase	0.33	0.32	0.19
Fast Iterative (10 iters)	2.20	0.04	0.04
Exact Iterative (10 iters)	128.16	0.04	0.04

Human Data: Field Correction



Acceleration using Toeplitz Approximations

In the presence of field inhomogeneity, the system matrix is:

$$a_{ij} = P(\vec{\kappa}_i)\, \mathrm{e}^{-\imath \omega(\vec{r}_j) t_i}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}$$

The Gram matrix $\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A}$ is *not* Toeplitz:

$$[\boldsymbol{A}'\boldsymbol{A}]_{jk} = \sum_{i=1}^{N} |P(\vec{\kappa}_i)|^2\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}\, \mathrm{e}^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i}.$$

Approximation (“time segmentation”):

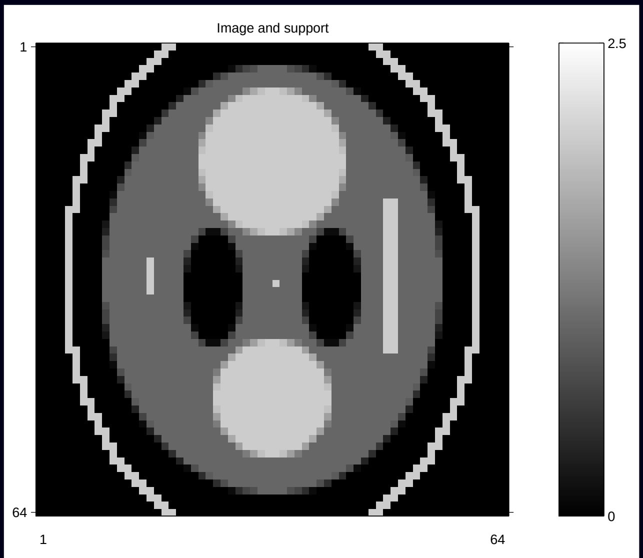
$$\mathrm{e}^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) t_i} \approx \sum_{l=1}^{L} b_{il}\, \mathrm{e}^{-\imath \left(\omega(\vec{r}_j) - \omega(\vec{r}_k)\right) \tau_l}$$

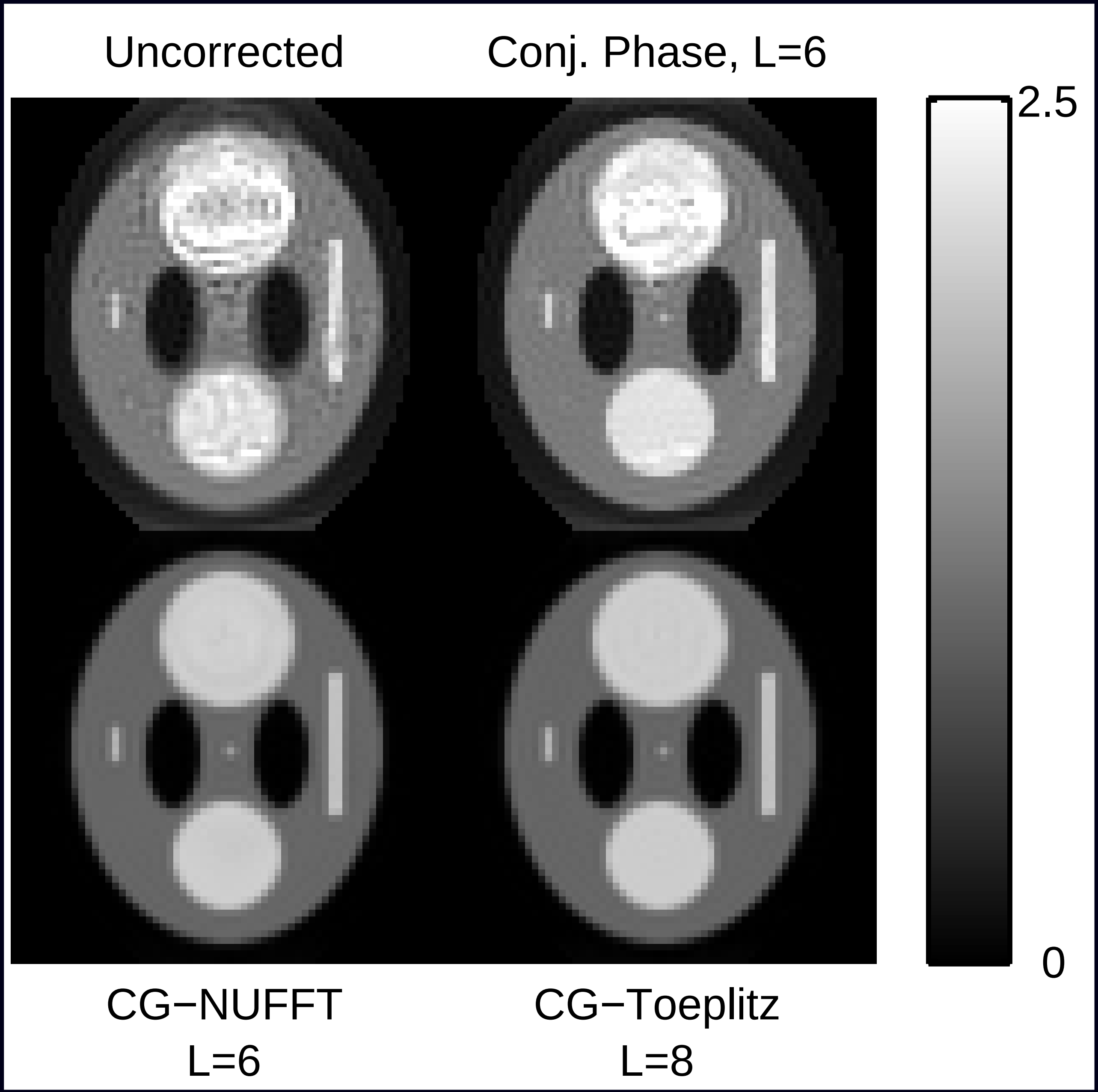
$$\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{A} \approx \sum_{l=1}^{L} \boldsymbol{D}_l' \boldsymbol{T}_l \boldsymbol{D}_l, \qquad \begin{array}{l} \boldsymbol{D}_l \triangleq \mathrm{diag}\left\{\mathrm{e}^{-\imath \omega(\vec{r}_j) \tau_l}\right\} \\ [\boldsymbol{T}_l]_{jk} \triangleq \sum_{i=1}^{N} |P(\vec{\kappa}_i)|^2\, b_{il}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot (\vec{r}_j - \vec{r}_k)}. \end{array}$$

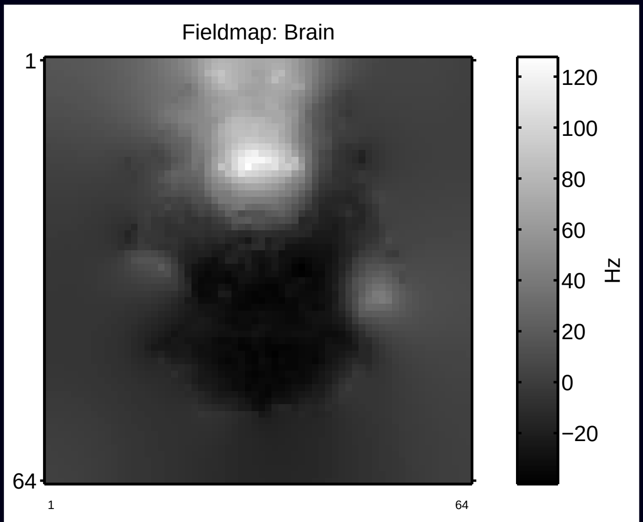
Each $\boldsymbol{T}_l$ is Toeplitz $\Longrightarrow \boldsymbol{T}\boldsymbol{f}$ using L pairs of FFTs.

