Nonuniform Fast Fourier Transforms and Applications in Imaging

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Outline

- Applications
  - 1. MRI
  - 2. Tomography
- Min-max framework for nonuniform FFT
- Examples
- Features / Limitations
- Future goals

MRI work with Brad Sutton, Doug Noll
Tomography work with Samuel Matej
MRI Application

MRI rosette k-space trajectory
Simplified MRI Signal Model

Ignoring *lots* of things:

\[ y_i = s(t_i) + \text{noise}_i, \quad i = 1, \ldots, N_{\text{samples}} \]

\[ s(t) = \int f(\vec{r}) \exp \left( -i2\pi \vec{k}(t) \cdot \vec{r} \right) \, d\vec{r}, \]

where \( \vec{k}(t) \) denotes the “k-space trajectory” of the MR pulse sequence.

- MRI measurements are (roughly) *samples of the Fourier transform* of the object’s transverse magnetization \( f(\vec{r}) \).
- Reconstruction problem: recover \( f(\vec{r}) \) from measurements \( \{y_i\} \)
Conventional MR Image Reconstruction

1. Interpolate measurements onto rectilinear grid ("gridding")
2. Apply inverse FFT to estimate samples of $f(\vec{r})$
Limitations of MR gridding-based reconstruction

1. Artifacts/inaccuracies due to interpolation
2. Contention about sample density “weighting”
3. Oversimplifications of Fourier transform signal model:
   - Magnetic field inhomogeneity
   - Magnetization decay ($T_2$)
   - Eddy currents
   - ...

Magnetic field inhomogeneity
MR Image Reconstruction as an Inverse Problem

1. Series expansion of unknown object:

\[ f(\vec{r}) \approx \sum_{j=1}^{n_p} x_j b(\vec{r} - \vec{r}_j) \]

2. Discrete-discrete measurement model:

\[ y = A x + \epsilon \]

\[ a_{ij} = \int b(\vec{r} - \vec{r}_j) \exp\left(-i2\pi \vec{k}(t_i) \cdot \vec{r}\right) \, d\vec{r} = B(\vec{k}(t_i)) \, e^{-i2\pi \vec{k}(t_i) \cdot \vec{r}_j} \]

3. \( \epsilon \) includes both measurement noise and model error

4. \( A \) can also include “non-Fourier” effects (inhomogeneity, decay, etc.)

5. Least-squares formulation (Gaussian noise model):

\[ \hat{x} = \arg \min_x \Psi(x), \quad \Psi(x) = \| y - Ax \|^2 \]

6. Regularization included when needed (depends on \( \vec{k}(t) \))

7. Preconditioned conjugate gradient gradient iteration for minimization.
Challenges for iterative MR image reconstruction

- Each PCG iteration requires calculation of $A'(y - Ax^{(n)})$
- $A$ is too large to store explicitly (not sparse)
- Even if $A$ were stored, directly computing $Ax$ is $O(n_p^2)$, per iteration, whereas FFT is only $O(n_p \log n_p)$

$\Rightarrow$ need fast algorithm for computing $Ax$, i.e., for computing

$$\sum_{n_1} \sum_{n_2} e^{-i2\pi (k_1(t_i)n_1 + k_2(t_i)n_2)} x(n_1, n_2),$$

assuming the $\tilde{r}_j$'s (basis centers) are unit spaced on a rectilinear grid.

Need: fast algorithm for 2D nonuniform Fourier transform
Simplified tomography measurement model (sinogram):

\[ y_i = (h \ast p_{\theta_i}(r; f))(r_i) + \text{noise}_i, \quad i = 1, \ldots, n_d, \quad n_d = n_r \cdot n_\theta. \]

Radon transform degraded by radially shift-invariant blur with PSF \( h(r) \).

