Nonuniform Fast Fourier Transforms and Applications in Imaging

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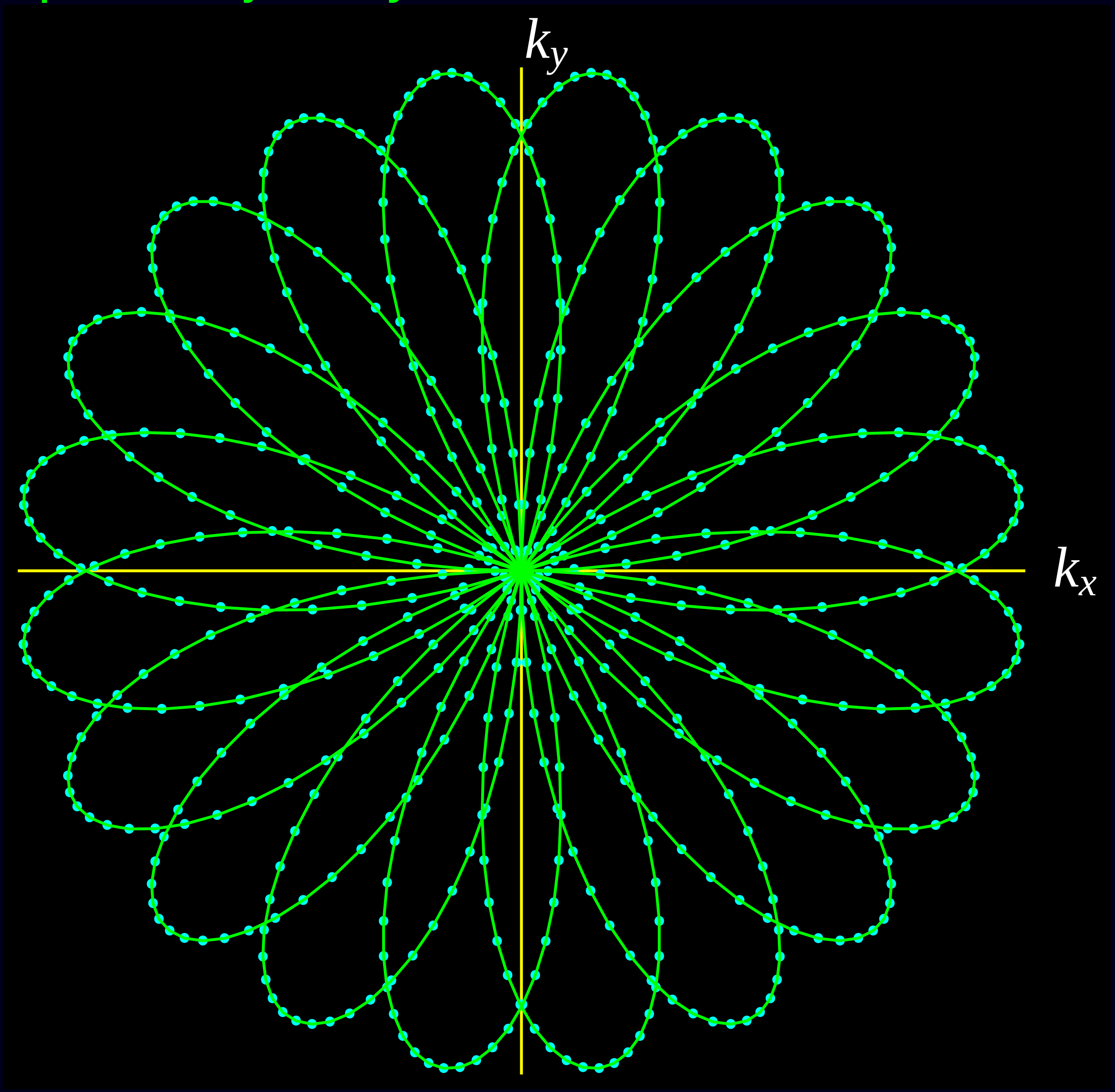
Outline

- Applications
  - 1. MRI
  - 2. Tomography
- Min-max framework for nonuniform FFT
- Examples
- Features / Limitations
- Future goals

MRI work with Brad Sutton, Doug Noll
Tomography work with Samuel Matej

MRI Application

MRI rosette k-space trajectory



Simplified MRI Signal Model

Ignoring *lots* of things:

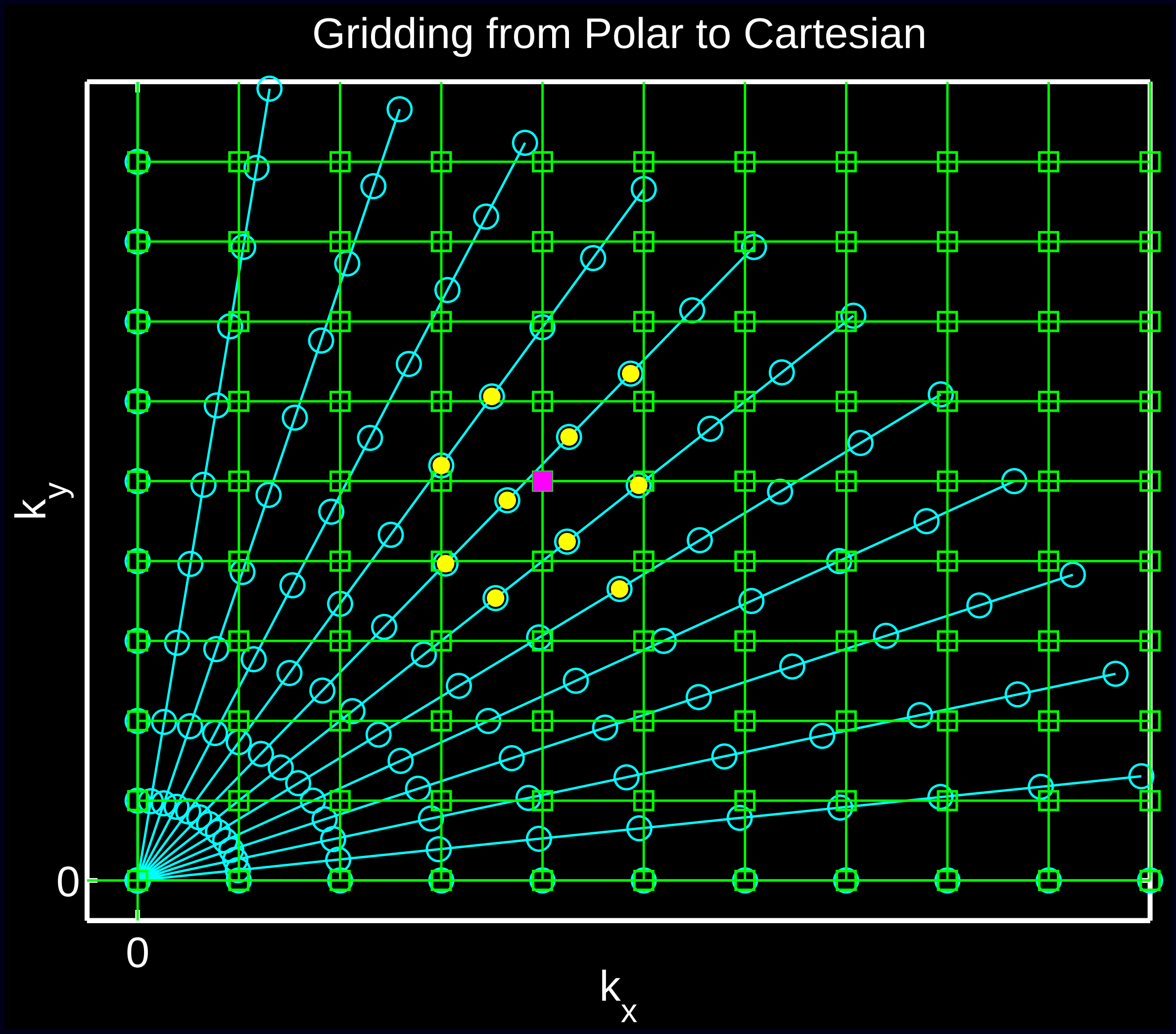
$$y_i = s(t_i) + \text{noise}_i, \qquad i = 1, \ldots, N_{\text{samples}}$$
$$s(t) = \int f(\vec{r}) \exp\left(-\imath 2\pi \vec{k}(t) \cdot \vec{r}\right) \, \mathrm{d}\vec{r},$$

where $\vec{k}(t)$ denotes the "k-space trajectory" of the MR pulse sequence.

- MRI measurements are (roughly) samples of the Fourier transform of the object's transverse magnetization $f(\vec{r})$.
- Reconstruction problem: recover $f(\vec{r})$ from measurements $\{y_i\}$

Conventional MR Image Reconstruction

1. Interpolate measurements onto rectilinear grid ("gridding")
2. Apply inverse FFT to estimate samples of $f(\vec{r})$



Limitations of MR gridding-based reconstruction

1. Artifacts/inaccuracies due to interpolation
2. Contention about sample density “weighting”
3. Oversimplifications of Fourier transform signal model:
   - Magnetic field inhomogeneity
   - Magnetization decay (T_2)
   - Eddy currents
   - ...

MR Image Reconstruction as an Inverse Problem

1. Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{n_\mathrm{p}} x_j \, b(\vec{r} - \vec{r}_j)$$

2. Discrete-discrete measurement model:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}$$

$$a_{ij} = \int b(\vec{r} - \vec{r}_j) \exp\left(-\imath 2\pi \vec{k}(t_i) \cdot \vec{r}\right) \, \mathrm{d}\vec{r} = B(\vec{k}(t_i)) \, \mathrm{e}^{-\imath 2\pi \vec{k}(t_i) \cdot \vec{r}_j}$$

3. $\boldsymbol{\varepsilon}$ includes both measurement noise and model error

4. $\boldsymbol{A}$ can also include “non-Fourier” effects (inhomogeneity, decay, etc.)