(Fessler *et al.*, IEEE T-SP, Sep. 2005, brain imaging special issue)

Toeplitz Results







Toeplitz Acceleration

		Precomputation						NRMS % vs SNR				
Method	L	$\boldsymbol{B},\boldsymbol{C}$	$\boldsymbol{A}'\boldsymbol{D}\boldsymbol{y}$	$\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{y}$	$\boldsymbol{T}_l$	15 iter	Total Time	∞	50 dB	40 dB	30 dB	20 dB
Conj. Phase	6	0.4	0.2				0.6	30.7	37.3	46.5	65.3	99.9
CG-NUFFT	6	0.4				5.0	5.4	5.6	16.7	26.5	43.0	70.4
CG-Toeplitz	8	0.4		0.2	0.6	1.3	2.5	5.5	16.7	26.4	42.9	70.4

- Reduces CPU time by $2\times$ on conventional workstation (Mac G5)
- No SNR compromise
- Eliminates k-space interpolations $\Longrightarrow$ ideal for FFT hardware

Joint Field-Map / Image Reconstruction

Signal model:

$$y_i = s(t_i) + \varepsilon_i, \qquad s(t) = \int f(\vec{r})\, \mathrm{e}^{-\imath \omega(\vec{r}) t}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t) \cdot \vec{r}}\, \mathrm{d}\vec{r}.$$

After discretization:

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{\omega})\, \boldsymbol{f} + \boldsymbol{\varepsilon}, \qquad a_{ij}(\boldsymbol{\omega}) = P(\vec{\kappa}_i)\, \mathrm{e}^{-\imath \omega_j t_i}\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}_i \cdot \vec{r}_j}.$$

Joint estimation via regularized (nonlinear) least-squares:

$$(\hat{\boldsymbol{f}}, \hat{\boldsymbol{\omega}}) = \operatorname*{arg\,min}_{\boldsymbol{f} \in \mathbb{C}^M, \boldsymbol{\omega} \in \mathbb{R}^M} \|\boldsymbol{y} - \boldsymbol{A}(\boldsymbol{\omega})\, \boldsymbol{f}\|^2 + \beta_1 R_1(\boldsymbol{f}) + \beta_2 R_2(\boldsymbol{\omega}).$$

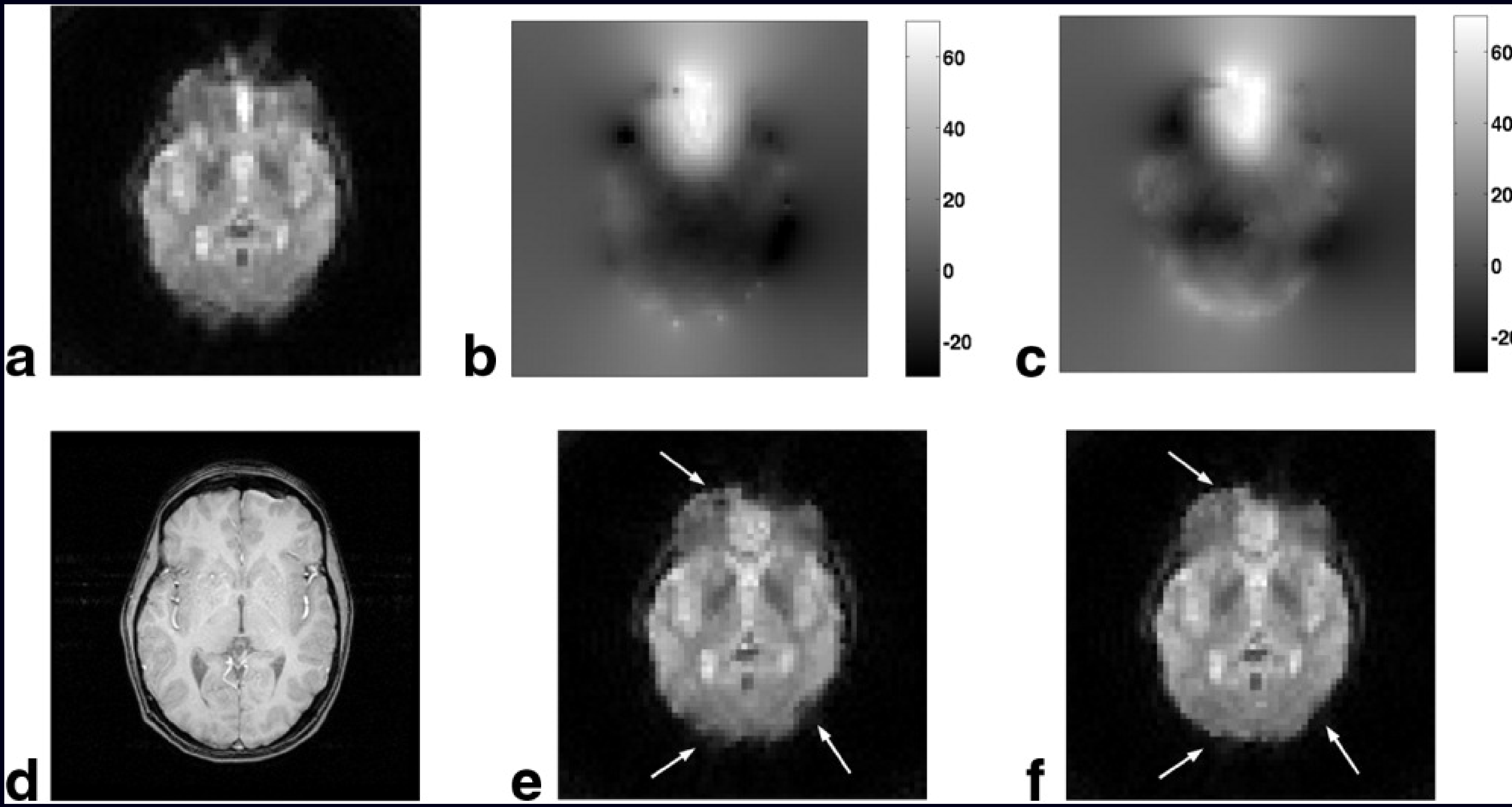
Alternating minimization:

- Using current estimate of fieldmap $\hat{\boldsymbol{\omega}}$, update $\hat{\boldsymbol{f}}$ using CG algorithm.
- Using current estimate $\hat{\boldsymbol{f}}$ of image, update fieldmap $\hat{\boldsymbol{\omega}}$ using gradient descent.

Use spiral-in / spiral-out sequence or “racetrack” EPI.