Radon transform (line integrals):

\[ p_{\theta}(r; f) = \int f(r \cos \theta - l \sin \theta, r \sin \theta + l \cos \theta) \, dl \]

Goal: reconstruct object \( f(\vec{r}) \) from sinogram measurements \( \{y_i\} \)
Classical Fourier-transform reconstruction

Fourier-slice theorem:

\[ p_{\theta}(r; f) \xrightarrow{1\text{D FT}} P_{\theta}(\rho) = F(\rho, \theta) \xrightarrow{2\text{D FT}} f(x, y) \]

- Compute 1D FFT of each row of sinogram.
- Possibly deconvolve blur \( h(r) \)
- Interpolate from polar samples onto rectilinear frequency samples
- Compute inverse 2D FFT

Limitations

- Artifacts due to polar-cartesian interpolation
- Suboptimal treatment of nonuniform-variance noise, e.g., Poisson
- Over-simplified measurement model
- Disregards nonnegativity constraint

Proposed approach partially overcomes first two limitations
Iterative Tomographic Image Reconstruction

1. Series expansion of unknown object:

\[ f(\vec{r}) \approx \sum_{j=1}^{n_p} x_j b(\vec{r} - \vec{r}_j) \]

2. Discrete-discrete measurement model

\[ y = Ax + \varepsilon, \quad a_{ij} = h(r) * p_{\theta_i}(r; b(\cdot - \vec{r}_j)) \bigg|_{r=r_i} \]

3. Penalized weighted least-squares (PWLS) formulation

\[ \hat{x} = \arg\min_{x} \Psi(x), \quad \Psi(x) = (y - Ax)'W(y - Ax) + \beta R(x) \]

4. Weighting matrix \( W \) for nonuniform noise variance
   (cf Delaney and Bresler, IEEE T-IP, May 1996)

5. Regularization essential due to ill-conditioned nature of tomography

6. Preconditioned conjugate gradient iteration for minimization.
Challenges for Iterative Tomographic Reconstruction

- Each PCG iteration requires calculation of $A'W(y - Ax^{(n)})$
- $A$ is sparse, but very large for 3D PET, too large to store in 2D X-ray CT
- Even if $A$ were stored, directly computing $Ax$ is $O(n_p^2)$, per iteration, whereas FFT is only $O(n_p \log n_p)$

Proposed approach for reprojection (computing $Ax$)

1. Apply nonuniform FFT to compute 2D FT on a polar grid accurately
2. Apply shift-invariant blur $h(r)$ in frequency domain
3. Compute inverse 1D FFT to form each row of reprojection
   - Avoids line-integral calculations!
   - Routine for $A'$ is the exact adjoint
Prior work on NUFFT

  *Fast Fourier transforms for nonequispaced data.*
  Gaussian based interpolation
  *On the fast Fourier transform of functions with singularities.*
  B-spline based interpolation in multiresolution framework (N-D)
  *Fast Fourier transforms for nonequispaced data, II.*
  fast multipole method
  *Rapid computation of the discrete Fourier transform.*
  Taylor series expansion, requiring multiple FFTs
  *The regular Fourier matrices and nonuniform fast Fourier transforms.*
  least-squares approach to shift-variant Fourier interpolation
  *Nonuniform fast Fourier transforms using min-max interpolation.*
NUFFT Problem Statement (1D)

Given signal \( x_n, n = 0, \ldots, N - 1 \) with (discrete-time) Fourier transform

\[
X(\omega) = \sum_{n=0}^{N-1} x_n e^{-i\omega n}
\]

and a collection of arbitrary frequencies \( \{\omega_m : m = 1, \ldots, M\} \), compute

\[
y_m = X(\omega_m), \quad m = 1, \ldots, M.
\]

Direct approach is \( O(NM) \); impractical for large \( M \).
NUFFT via linear interpolation

1. Compute $K$-point FFT of $x_n$ (where $K \geq N$, possibly oversampled)

$$X_k = X \left( \frac{2\pi k}{K} \right), \quad k = 0, \ldots, K - 1$$

2. Interpolate from set $\{2\pi k/K\}$ to set $\{\omega_m\}$

$$\hat{y}_m = \sum_{k=0}^{K-1} v_{mk} X_k$$

Design question: how to choose interpolation coefficients $\{v_{mk}\}$?