5. Least-squares formulation (Gaussian noise model):

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \Psi(\boldsymbol{x}), \qquad \Psi(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|^2$$

6. Regularization included when needed (depends on $\vec{k}(t)$)

7. Preconditioned conjugate gradient iteration for minimization.

Challenges for iterative MR image reconstruction

- Each PCG iteration requires calculation of $\boldsymbol{A}'(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(n)})$
- $\boldsymbol{A}$ is too large to store explicitly (not sparse)
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{x}$ is $O(n_{\mathrm{p}}^2)$, per iteration, whereas FFT is only $O(n_{\mathrm{p}} \log n_{\mathrm{p}})$

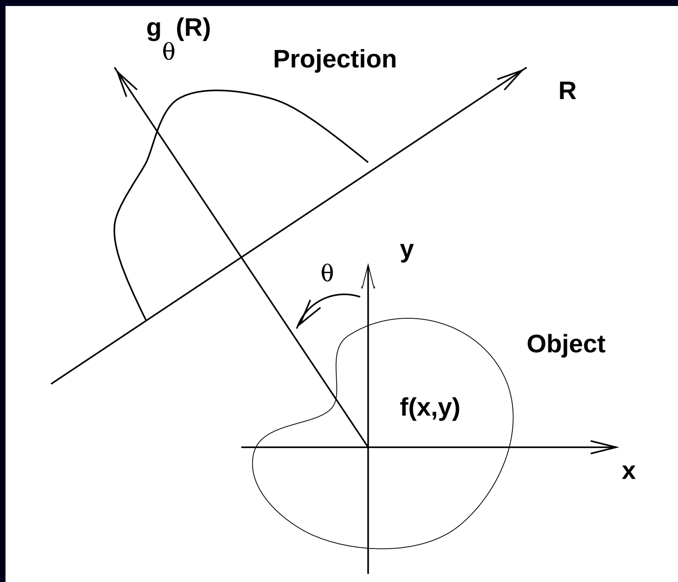
$\Rightarrow$ need fast algorithm for computing $\boldsymbol{A}\boldsymbol{x}$, *i.e.*, for computing

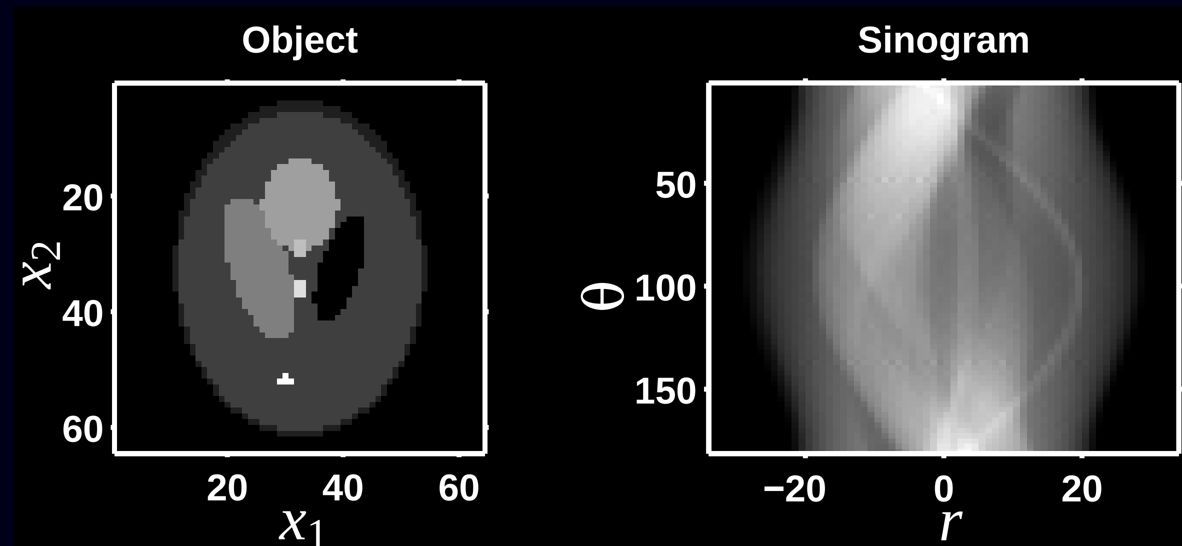
$$\sum_{n_1} \sum_{n_2} \mathrm{e}^{-\imath 2\pi (k_1(t_i) n_1 + k_2(t_i) n_2)} x(n_1, n_2),$$

assuming the $\vec{r}_j$'s (basis centers) are unit spaced on a rectilinear grid.

Need: fast algorithm for 2D nonuniform Fourier transform

Tomography Application





Simplified tomography measurement model (sinogram):

$$y_i = (h * p_{\theta_i}(\cdot; f))(r_i) + \text{noise}_i, \qquad i = 1, \ldots, n_\text{d}, \qquad n_\text{d} = n_r \cdot n_\theta.$$

Radon transform degraded by radially shift-invariant blur with PSF $h(r)$.

Radon transform (line integrals):

$$p_\theta(r; f) = \int f(r\cos\theta - l\sin\theta, r\sin\theta + l\cos\theta)\, dl$$

Goal: reconstruct object $f(\vec{r})$ from sinogram measurements $\{y_i\}$

Classical Fourier-transform reconstruction

Fourier-slice theorem:

$$p_\theta(r;f) \overset{\text{1D FT}}{\longleftrightarrow} P_\theta(\rho) = F(\rho,\theta) \overset{\text{2D FT}}{\longleftrightarrow} f(x,y)$$

- Compute 1D FFT of each row of sinogram.
- Possibly deconvolve blur $h(r)$
- Interpolate from polar samples onto rectilinear frequency samples
- Compute inverse 2D FFT

Limitations

- Artifacts due to polar-cartesian interpolation
- Suboptimal treatment of nonuniform-variance noise, *e.g.*, Poisson
- Over-simplified measurement model
- Disregards nonnegativity constraint

Proposed approach partially overcomes first two limitations

Iterative Tomographic Image Reconstruction

1. Series expansion of unknown object:

$$f(\vec{r}) \approx \sum_{j=1}^{n_\mathrm{p}} x_j \, b(\vec{r} - \vec{r}_j)$$

2. Discrete-discrete measurement model

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}, \qquad a_{ij} = h(r) * p_{\theta_i}(r; b(\cdot - \vec{r}_j))\Big|_{r=r_i}$$

3. Penalized weighted least-squares (PWLS) formulation

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \Psi(\boldsymbol{x}), \qquad \Psi(\boldsymbol{x}) = (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x})' \boldsymbol{W} (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}) + \beta R(\boldsymbol{x})$$

4. Weighting matrix $\boldsymbol{W}$ for nonuniform noise variance
   (cf Delaney and Bresler, IEEE T-IP, May 1996)
5. Regularization essential due to ill-conditioned nature of tomography
6. Preconditioned conjugate gradient iteration for minimization.

Challenges for Iterative Tomographic Reconstruction

- Each PCG iteration requires calculation of $\boldsymbol{A}'\boldsymbol{W}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^{(n)})$
- $\boldsymbol{A}$ is sparse, but very large for 3D PET, too large to store in 2D X-ray CT
- Even if $\boldsymbol{A}$ were stored, directly computing $\boldsymbol{A}\boldsymbol{x}$ is $O(n_\mathrm{p}^2)$, per iteration, whereas FFT is only $O(n_\mathrm{p} \log n_\mathrm{p})$

Proposed approach for reprojection (computing $\boldsymbol{A}\boldsymbol{x}$)

1. Apply nonuniform FFT to compute 2D FT *on a polar grid* accurately
2. Apply shift-invariant blur $h(r)$ in frequency domain
3. Compute inverse 1D FFT to form each row of reprojection

- Avoids line-integral calculations!
- Routine for $\boldsymbol{A}'$ is the *exact* adjoint

Prior work on NUFFT

- [1] Dutt & Rokhlin, SIAM JSC, 1993
  Fast Fourier transforms for nonequispaced data.
  Gaussian based interpolation
- [2] Beylkin, ACHA, 1995
  On the fast Fourier transform of functions with singularities.
  B-spline based interpolation in multiresolution framework (N-D)
- [3] Dutt & Rokhlin, ACHA, 1995
  Fast Fourier transforms for nonequispaced data, II.
  fast multipole method
- [4] Anderson & Dahleh, SIAM JSC, 1996
  Rapid computation of the discrete Fourier transform.
  Taylor series expansion, requiring multiple FFTs
- [5] Nguyen & Liu, SIAM JSC, 1999
  The regular Fourier matrices and nonuniform fast Fourier transforms.
  least-squares approach to shift-variant Fourier interpolation

- [7] Fessler & Sutton, IEEE T-SP, 2003
  Nonuniform fast Fourier transforms using min-max interpolation.

NUFFT Problem Statement (1D)

Given signal x_n, $n = 0, \ldots, N-1$ with (discrete-time) Fourier transform

$$X(\omega) = \sum_{n=0}^{N-1} x_n \, \mathrm{e}^{-\imath \omega n}$$

and a collection of arbitrary frequencies $\{\omega_m : m = 1, \ldots, M\}$, compute

$$y_m = X(\omega_m), \qquad m = 1, \ldots, M.$$

Direct approach is $O(NM)$; impractical for large M.

NUFFT via linear interpolation

1. Compute K-point FFT of x_n (where $K \geq N$, possibly oversampled)

$$X_k = X\left(\frac{2\pi k}{K}\right), \qquad k = 0, \ldots, K-1$$

2. Interpolate from set $\{2\pi k/K\}$ to set $\{\omega_m\}$

$$\hat{y}_m = \sum_{k=0}^{K-1} v_{mk} X_k$$

Design question: how to choose interpolation coefficients $\{v_{mk}\}$?

Scaled variation

1. Start with "weighted" K-point FFT:

$$Y_k = \sum_{n=0}^{N-1} s_n x_n \, e^{-\imath \frac{2\pi k}{K} n}$$

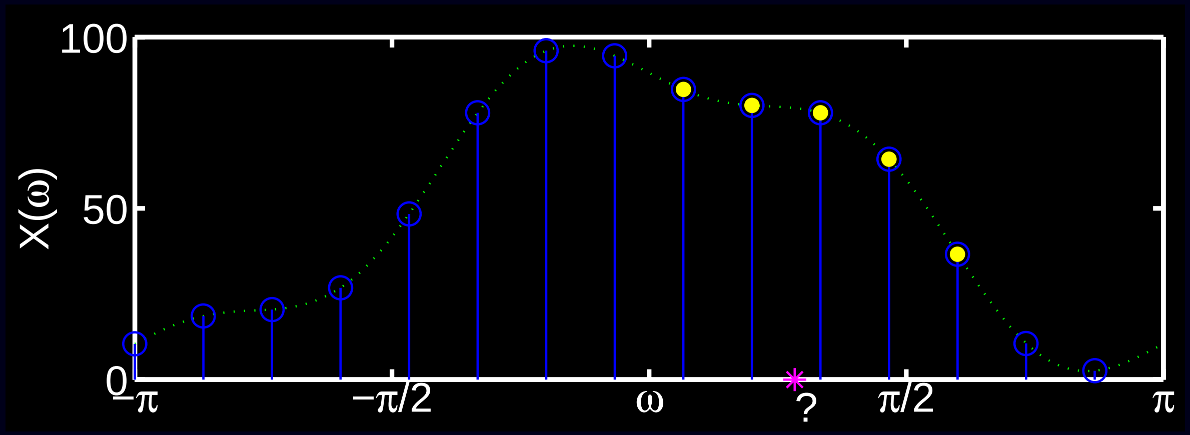
2. Design problem includes choosing scaling factors $\{s_n\}$. (Important!)