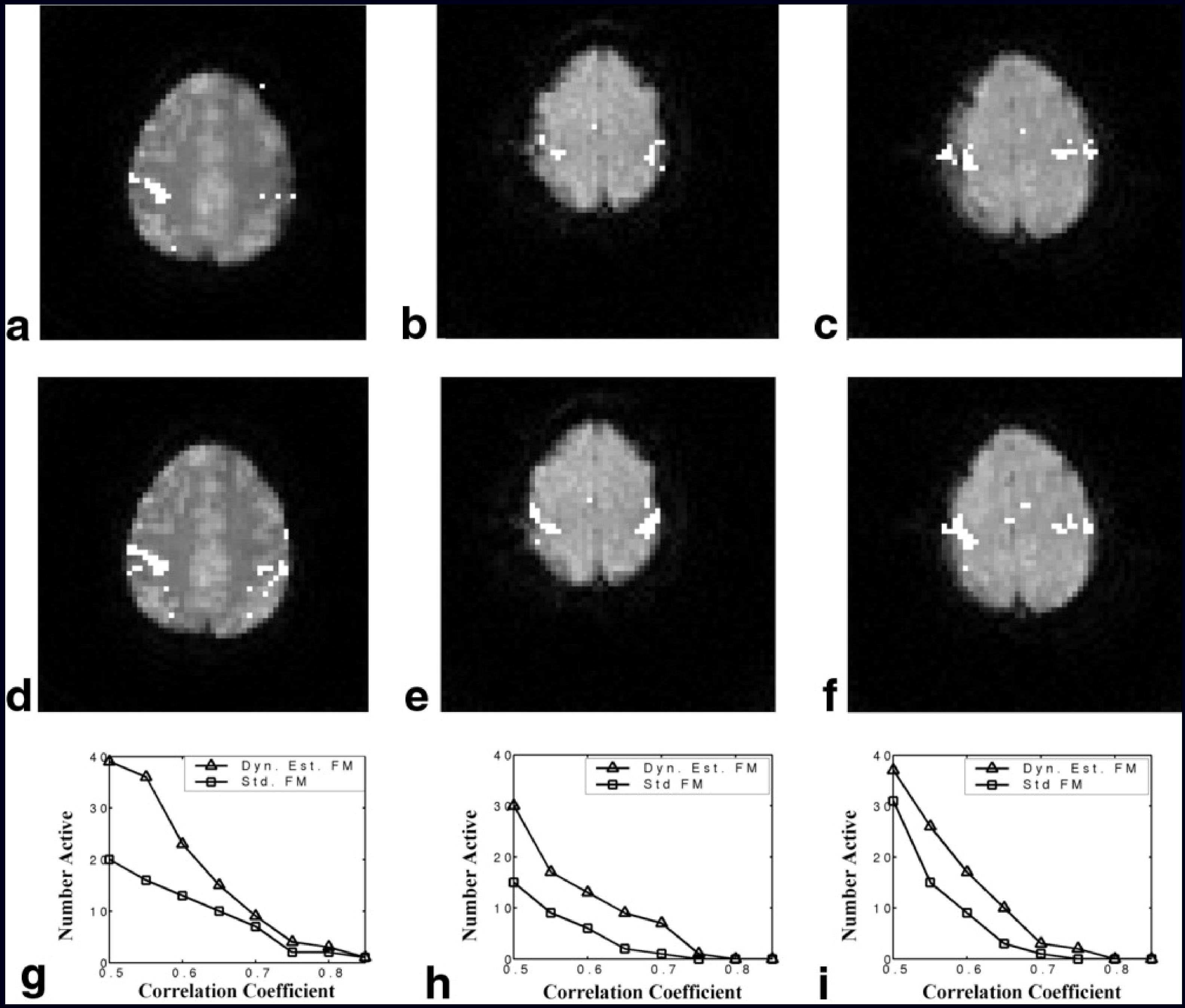
(Sutton *et al.*, MRM, 2004)

Joint Estimation Example



(a) uncorr., (b) std. map, (c) joint map, (d) T1 ref, (e) using std, (f) using joint.

Activation Results: Static vs Dynamic Field Maps



Functional results for the two reconstructions for 3 human subjects.

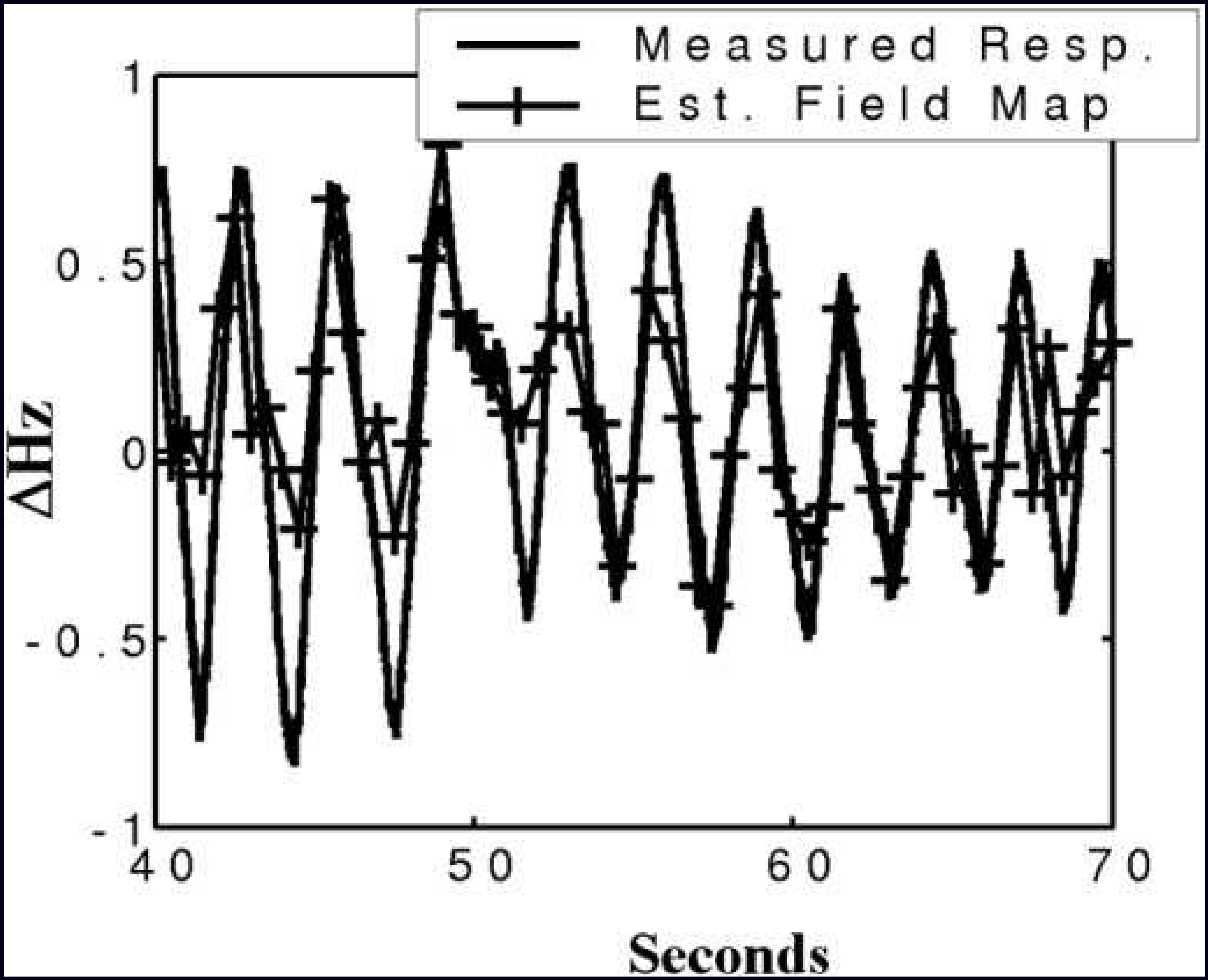
Reconstruction using the standard field map
for (a) subject 1, (b) subject 2, and (c) subject 3.

Reconstruction using the jointly estimated field map
for (d) subject 1, (e) subject 2, and (f) subject 3.

Number of pixels with correlation coefficients higher than thresholds
for (g) subject 1, (h) subject 2, and (i) subject 3.

Take home message: dynamic field mapping is possible, using iterative reconstruction as an essential tool.
(Standard field maps based on echo-time differences work poorly for spiral-in / spiral-out sequences due to phase discrepancies.)

Tracking Respiration-Induced Field Changes



Regularization Variations

Regularization Revisited

- Conventional regularization for MRI uses a roughness penalty for the *complex* voxel values:

$$\mathsf{R}(\boldsymbol{f}) \approx \sum_{j=1}^{M} |f_j - f_{j-1}|^2 \qquad \text{(in 1D).}$$

- Regularizes the real and imaginary image components equally.
- In some MR studies, including BOLD fMRI:
  - magnitude of f_j carries the information of interest,
  - phase of f_j should be spatially smooth.
  - This *a priori* information is ignored by $\mathsf{R}(\boldsymbol{f})$.
- Alternatives to $\mathsf{R}(\boldsymbol{f})$:
  - Constrain $\boldsymbol{f}$ to be real?
    (Unrealistic: RF phase inhomogeneity, eddy currents, ...)
  - Determine phase of $\boldsymbol{f}$ “somehow,” then estimate its magnitude.
    - Non-iteratively (Noll, Nishimura, Macovski, IEEE T-MI, 1991)
    - Iteratively (Lee, Pauly, Nishimura, ISMRM, 2003)

Separate Magnitude/Phase Regularization

Decompose $\boldsymbol{f}$ into its "magnitude" $\boldsymbol{m}$ and phase $\boldsymbol{x}$:

$$f_j(\boldsymbol{m},\boldsymbol{x}) = m_j \mathrm{e}^{\imath x_j}, \qquad m_j \in \mathbb{R}, \qquad x_j \in \mathbb{R}, \qquad j = 1,\ldots,M.$$

(Allow "magnitude" m_j to be negative.)

Proposed cost function with separate regularization of $\boldsymbol{m}$ and $\boldsymbol{x}$:

$$\Psi(\boldsymbol{m},\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})\|^2 + \gamma \mathsf{R}_1(\boldsymbol{m}) + \beta \mathsf{R}_2(\boldsymbol{x}).$$

Choose $\beta \gg \gamma$ to strongly smooth phase estimate.

Joint estimation of magnitude and phase via regularized LS:

$$(\hat{\boldsymbol{m}},\hat{\boldsymbol{x}}) = \operatorname*{arg\,min}_{\boldsymbol{m}\in\mathbb{R}^M,\ \boldsymbol{x}\in\mathbb{R}^M} \Psi(\boldsymbol{m},\boldsymbol{x})$$

Ψ is not convex $\Longrightarrow$ need good initial estimates $(\boldsymbol{m}^{(0)},\boldsymbol{x}^{(0)})$.

Alternating Minimization

Magnitude Update:

$$\boldsymbol{m}^{\text{new}} = \arg\min_{\boldsymbol{m}\in\mathbb{R}^M} \Psi\left(\boldsymbol{m}, \boldsymbol{x}^{\text{old}}\right)$$

Phase Update:

$$\boldsymbol{x}^{\text{new}} = \arg\min_{\boldsymbol{x}\in\mathbb{R}^M} \Psi(\boldsymbol{m}^{\text{new}}, \boldsymbol{x}),$$

Since $f_j = m_j e^{\imath x_j}$ is linear in m_j, the magnitude update is easy. Apply a few iterations of slightly modified CG algorithm (constrain $\boldsymbol{m}$ to be real)

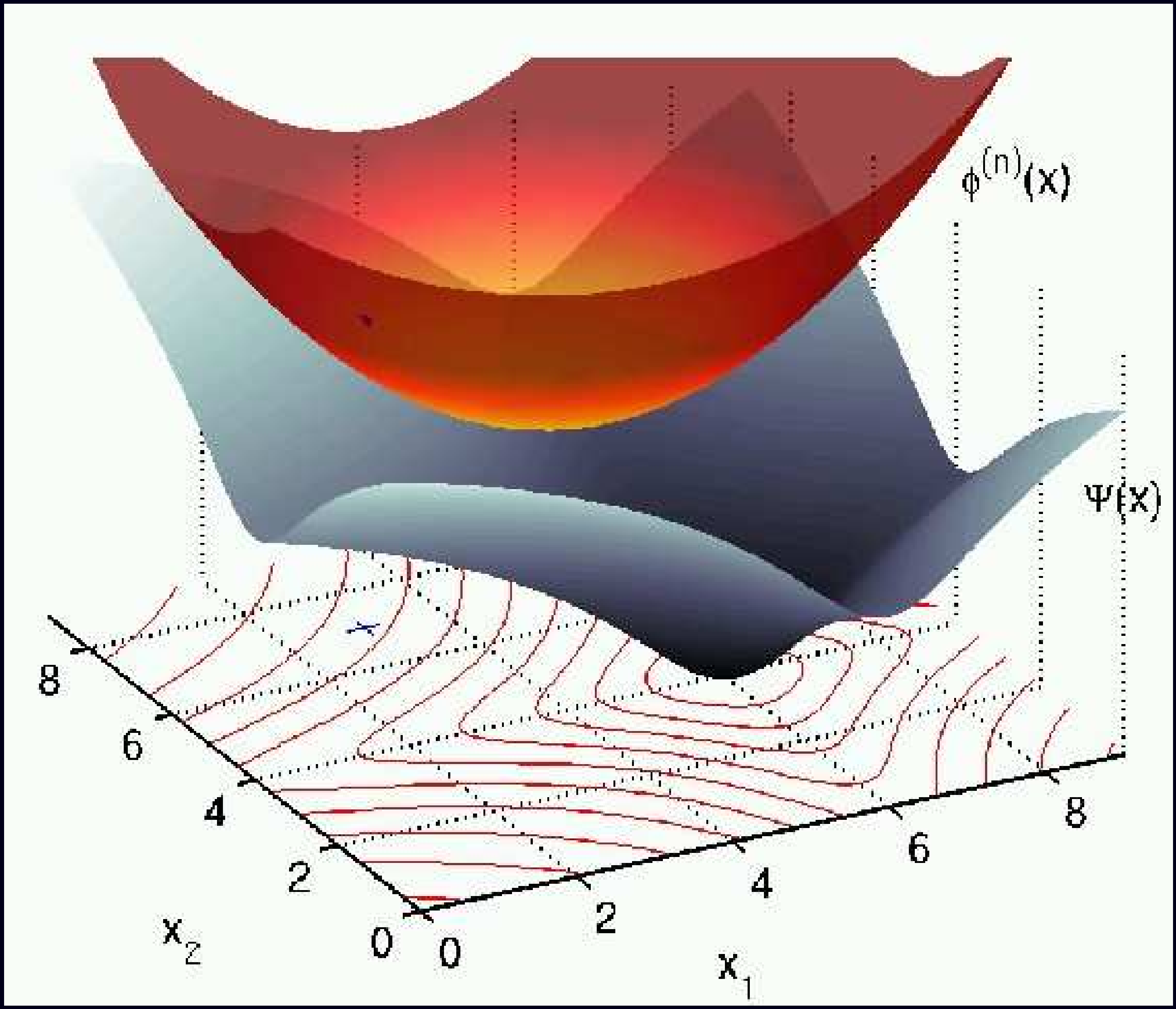
But $f_j = m_j e^{\imath x_j}$ is highly nonlinear in $\boldsymbol{x}$. Complicates “argmin.”

Steepest descent?

$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \lambda \nabla_{\boldsymbol{x}} \Psi\left(\boldsymbol{m}^{\text{old}}, \boldsymbol{x}^{(n)}\right).$$

Choosing the stepsize λ is difficult.

Optimization Transfer



Surrogate Functions

To minimize a cost function $\Phi(\boldsymbol{x})$, choose surrogate functions $\phi^{(n)}(\boldsymbol{x})$ that satisfy the following *majorization* conditions:

$$\begin{aligned} \phi^{(n)}\left(\boldsymbol{x}^{(n)}\right) &= \Phi(\boldsymbol{x}^{(n)}) \\ \phi^{(n)}(\boldsymbol{x}) &\geq \Phi(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathbb{R}^M. \end{aligned}$$

Iteratively minimize the surrogates as follows:

$$\boldsymbol{x}^{(n+1)} = \underset{\boldsymbol{x}^{(n)} \in \mathbb{R}^M}{\arg\min}\, \phi^{(n)}(\boldsymbol{x})\,.$$

This will decrease Φ monotonically; $\Phi(\boldsymbol{x}^{(n+1)}) \leq \Phi(\boldsymbol{x}^{(n)})$.

The art is in the design of surrogates.
Tradeoffs:
- ◦ complexity
- ◦ computation per iteration
- ◦ convergence rate / number of iterations.

Surrogate Functions for MR Phase

$$\mathsf{L}(\boldsymbol{x}) \triangleq \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})\|^2 = \sum_{i=1}^{N} \mathsf{h}_i([\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i),$$

where $\mathsf{h}_i(t) \triangleq |y_i - t|^2$ is convex.