Interpolators

1. Shift invariant:
   - Gaussian
   - B-spline
   - ...
   - Rarely precomputed
   - Less memory
   - More in-line work

2. Shift variant
   - Constraint: use the J nearest FFT samples for each ω_m

$$\hat{y}_m = \sum_{j=1}^{J} u_{mj}^{\star} X_{k_0(\omega_m)+j}, \text{ where } k_0(\omega) \triangleq \begin{cases} \left(\arg\min_k \left|\omega - \frac{2\pi}{K}k\right|\right) - \frac{J+1}{2}, & J \text{ odd} \\ \left(\max\left\{k : \omega \geq \frac{2\pi}{K}k\right\}\right) - \frac{J}{2}, & J \text{ even.} \end{cases}$$



   - $O(JM)$ memory if interpolation coefficients are precomputed
   - $O(K \log K) + O(JM)$ computation

Min-Max Criterion

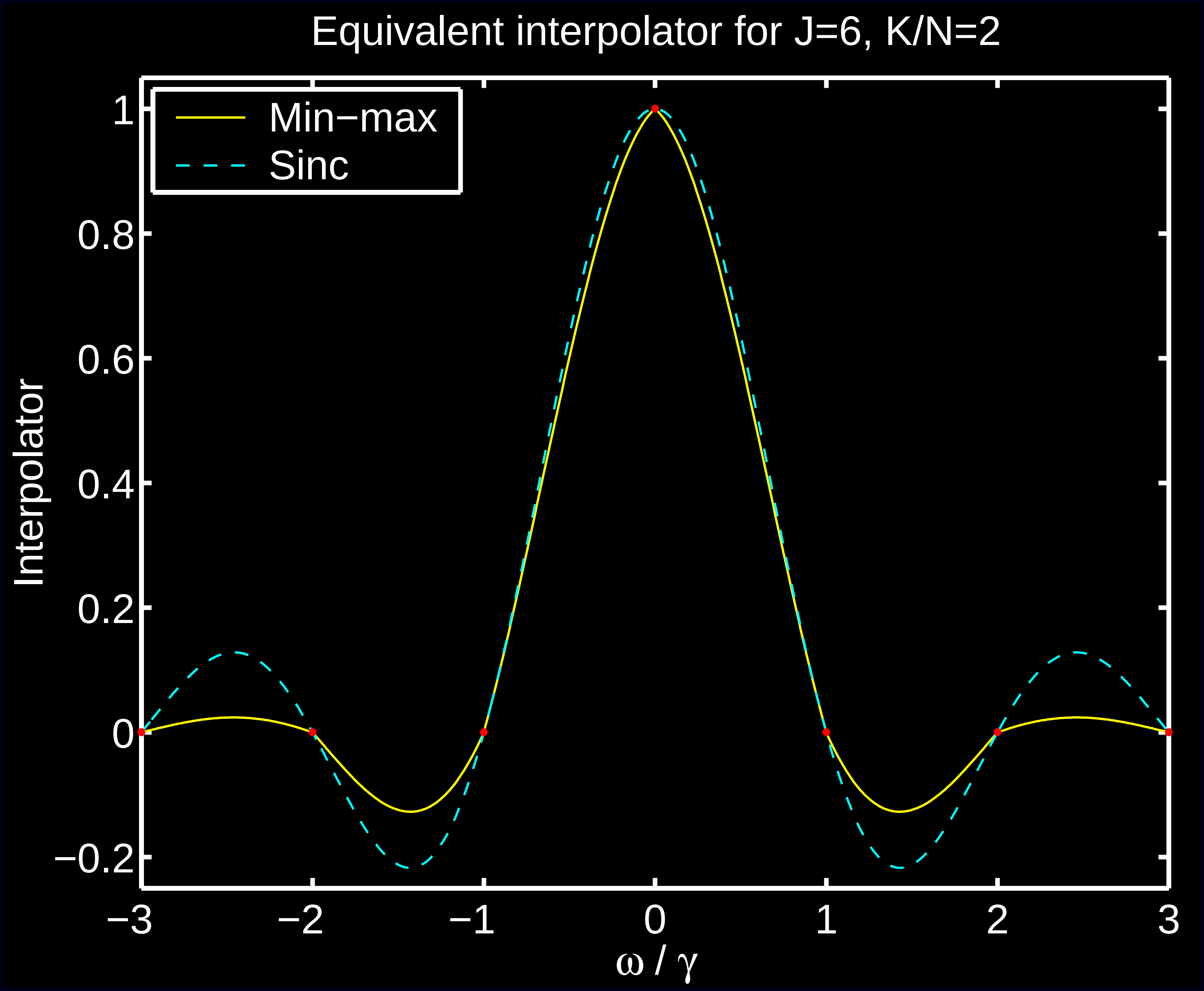
$\boxed{\text{Choose interpolation cofficients } \{u_{mj}\} \text{ to minimize worst-case error.}}$

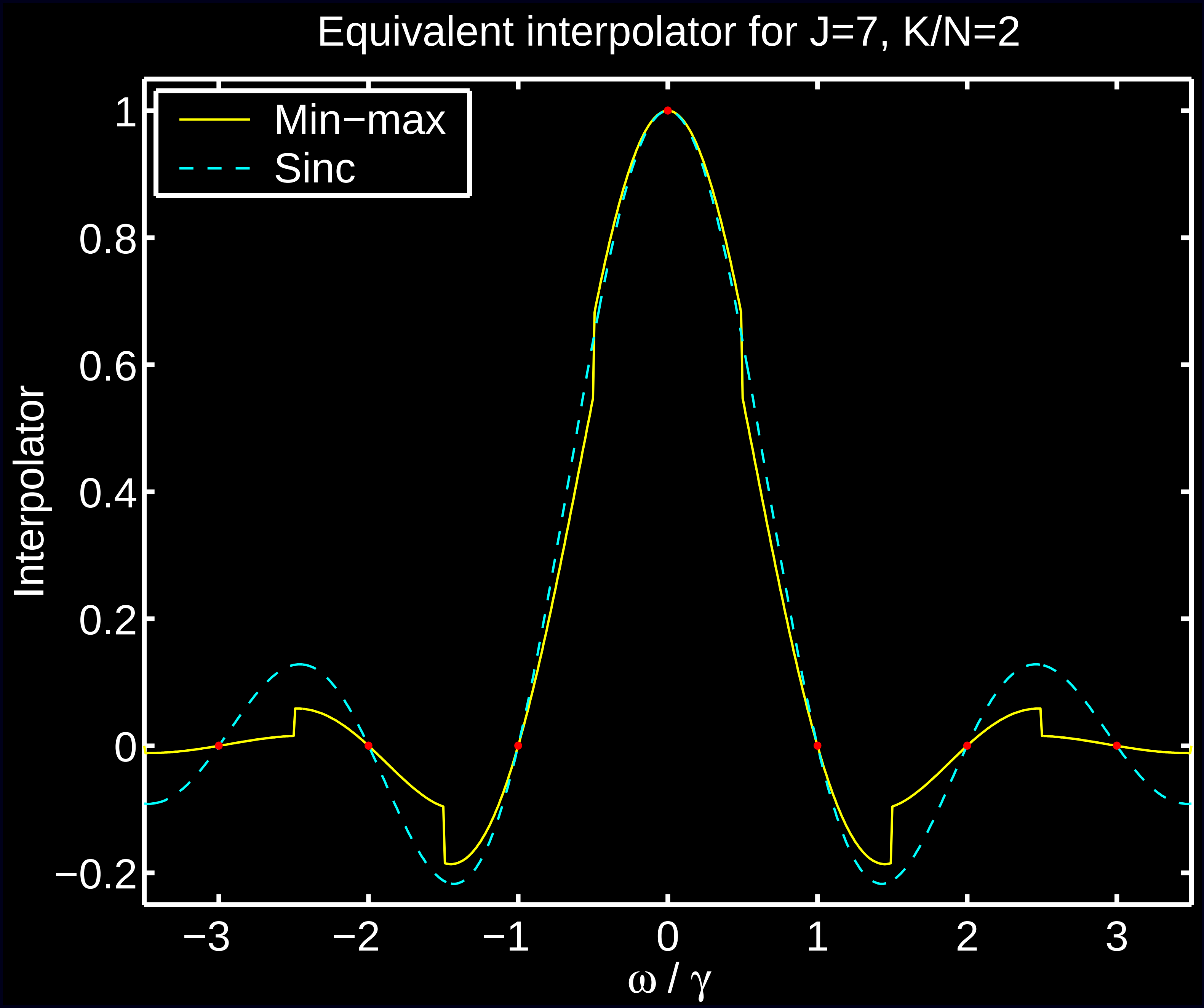
$$\min_{\boldsymbol{u}_m \in \mathbb{C}^J} \max_{\boldsymbol{x} \in \mathbb{C}^N : \|\boldsymbol{x}\| \leq 1} |\hat{y}_m - y_m|, \text{ where } \boldsymbol{u}_m = (u_{m1}, \ldots, u_{mJ}).$$

Solution (data independent!):

$$\boldsymbol{u}_m = \boldsymbol{\Lambda}'(\omega) \boldsymbol{T} \boldsymbol{r}(\omega_m), \quad \text{where:} \quad \begin{aligned} \Lambda_{jj}(\omega) &= e^{-\imath[\omega - \frac{2\pi}{K}(k_0(\omega)+j)]\frac{N-1}{2}} \\ \boldsymbol{T} &= [\boldsymbol{C}'\boldsymbol{C}]^{-1} \in \mathbb{R}^{J \times J} \\ [\boldsymbol{C}'\boldsymbol{C}]_{l,j} &= \delta_0(j-l) \\ r_j(\omega) &= \delta_0(\omega/(2\pi/K) - k_0(\omega) - j) \\ \delta_0(t) &\triangleq \frac{\sin(\pi t N/K)}{N \sin(\pi t/K)}. \end{aligned}$$

"Modified truncated-Dirichlet interpolation of oversampled FFT"

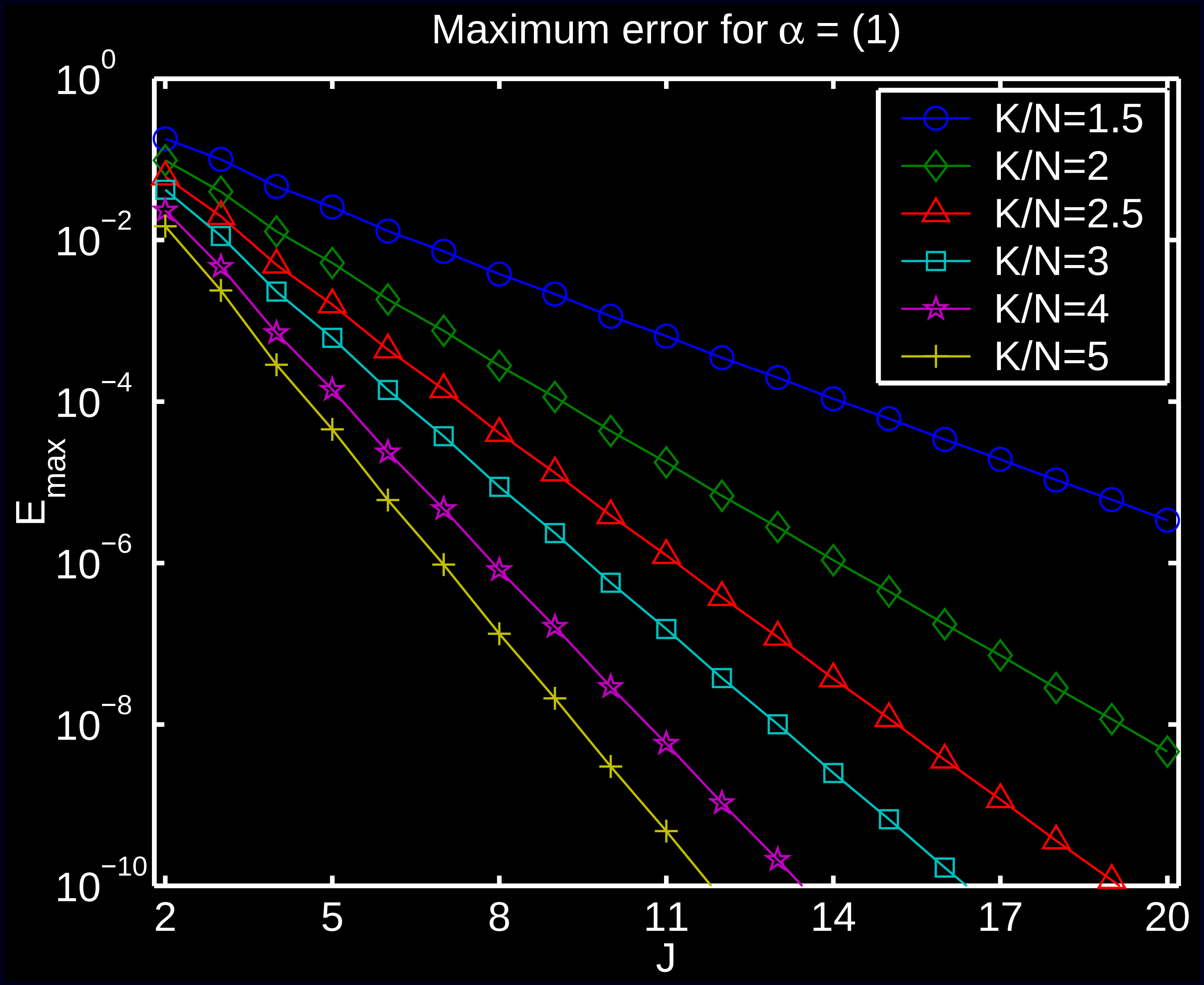

Equivalent interpolator for J=6, K/N=2
Min−max
Sinc
Interpolator
1
0.8
0.6
0.4
0.2
0
−0.2
−3
−2
−1
0
1
2
3
ω / γ