Extending De Pierro (IEEE T-MI, 1995), for $\pi_{ij} \geq 0$ and $\sum_{j=1}^{M} \pi_{ij} = 1$:

$$[\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i = \sum_{j=1}^{M} b_{ij}\, \mathrm{e}^{\imath x_j} = \sum_{j=1}^{M} \pi_{ij} \left[\frac{b_{ij}}{\pi_{ij}} \left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right],$$

where $b_{ij} \triangleq a_{ij} m_j$, $\bar{y}_i^{(n)} \triangleq [\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x}^{(n)})]_i$. Choose $\pi_{ij} \geq 0$ and $\sum_{j=1}^{M} \pi_{ij} = 1$.

Since h_i is convex:

$$\begin{aligned} \mathsf{h}_i([\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i) &= \mathsf{h}_i\left(\sum_{j=1}^{M} \pi_{ij} \left[\frac{b_{ij}}{\pi_{ij}} \left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right]\right) \\ &\leq \sum_{j=1}^{M} \pi_{ij}\, \mathsf{h}_i\left(\frac{b_{ij}}{\pi_{ij}} \left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right), \end{aligned}$$

with equality when $\boldsymbol{x} = \boldsymbol{x}^{(n)}$.

Separable Surrogate Function

$$\mathsf{L}(\boldsymbol{x}) = \sum_{i=1}^{N} \mathsf{h}_i([\boldsymbol{A}\boldsymbol{f}(\boldsymbol{m},\boldsymbol{x})]_i) \leq \sum_{i=1}^{N}\sum_{j=1}^{M} \pi_{ij}\,\mathsf{h}_i\left(\frac{b_{ij}}{\pi_{ij}}\left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right)$$

$$= \sum_{j=1}^{M} \underbrace{\sum_{i=1}^{N} \pi_{ij}\,\mathsf{h}_i\left(\frac{b_{ij}}{\pi_{ij}}\left(\mathrm{e}^{\imath x_j} - \mathrm{e}^{\imath x_j^{(n)}}\right) + \bar{y}_i^{(n)}\right)}_{Q_j(x_j;\boldsymbol{x}^{(n)})}.$$

Construct similar surrogates $\{S_j\}$ for (convex) roughness penalty...

$$\text{Surrogate: } \phi^{(n)}(\boldsymbol{x}) = \sum_{j=1}^{M} Q_j(x_j;\boldsymbol{x}^{(n)}) + \beta S_j(x_j;\boldsymbol{x}^{(n)}).$$

Parallelizable (simultaneous) update, with 1D minimizations:

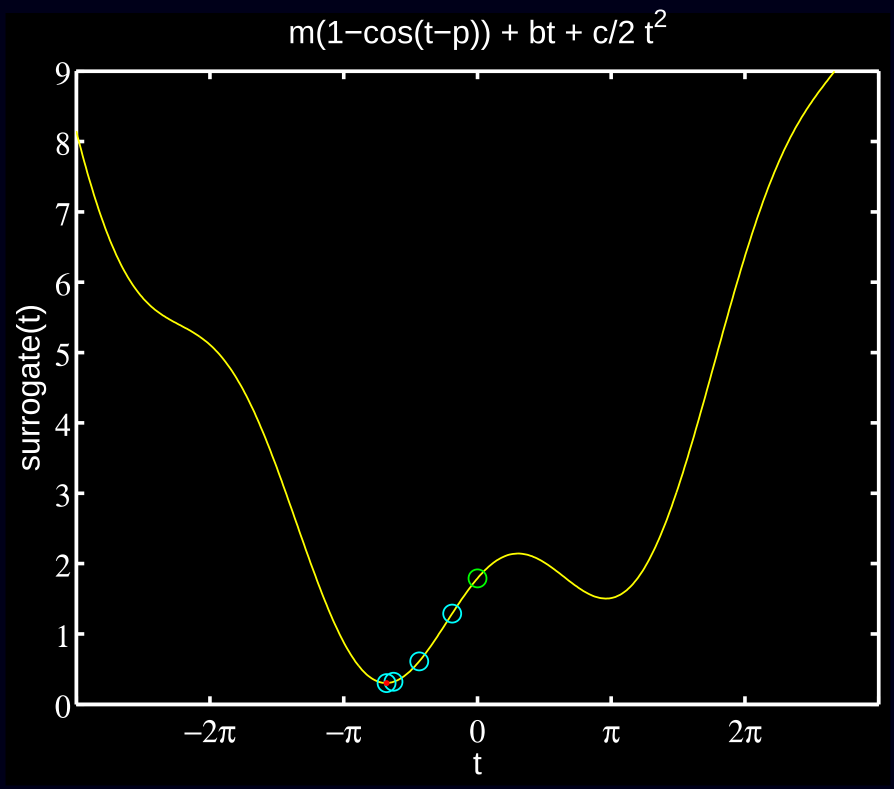
$$\boldsymbol{x}^{(n+1)} = \operatorname*{arg\,min}_{\boldsymbol{x}^{(n)}\in\mathbb{R}^M} \phi^{(n)}(\boldsymbol{x}) \implies x_j^{(n+1)} = \operatorname*{arg\,min}_{x_j\in\mathbb{R}} Q_j(x_j;\boldsymbol{x}^{(n)}) + \beta S_j(x_j;\boldsymbol{x}^{(n)}).$$

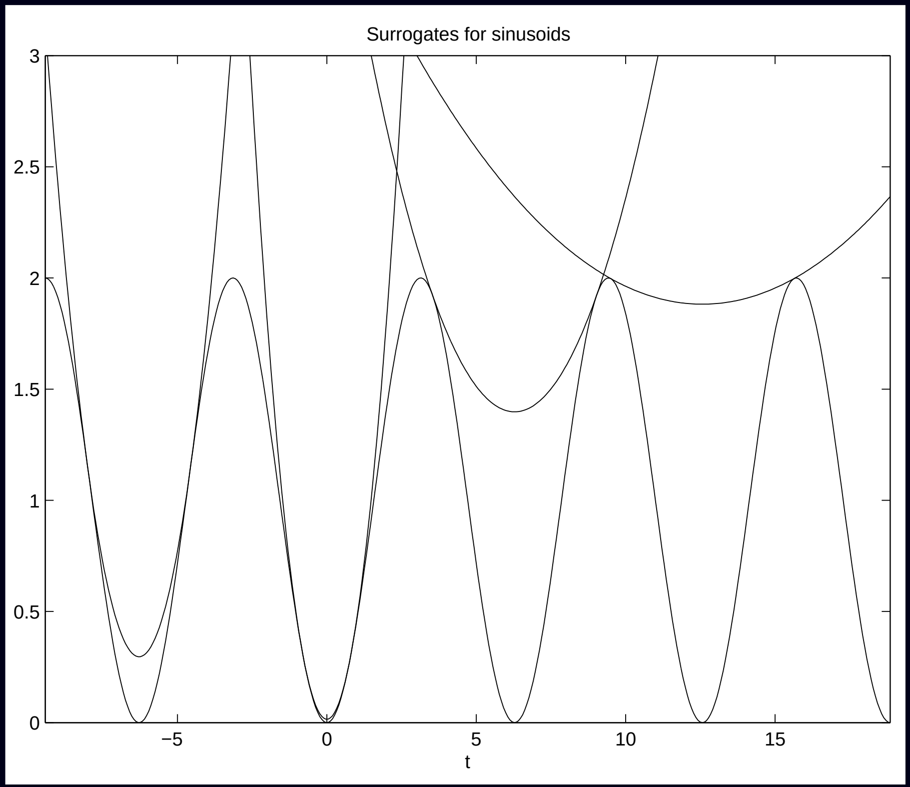
Intrinsically guaranteed to monotonically decrease the cost function.