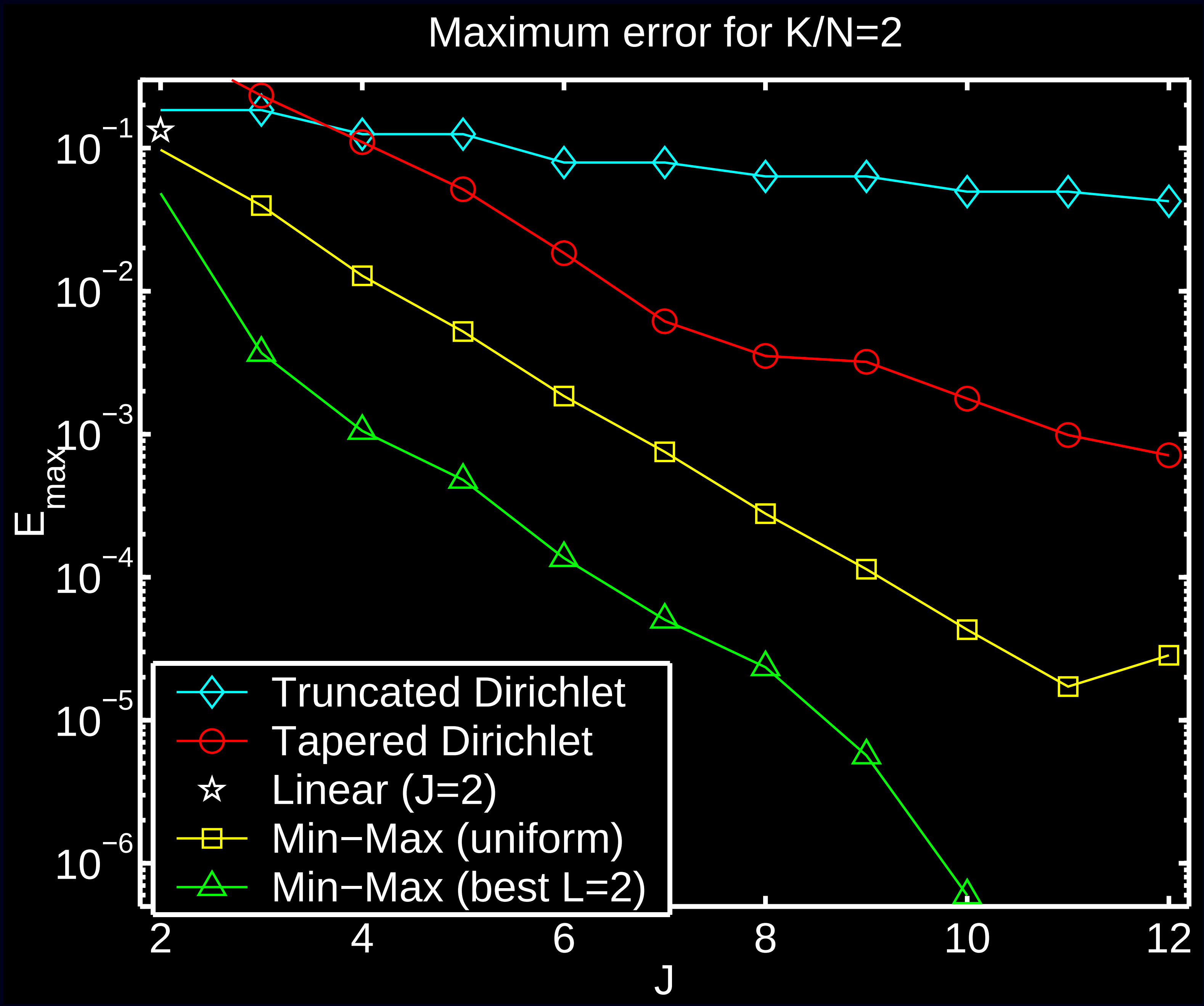
Equivalent interpolator for J=7, K/N=2
Min−max
Sinc
Interpolator
1
0.8
0.6
0.4
0.2
0
−0.2
−3
−2
−1
0
1
2
3
ω / γ

Accuracy

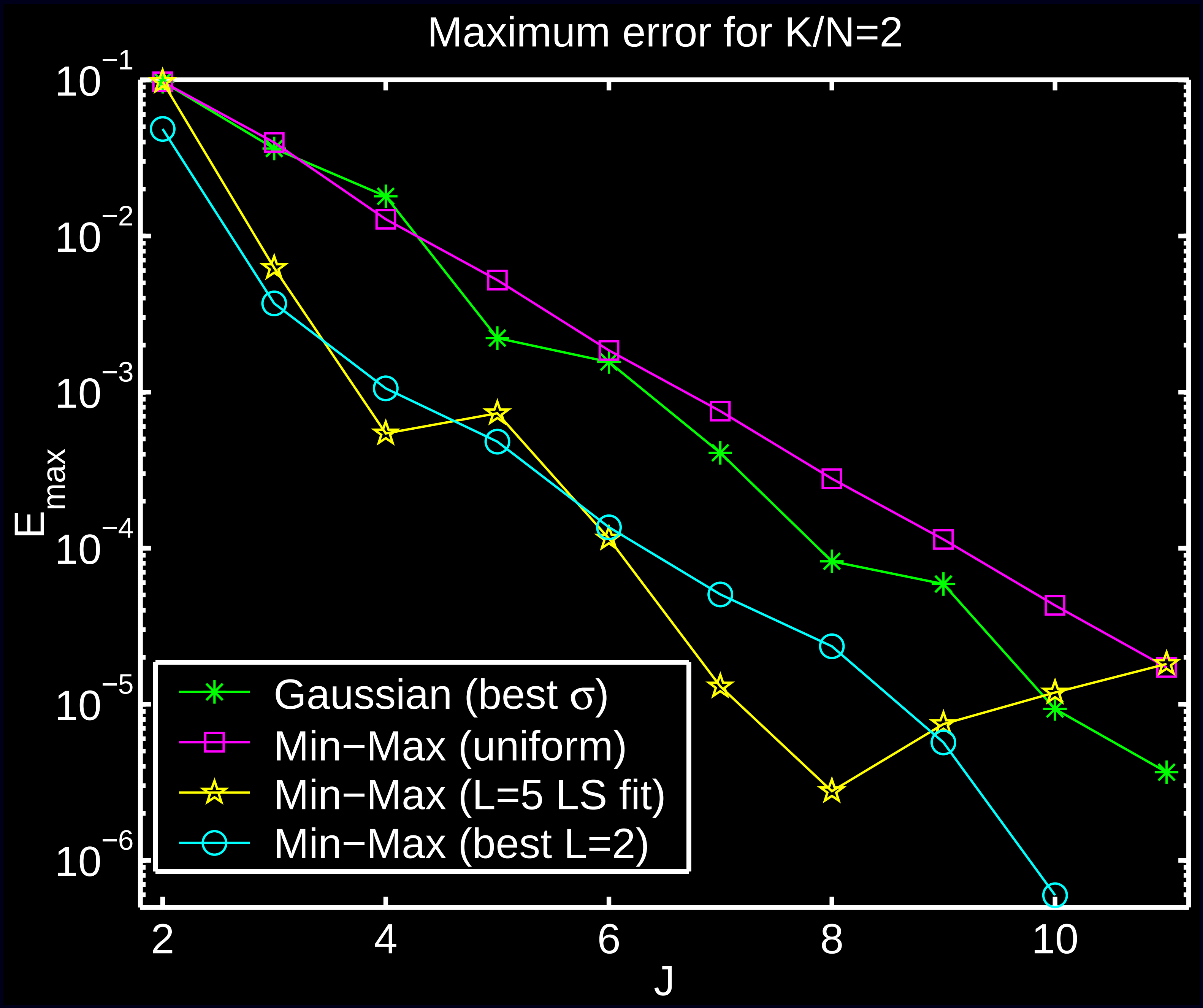
Worst-case error for unit-norm signal is $\frac{E_{\max}(\omega)}{\sqrt{N}} = \sqrt{1 - \boldsymbol{r}'(\omega)\boldsymbol{T}\boldsymbol{r}(\omega)}$.



Comparison with Dirichlet



Comparison with Gaussian (Dutt/Rokhlin)



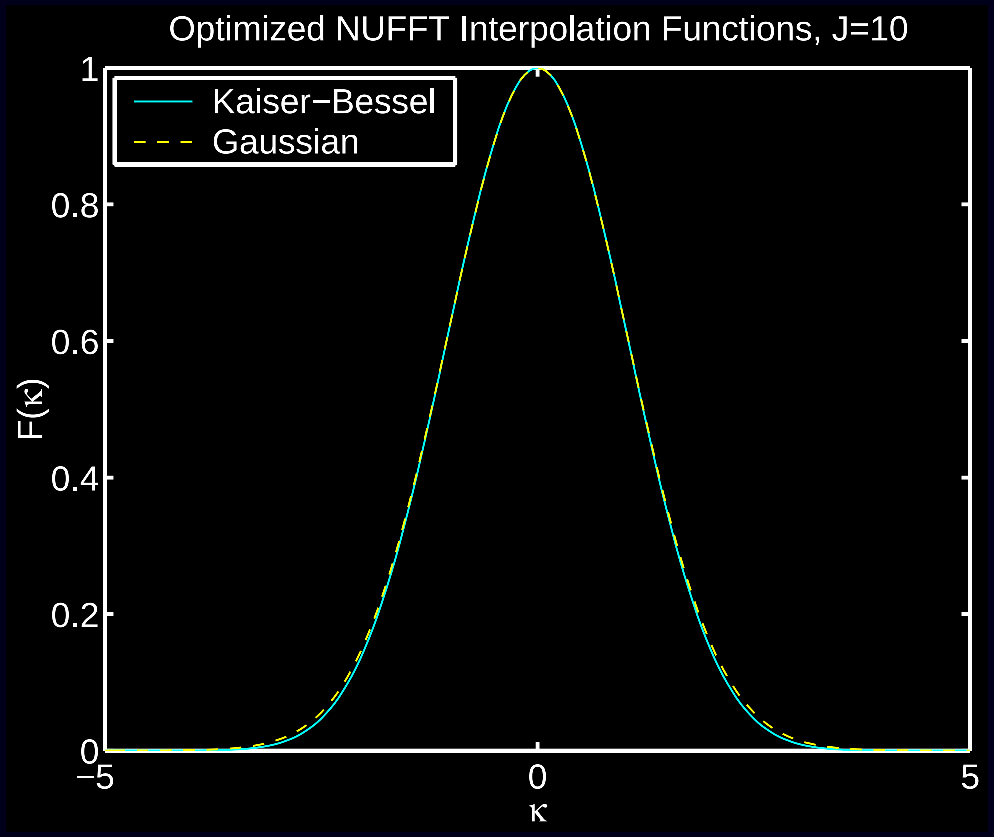
Extensions

- Multidimensional NUFFT
  Use $J \times J$ neighborhood (in 2D, *e.g.*) around each spatial frequency location of interest. Straightforward generalization.
- Adjoint operator
  1. Hermitian transpose of interpolation matrix
  2. K-point inverse FFT
- Adaptive neighborhoods
  Neighborhood size J vs distance between ω_m and nearest neighbor.
- Free software: `http://www.eecs.umich.edu/~fessler`

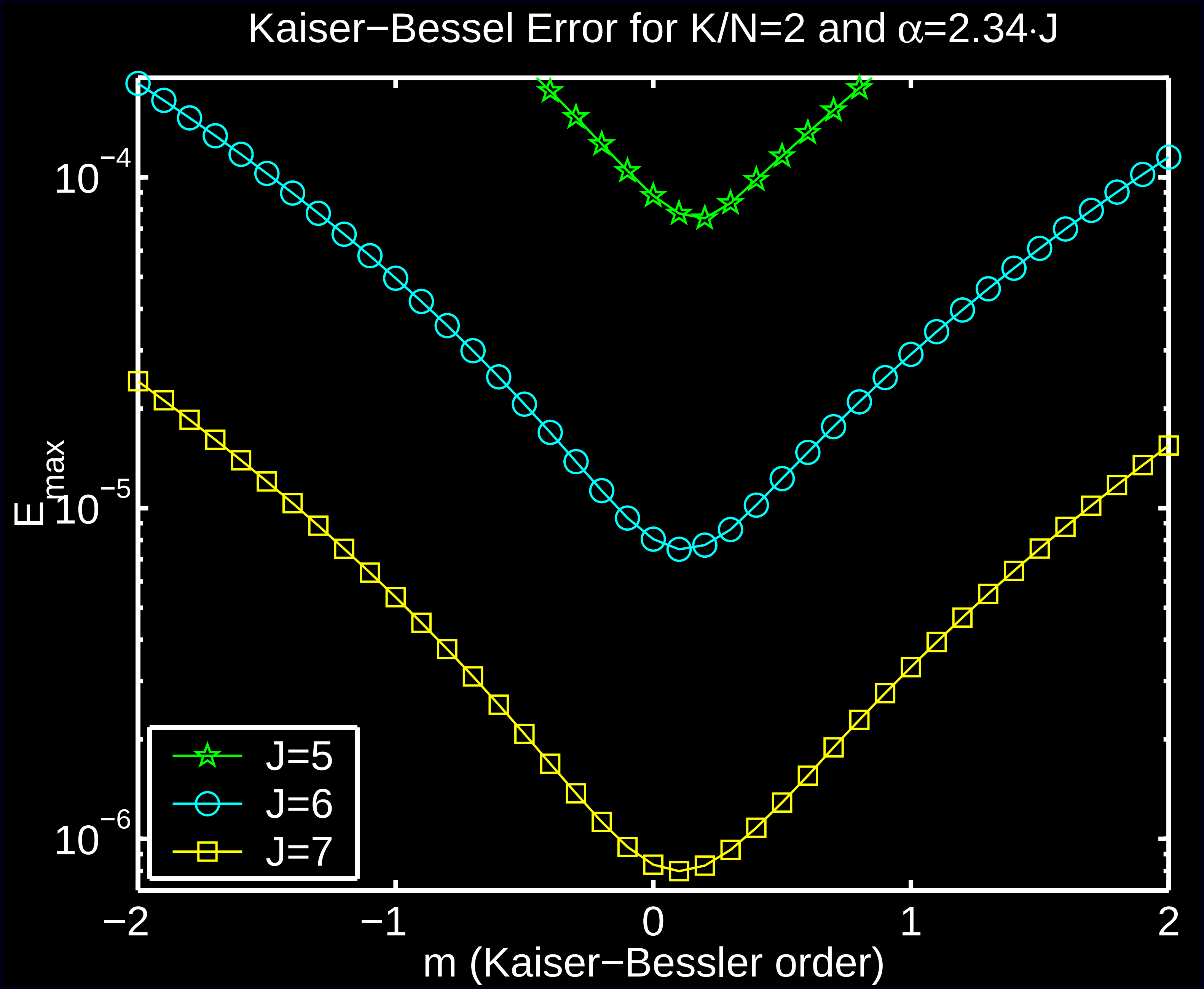
Kaiser-Bessel Interpolator

$$F(\kappa) = f_J^m(\kappa)\frac{I_m(\alpha f_J(\kappa))}{I_m(\alpha)}, \text{ where } f_J(\kappa) \triangleq \begin{cases} \sqrt{1-\left(\frac{\kappa}{J/2}\right)^2}, & |\kappa| < J/2 \\ 0, & \text{otherwise.} \end{cases}$$

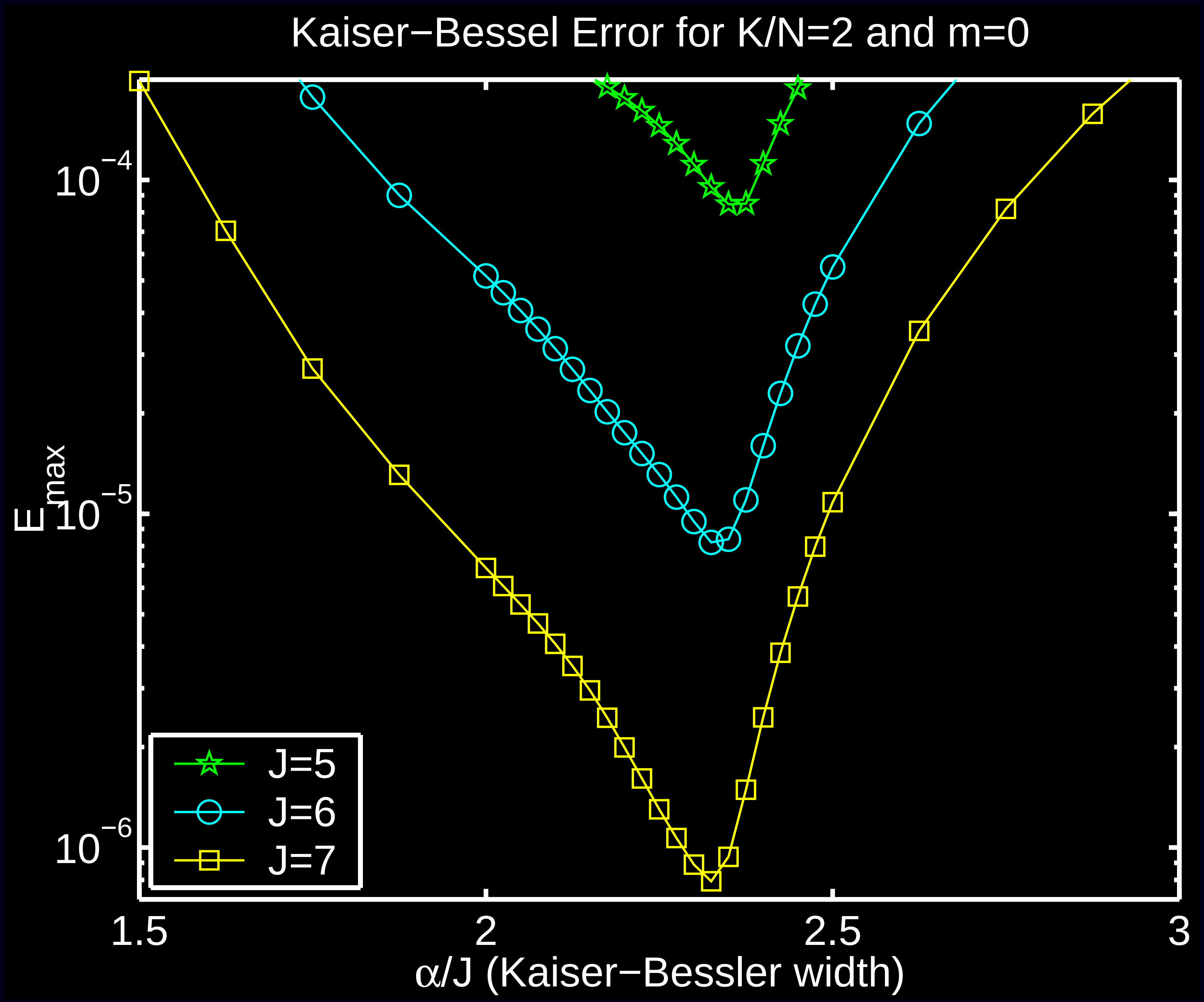
- Optimality properties?
- Usually $m = 2$ so continuous and differentiable on boundaries.