1D Minimization: cos + quadratic

$$\ldots\; Q_j(x_j; \boldsymbol{x}^{(n)}) \equiv -\left|r_j^{(n)}\right| \cos\left(x_j - x_j^{(n)} - \angle r_j^{(n)}\right),$$

$$r_j^{(n)} = \left(f_j^{(n)}\right)^* [\boldsymbol{A}'(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(n)})]_j + |m_j|^2 M \sum_{i=1}^{N} |P(\vec{\kappa}_i)|^2$$





Simple 1D optimization transfer iterations...

Final Algorithm for Phase Update

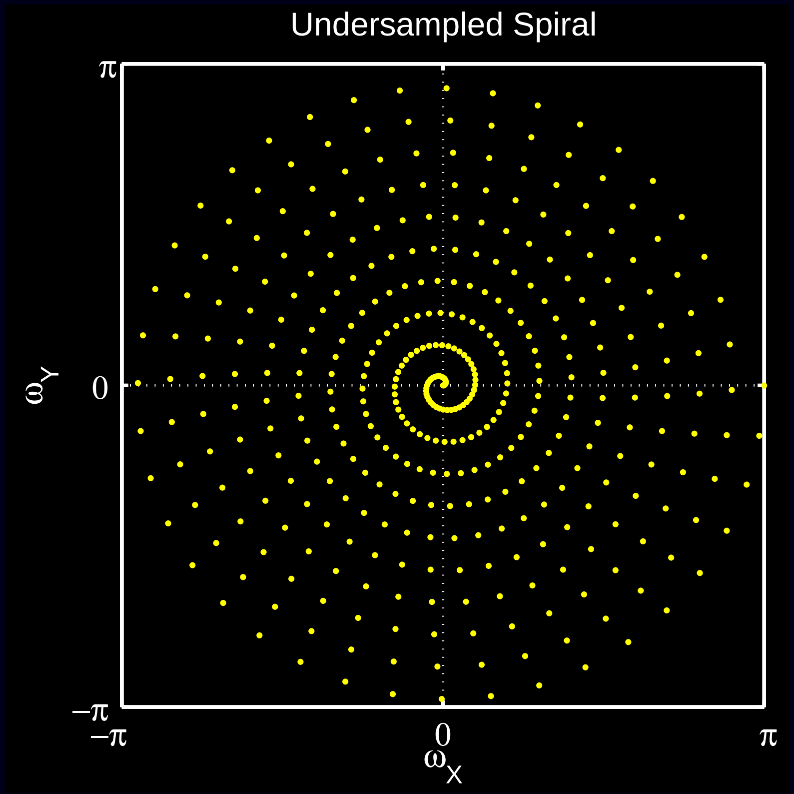
Diagonally preconditioned gradient descent:

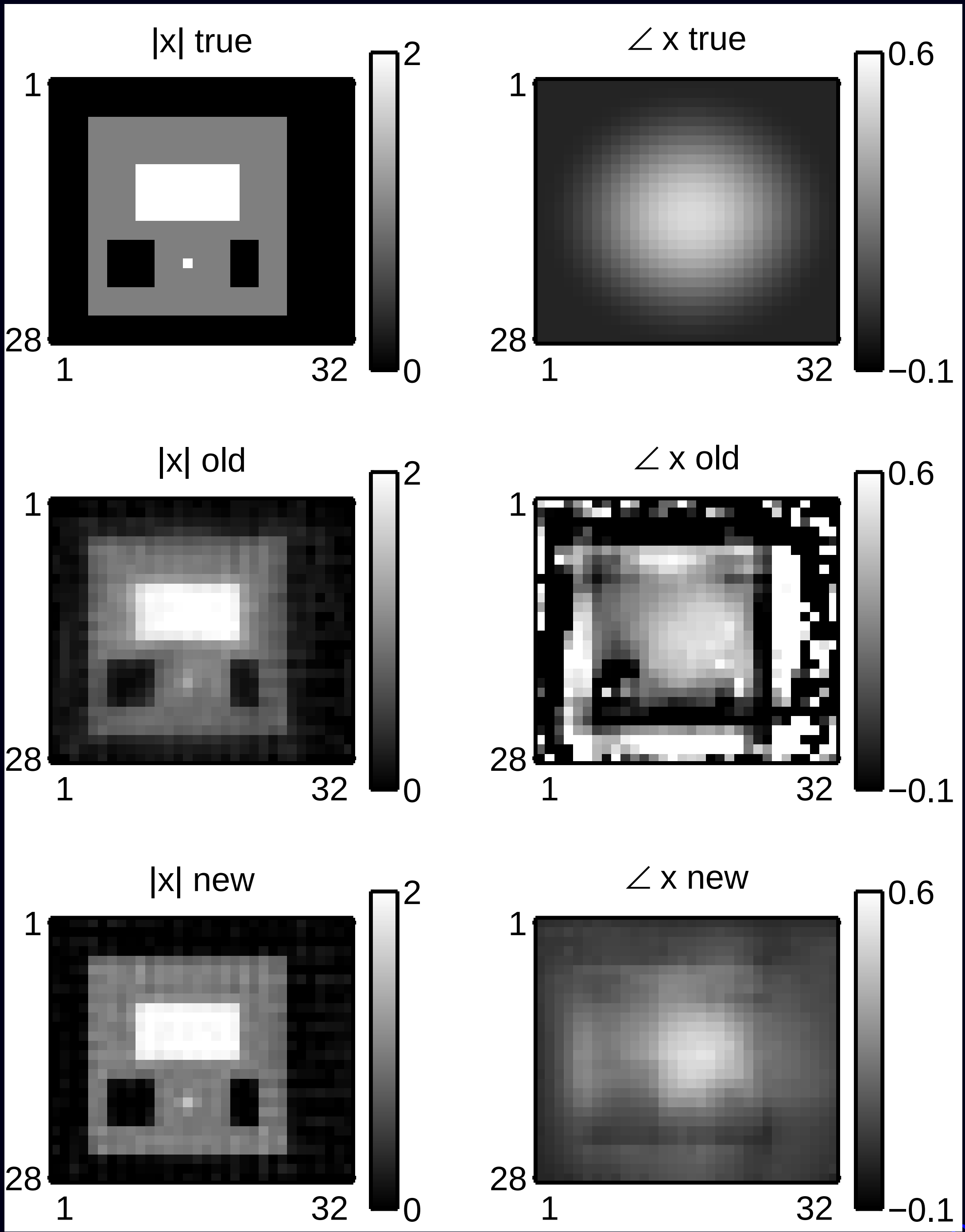
$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \boldsymbol{D}(\boldsymbol{x}^{(n)})\nabla\Phi(\boldsymbol{x}^{(n)})$$

where the diagonal matrix $\boldsymbol{D}$ has elements that ensure Φ decreases monotonically.

Alternate between magnitude and phase updates...