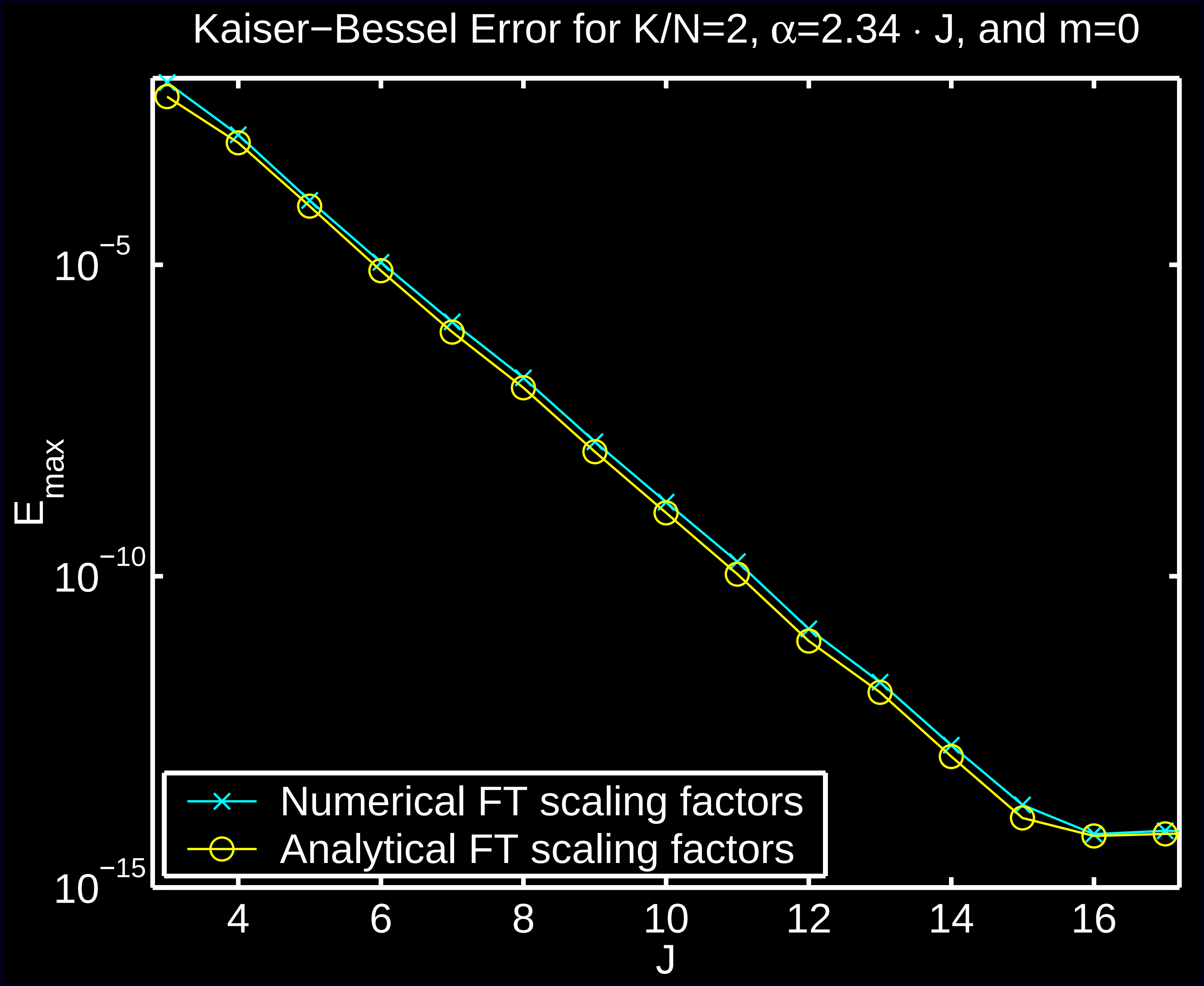
Kaiser-Bessel: Optimizing Order



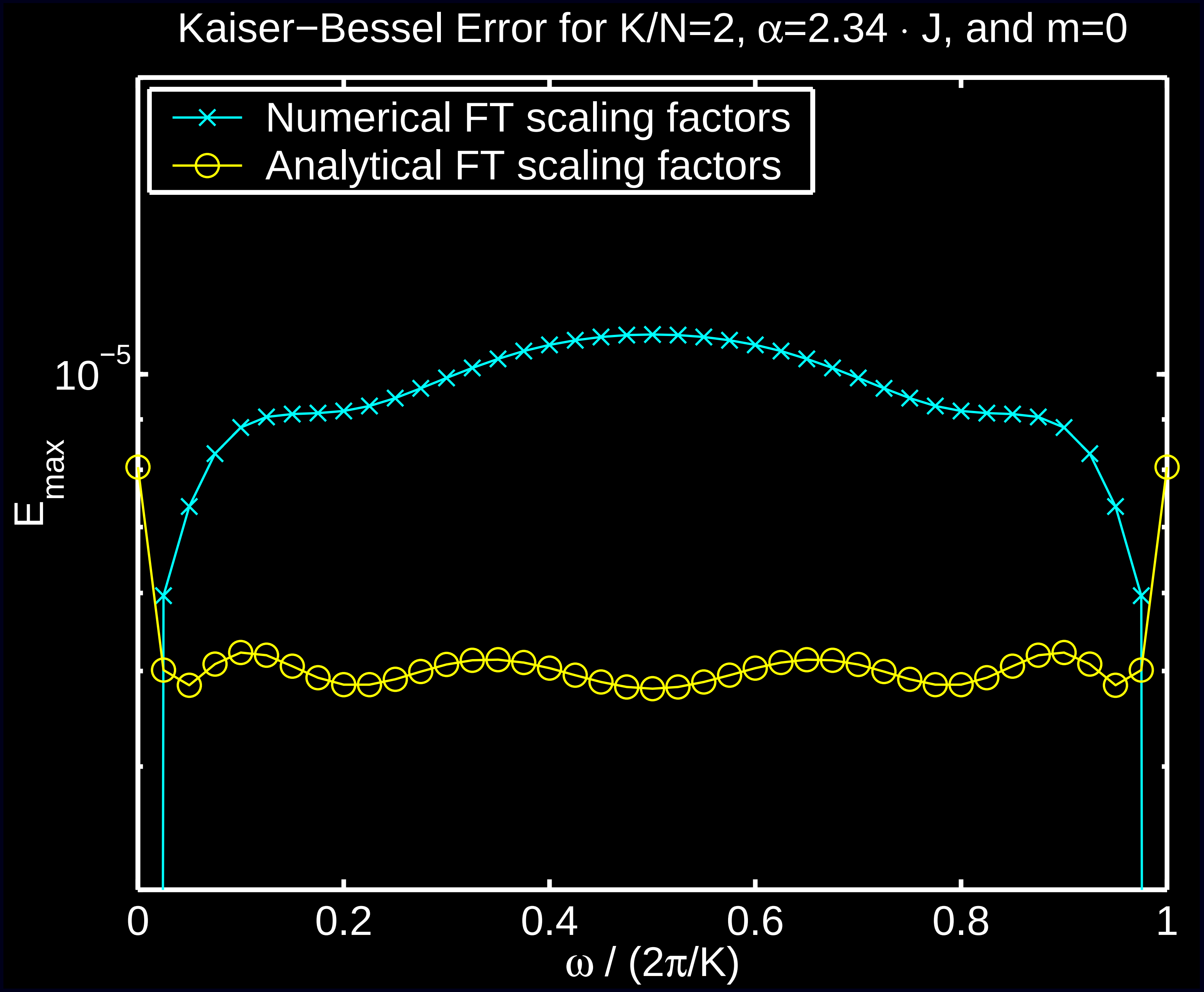
Kaiser-Bessel: Optimizing Width



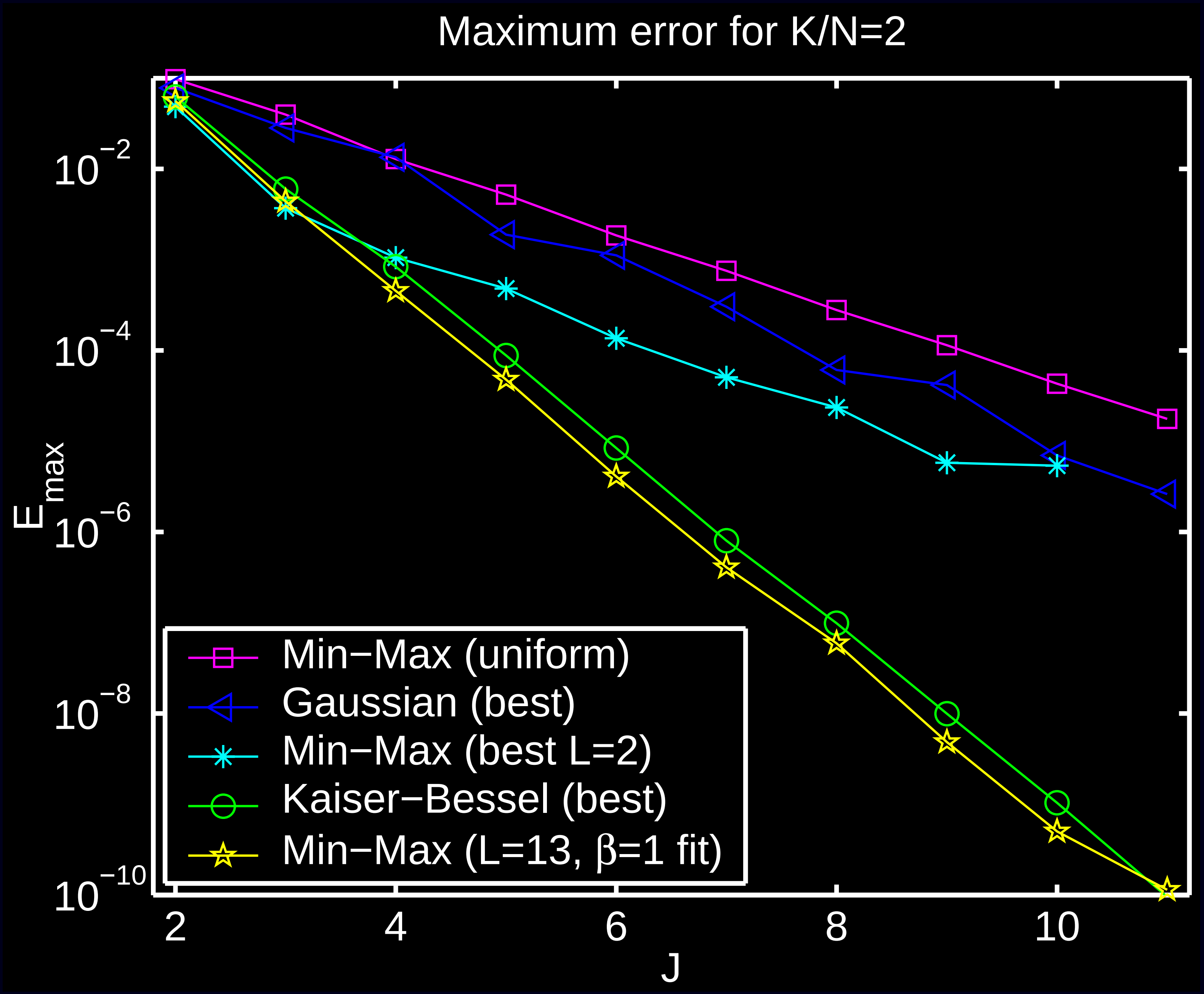
Kaiser-Bessel: Optimizing Scaling Factors



Kaiser-Bessel: Scaling Factors Tradeoff



Kaiser-Bessel vs Min-Max Interpolators



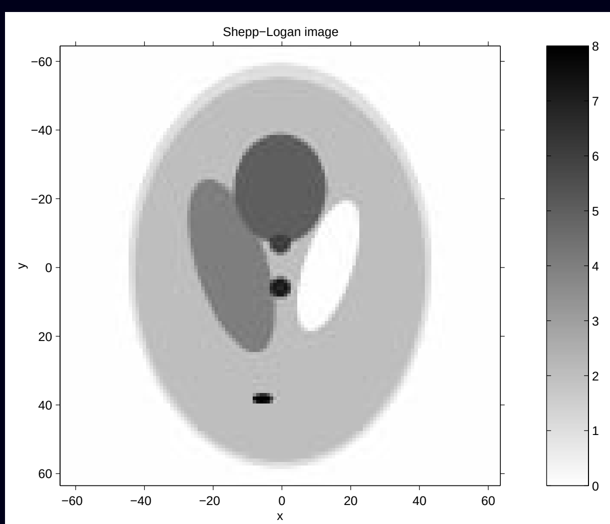
Fourier-Based Tomographic Projection

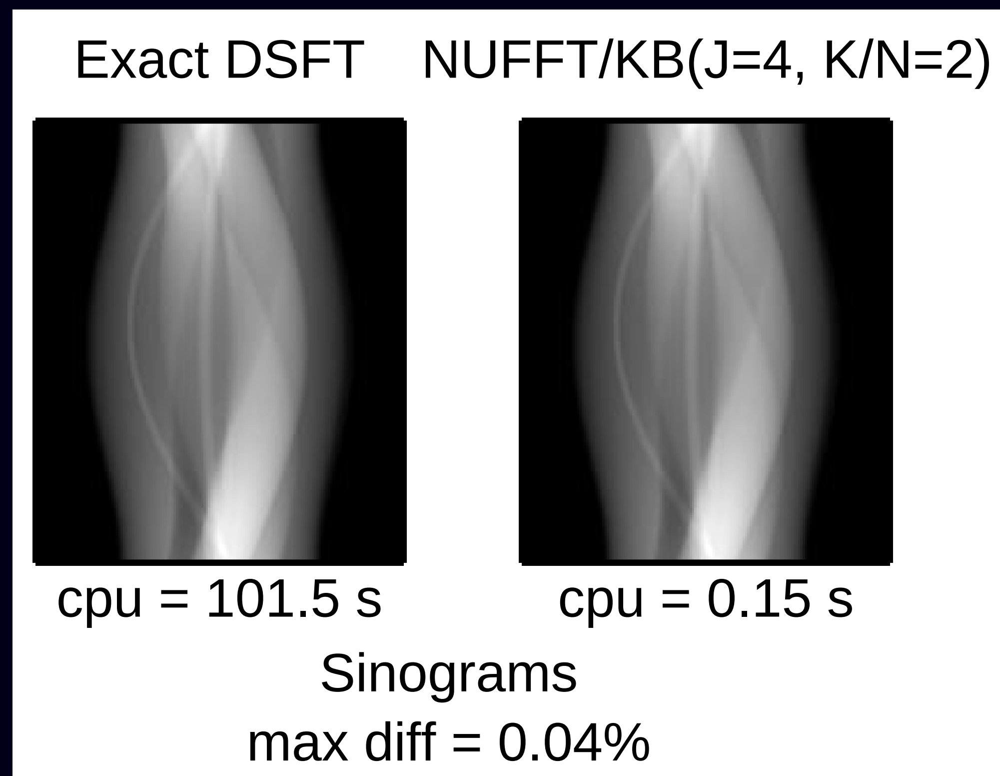
(Radon Transform)

1. Compute $2\times$ oversampled 2D FFT of object
2. Min-max interpolation onto polar coordinates (5×5 neighborhood)
3. Multiply spectrum by effects of
   - shift-invariant detector blur
   - and (square) pixel basis.
4. 1D inverse FFT for each sinogram row

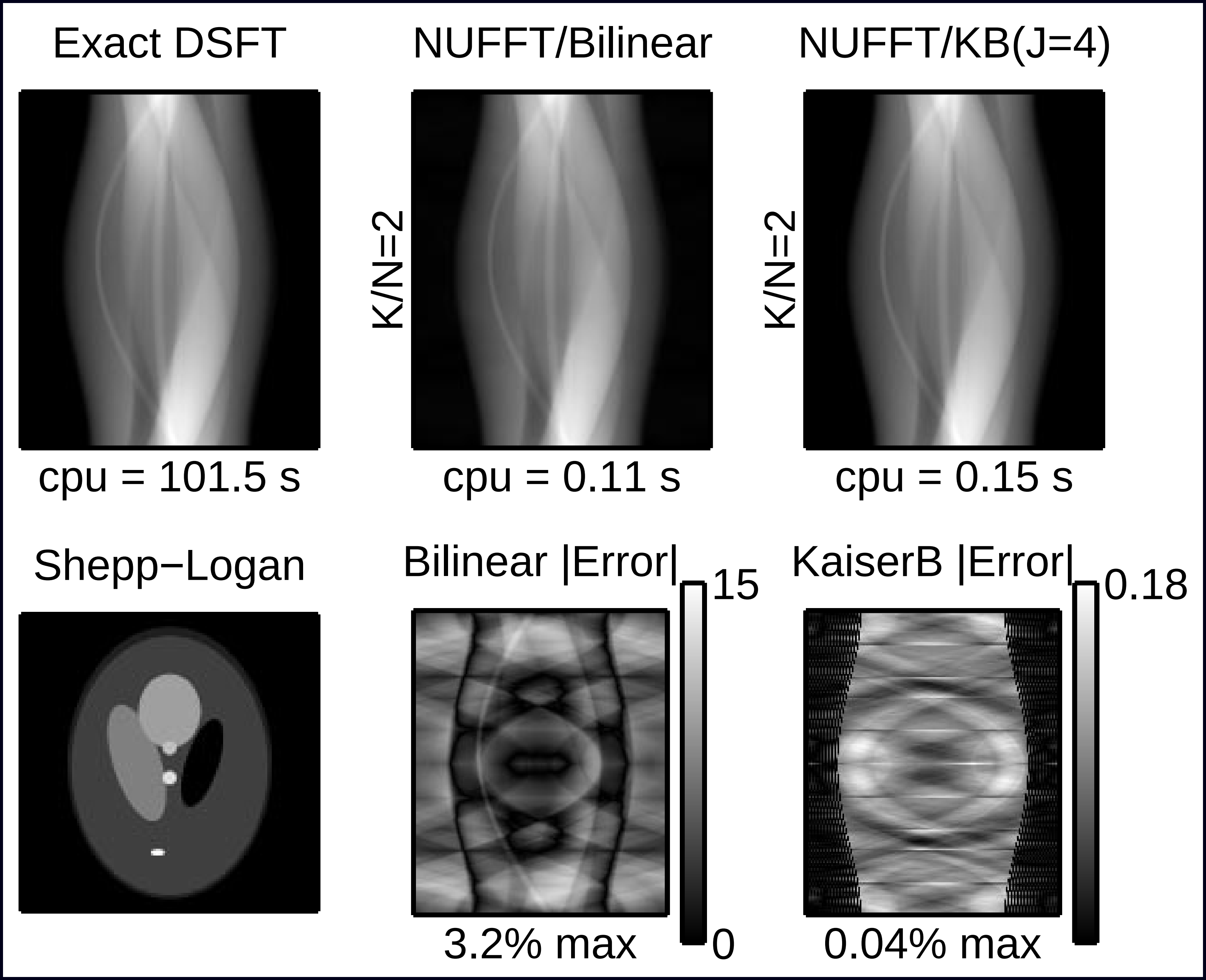
Forward Projector Simulation

- 128×128 Shepp-Logan digital phantom
- 160 bins $\times$ 192 angles sinogram
- 1-bin rectangular detector PSF
- Exact DSFT-based Fourier projector (no interpolation) vs NUFFT based on min-max interpolator
- 6.3s precompute time on 1GHz Pentium III / Linux

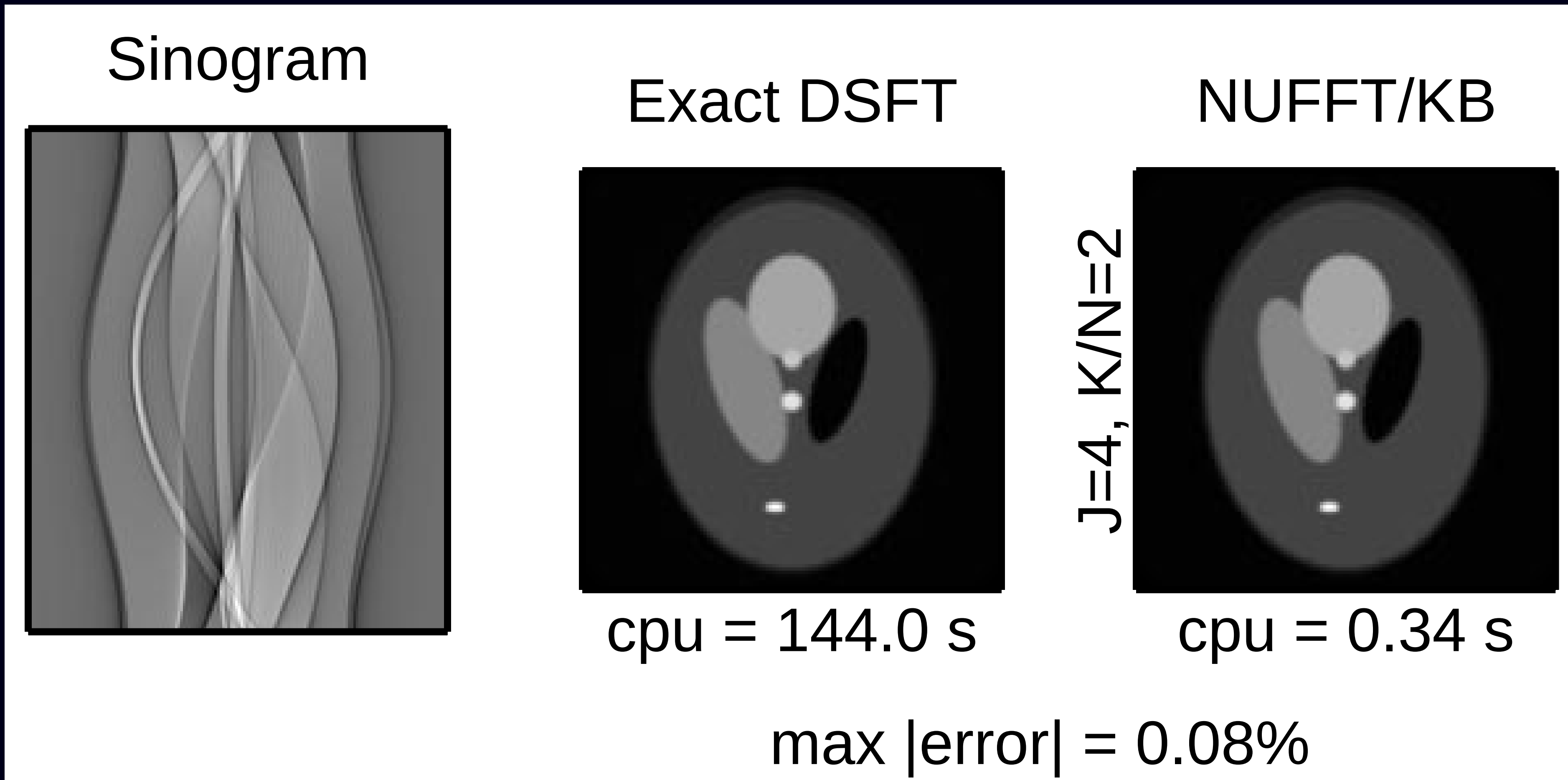




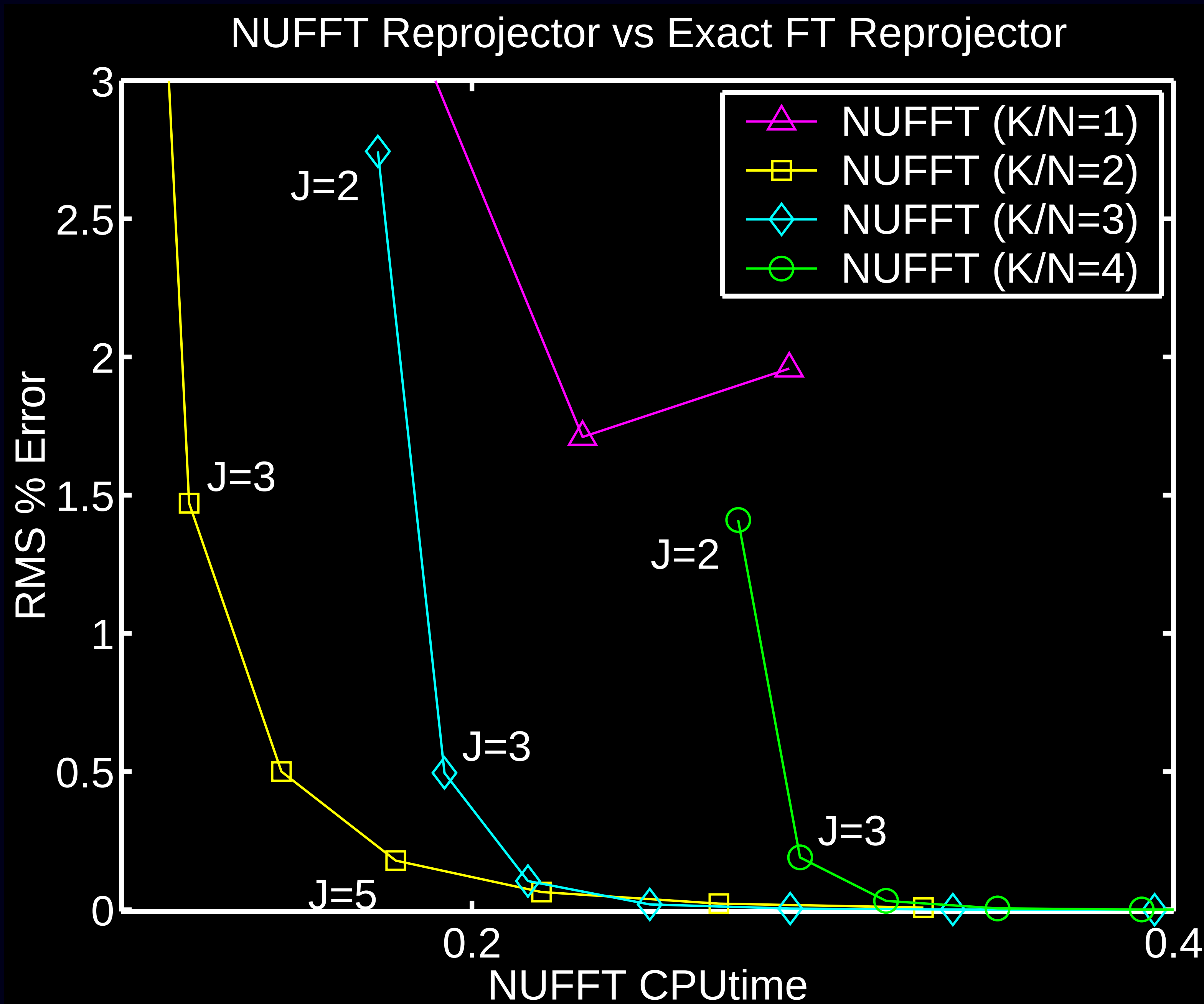
Bilinear Interpolation (“Gridding”) Comparison