Preliminary Simulation Example





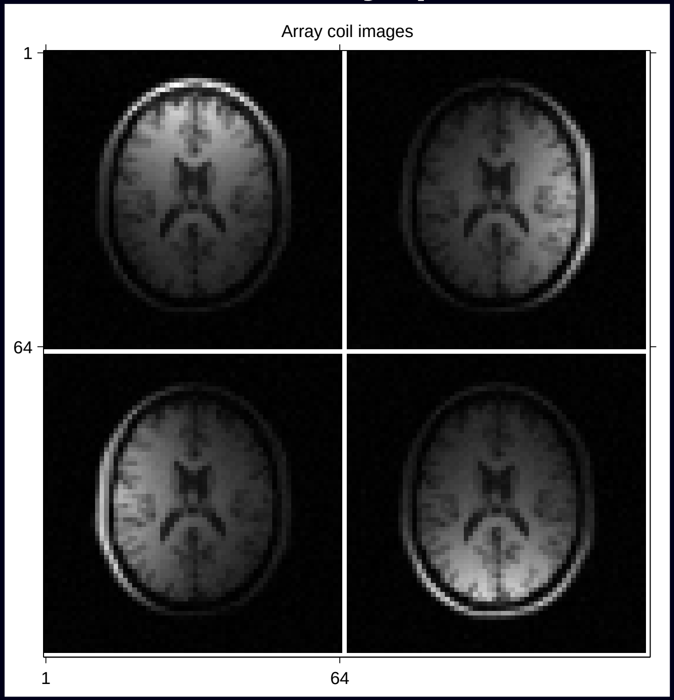
Parallel Imaging

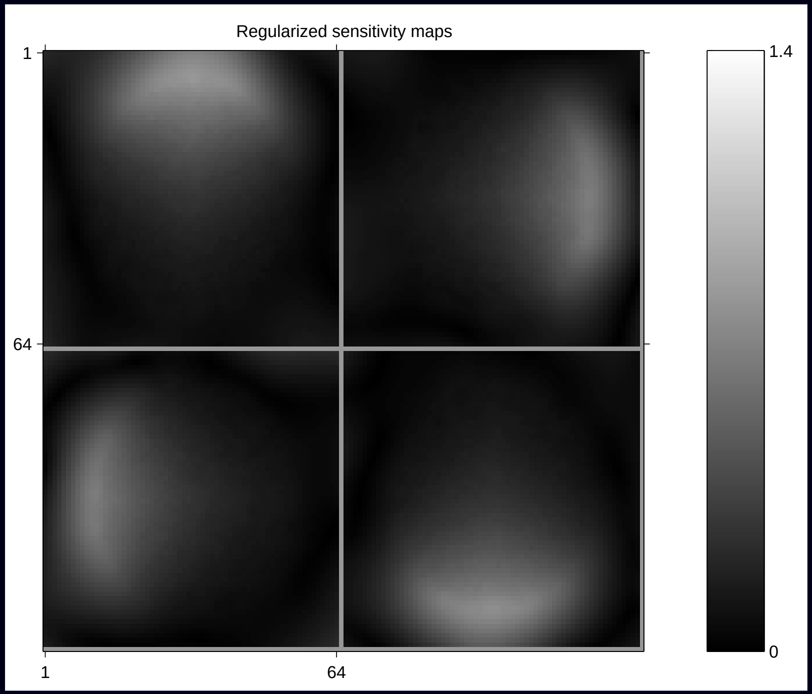
Sensitivity encoded (SENSE) imaging

Use multiple receive coils (requires multiple RF channels).
Exploit spatial localization of sensitivity pattern of each coil.

Note: at 1.5T, RF is about 60MHz.
$\Longrightarrow$ RF wavelength is about $3 \cdot 10^8$m/s$/60 \cdot 10^6$Hz $= 5$ meters

RF coil sensitivity patterns





Pruessmann *et al.*, MRM, 1999

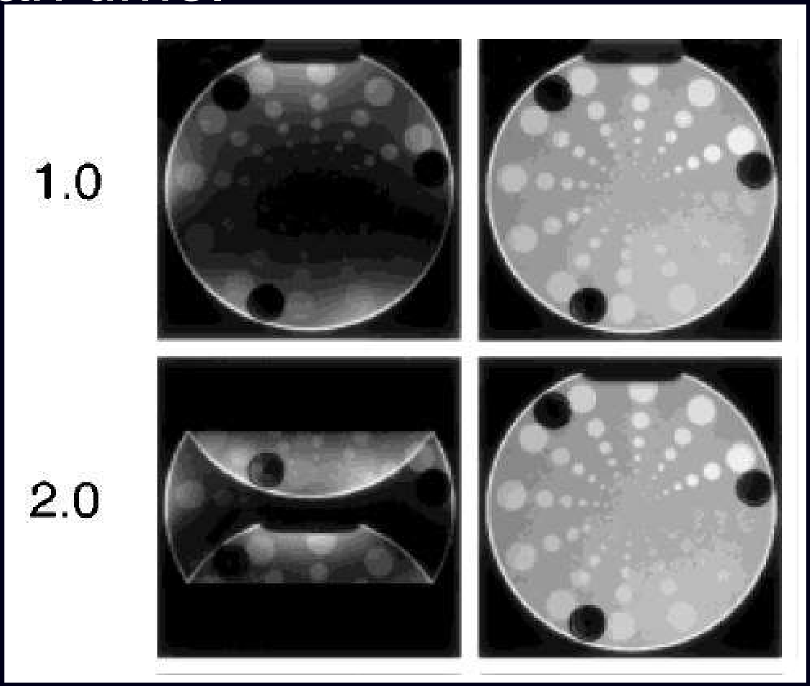
SENSE Model

Multiple coil data:

$$y_{li} = s_l(t_i) + \varepsilon_{li}, \quad s_l(t) = \int f(\vec{r})\, s_l^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}, \quad l = 1, \ldots, L = N_{\text{coil}}$$

Goal: reconstruct $f(\vec{r})$ from coil data $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$
"given" sensitivity maps $\left\{ s_l^{\text{coil}}(\vec{r}) \right\}_{l=1}^{L}$.

Benefit: reduced scan time.



Left: sum of squares; right: SENSE.

SENSE Reconstruction

Signal model:

$$s_l(t) = \int f(\vec{r})\, s_l^{\text{coil}}(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\kappa}(t)\cdot\vec{r}}\, \mathrm{d}\vec{r}$$

Discretized form:

$$\boldsymbol{y}_l = \boldsymbol{A}\boldsymbol{D}_l\boldsymbol{f} + \boldsymbol{\varepsilon}_l, \quad l = 1,\ldots,L,$$

where $\boldsymbol{A}$ is the usual frequency/phase encoding matrix and $\boldsymbol{D}_l$ contains the sensitivity pattern of the lth coil: $\boldsymbol{D}_l = \mathrm{diag}\left\{s_l^{\text{coil}}(\vec{r}_j)\right\}$.

Regularized least-squares estimation:

$$\hat{\boldsymbol{f}} = \arg\min_f \sum_{l=1}^{L} \|\boldsymbol{y}_l - \boldsymbol{A}\boldsymbol{D}_l\boldsymbol{f}\|^2 + \beta R(\boldsymbol{f}).$$