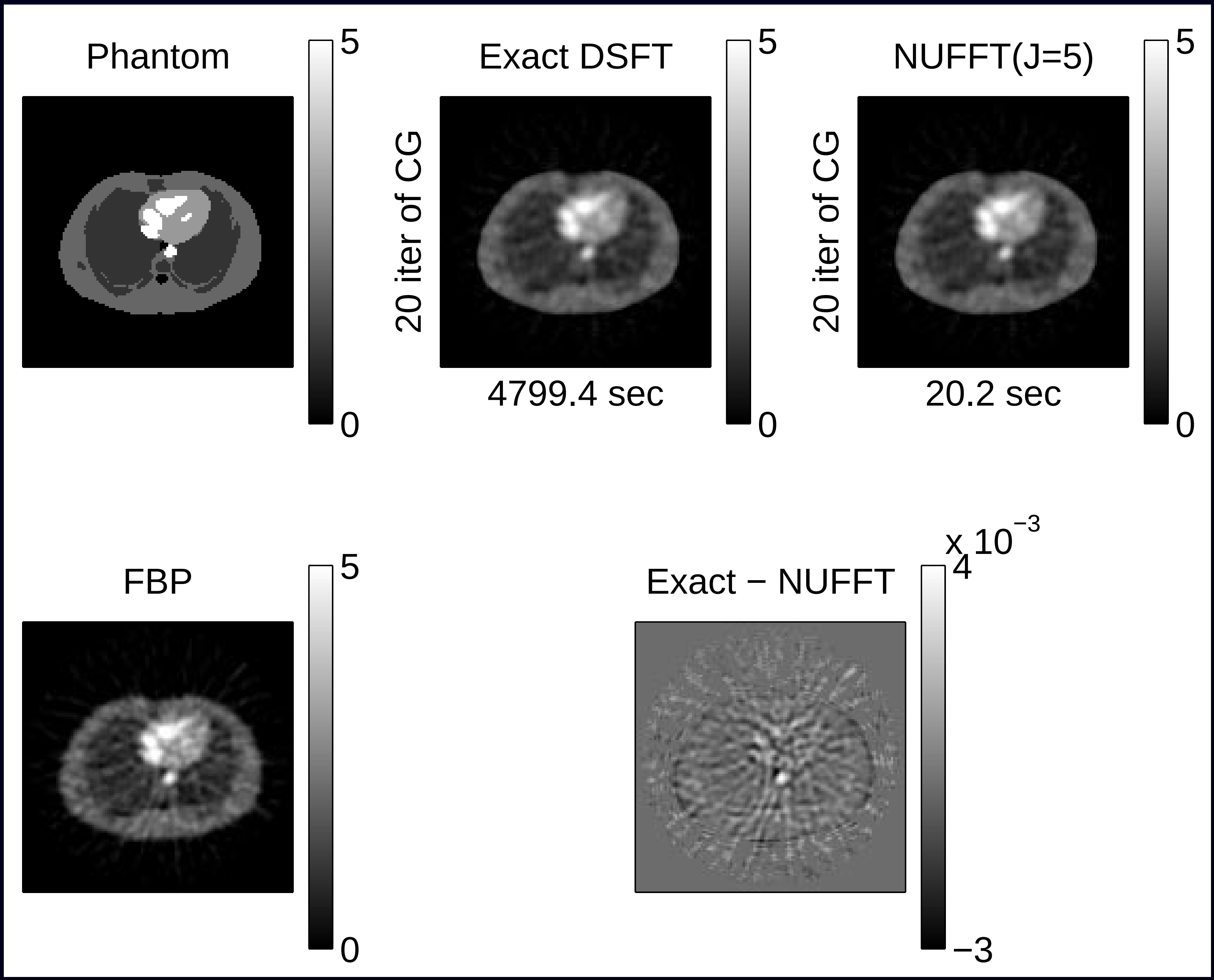
Back-projector (Adjoint) Test



NUFFT Projector Time/Accuracy Tradeoff



QPWLS Iterative Reconstruction



Summary

Min-max interpolation approach for NUFFT:
minimizes worst-case interpolation error.
Accurate and fast projector/backprojector for 2D tomography.

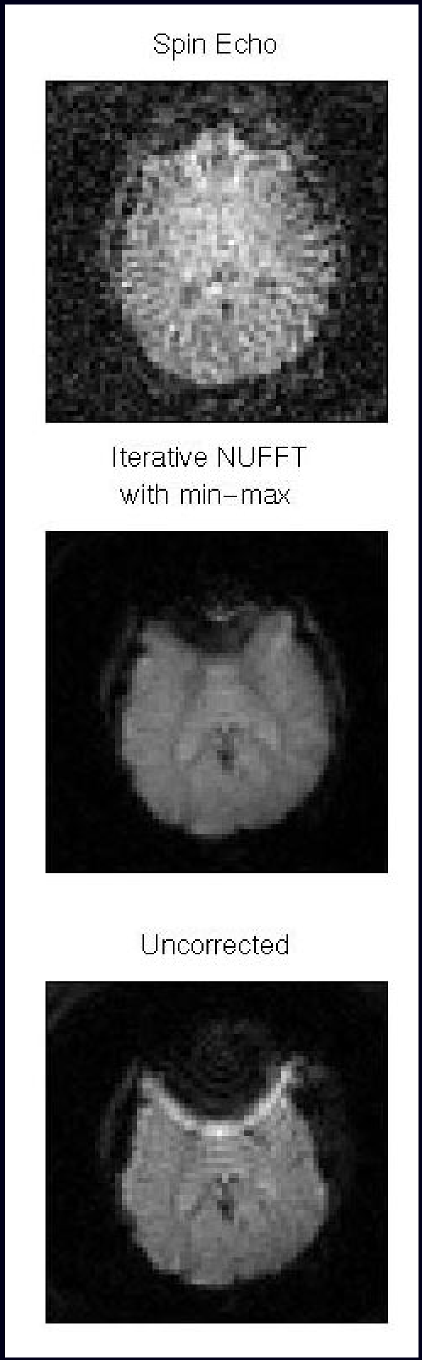
Future Applications

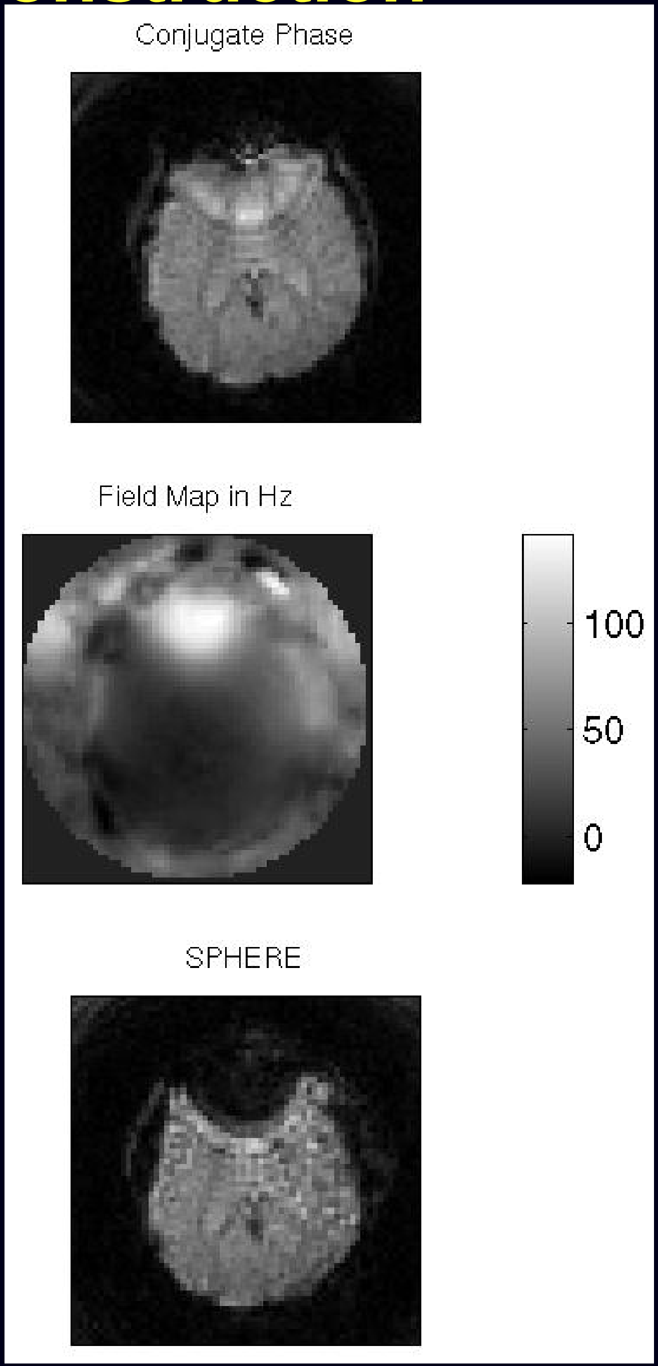
- MRI with field inhomogeneity
- MRI with multiple coils
- 3D PET

Limitations / Challenges

- Slightly negative a_{ij}'s (in tomography)
- Shift-invariant PSF
- Parallel-beam geometry
- Non-uniform radial sampling in ring PET geometry
- Numerical conditioning for large J
- Ordered-subsets

Iterative MRI Reconstruction





References

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[2] G. Beylkin. On the fast Fourier transform of functions with singularities. *Applied and Computational Harmonic Analysis*, 2(4):363–81, October 1995.

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[7] J. A. Fessler and B. P. Sutton. Nonuniform fast Fourier transforms using min-max interpolation. *IEEE Tr. Sig. Proc.*, 51(2), February 2003. To appear.