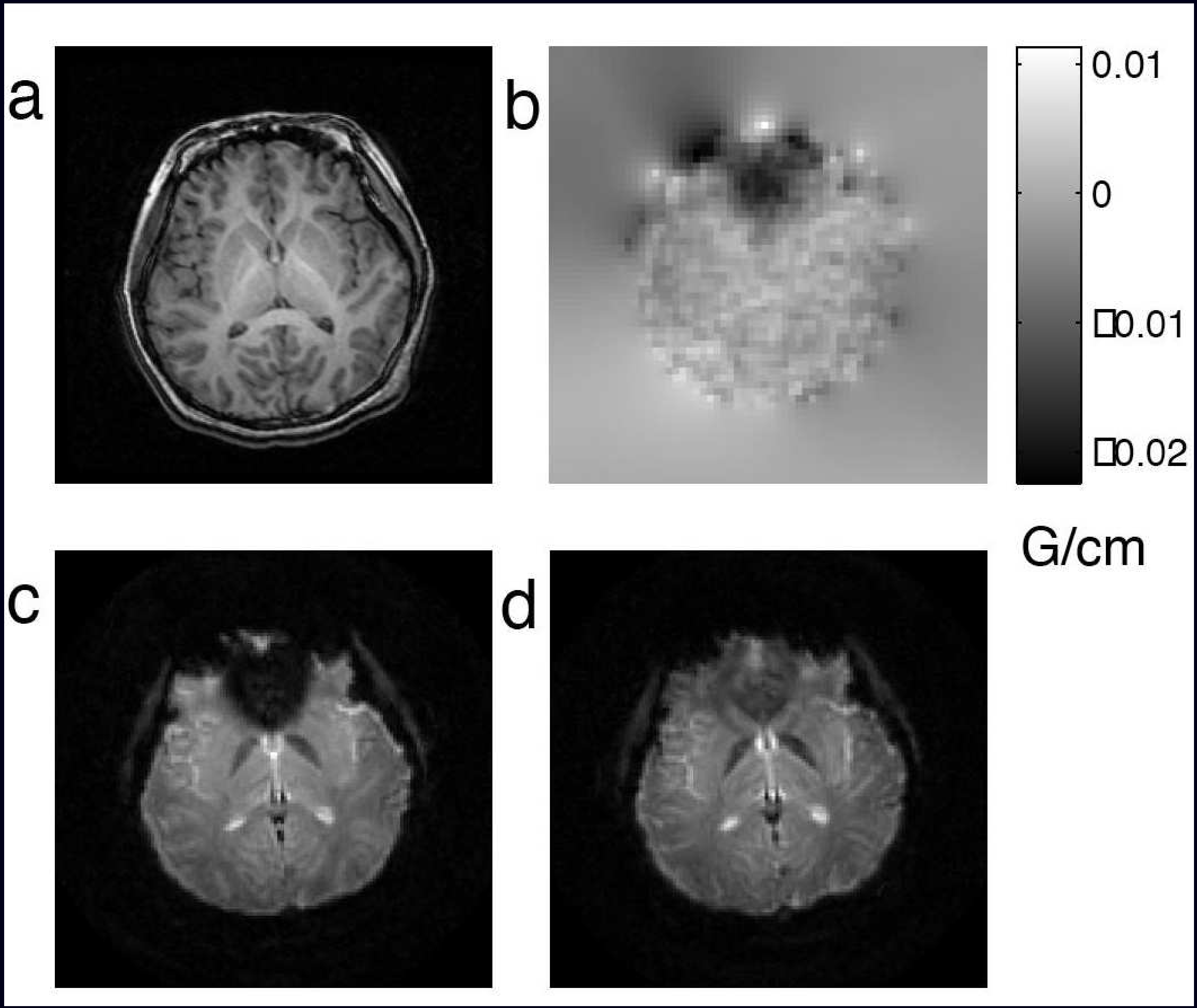
Can generalize to account for noise correlation due to coil coupling.
Easy to apply CG algorithm, including Toeplitz/NUFFT acceleration.

For Cartesian SENSE, iterations are not needed.
(Solve small system of linear equations for each voxel.)

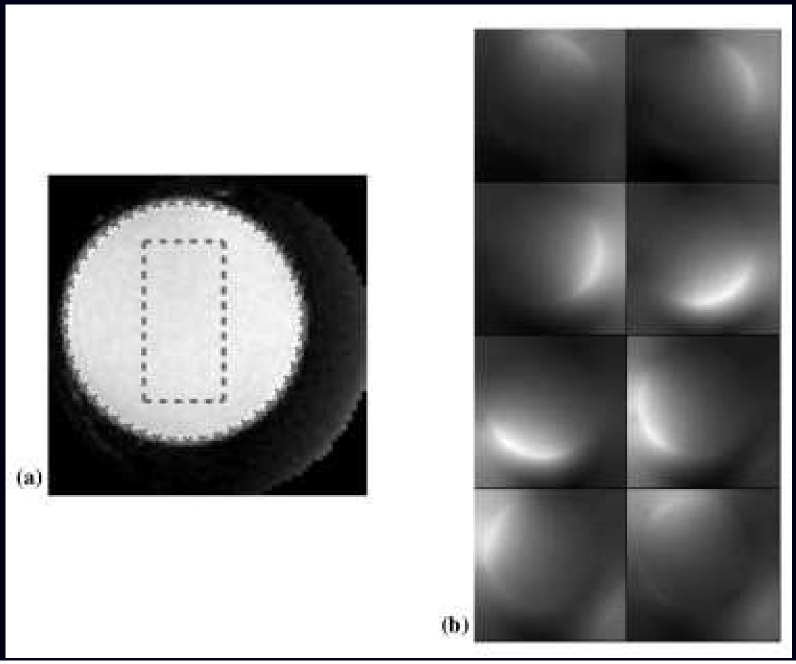
RF Pulse Design

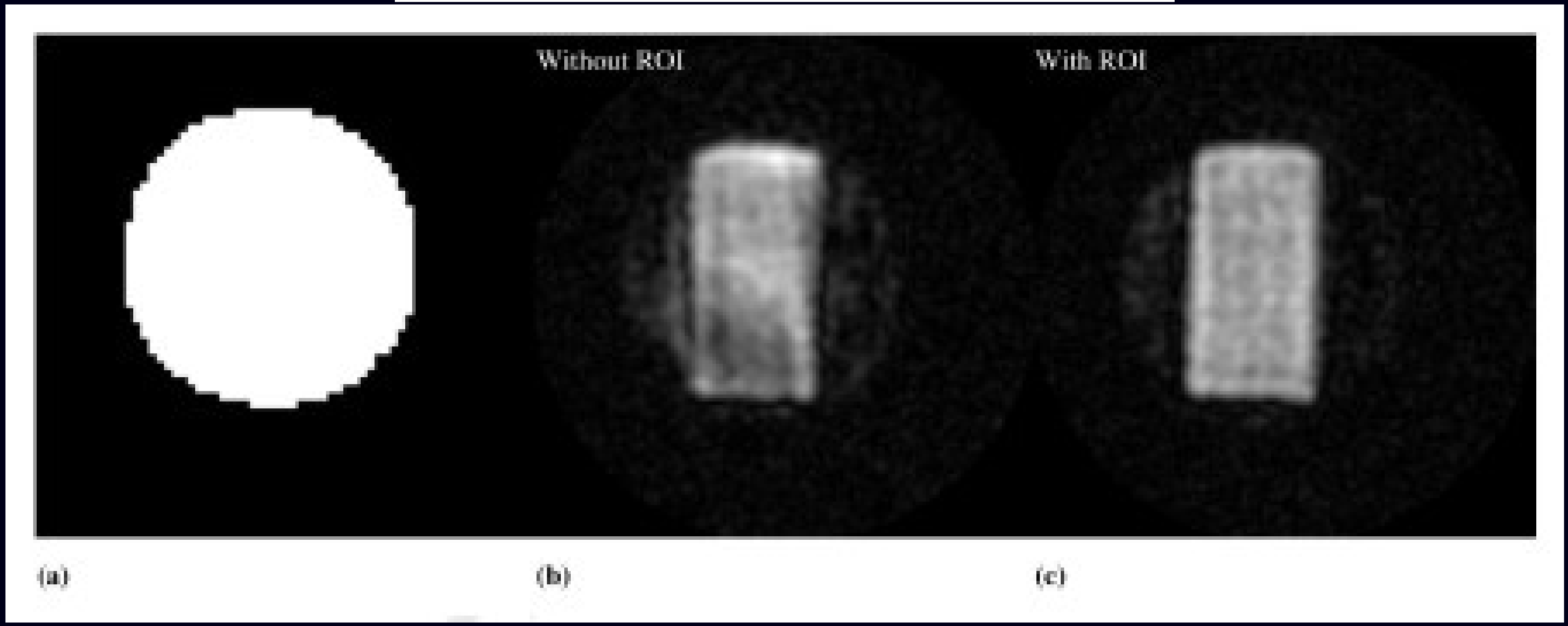
Example: Iterative RF Pulse Design

(3D tailored RF pulses for through-plane dephasing compensation)



Multiple-coil Transmit Imaging Pulses (Mc-TIP)





Summary

- Iterative reconstruction: much potential in MRI
- Even nonlinear problems involving phase terms $e^{\imath x}$ are tractable by using optimization transfer.
- Computation: reduced by tools like NUFFT / Toeplitz
- Optimization algorithm design remains important (*cf.* Shepp and Vardi, 1982, PET)

Some current challenges

- Sensitivity pattern mapping for SENSE
- Through-voxel field inhomogeneity gradients
- Motion / dynamics / partial k-space data
- Establishing diagnostic efficacy with clinical data...