Scaled variation

1. Start with “weighted” $K$-point FFT:

$$Y_k = \sum_{n=0}^{N-1} s_n x_n e^{-i \frac{2\pi k}{K} n}$$

2. Design problem includes choosing scaling factors $\{s_n\}$. (Important!)
Interpolators

1. Shift invariant:
   - Gaussian
   - B-spline
   - Rarely precomputed
   - Less memory
   - More in-line work

2. Shift variant
   - Constraint: use the $J$ nearest FFT samples for each $\omega_m$

$$\hat{y}_m = \sum_{j=1}^{J} u_{m,j}^* X_{k_0(\omega_m)+j}, \text{ where } k_0(\omega) \triangleq \begin{cases} \left( \arg\min_k \left| \omega - \frac{2\pi k}{2}\right| \right) - \frac{J+1}{2}, & J \text{ odd} \\ \left( \max \{ k : \omega \geq \frac{2\pi k}{2}\} \right) - \frac{J}{2}, & J \text{ even} \end{cases}$$

- $O(JM)$ memory if interpolation coefficients are precomputed
- $O(K \log K) + O(JM)$ computation
Min-Max Criterion

Choose interpolation coefficients \( \{u_{mj}\} \) to minimize worst-case error.

\[
\min_{u_m \in \mathbb{C}^J} \max_{x \in \mathbb{C}^N: \|x\| \leq 1} |\hat{y}_m - y_m|, \text{ where } u_m = (u_{m1}, \ldots, u_{mJ}).
\]

Solution (data independent!):

\[ u_m = \Lambda'(\omega) Tr(\omega_m), \text{ where:} \]

\[
\Lambda_{jj}(\omega) = e^{-i[\omega - \frac{2\pi}{K}(k_0(\omega) + j)]\frac{N-1}{2}}
\]

\[
T = [C' C]^{-1} \in \mathbb{R}^{J \times J}
\]

\[
[C' C]_{l,j} = \delta_0(j - l)
\]

\[
r_j(\omega) = \delta_0(\omega / (2\pi / K) - k_0(\omega) - j)
\]

\[
\delta_0(t) \triangleq \frac{\sin(\pi t N / K)}{N \sin(\pi t / K)}.
\]

“Modified truncated-Dirichlet interpolation of oversampled FFT”
Equivalent interpolator for $J=6$, $K/N=2$

- Min–max
- Sinc
Equivalent interpolator for $J=7$, $K/N=2$

- Min–max
- Sinc
**Accuracy**

Worst-case error for unit-norm signal is \( \frac{E_{\text{max}}(\omega)}{\sqrt{N}} = \sqrt{1 - r'(\omega) T r(\omega)}. \)

Maximum error for \( \alpha = 1 \):

- \( K/N = 1.5 \)
- \( K/N = 2 \)
- \( K/N = 2.5 \)
- \( K/N = 3 \)
- \( K/N = 4 \)
- \( K/N = 5 \)
Comparison with Dirichlet

Maximum error for $K/N=2$

- Truncated Dirichlet
- Tapered Dirichlet
- Linear ($J=2$)
- Min–Max (uniform)
- Min–Max (best $L=2$)
Comparison with Gaussian (Dutt/Rokhlin)

Maximum error for $K/N=2$

- Gaussian (best $\sigma$)
- Min–Max (uniform)
- Min–Max (L=5 LS fit)
- Min–Max (best L=2)
Extensions

- Multidimensional NUFFT
  Use $J \times J$ neighborhood (in 2D, e.g.) around each spatial frequency location of interest. Straightforward generalization.

- Adjoint operator
  1. Hermitian transpose of interpolation matrix
  2. $K$-point inverse FFT

- Adaptive neighborhoods
  Neighborhood size $J$ vs distance between $\omega_m$ and nearest neighbor.

- Free software: http://www.eecs.umich.edu/~fessler
Kaiser-Bessel Interpolator

\[ F(\kappa) = f_J^m(\kappa) \frac{I_m(\alpha f_J(\kappa))}{I_m(\alpha)}, \text{ where } f_J(\kappa) \triangleq \begin{cases} \sqrt{1 - \left(\frac{\kappa}{J/2}\right)^2}, & |\kappa| < J/2 \\ 0, & \text{otherwise.} \end{cases} \]

- Optimality properties?
- Usually \( m = 2 \) so continuous and differentiable on boundaries.
Kaiser-Bessel: Optimizing Order

Kaiser-Bessel Error for $K/N=2$ and $\alpha=2.34\cdot J$

$E_{\text{max}}$ vs $m$ (Kaiser-Bessel order)
Kaiser-Bessel: Optimizing Width

Kaiser–Bessel Error for K/N=2 and m=0

\[ E_{\text{max}} \]

\[ \frac{\alpha}{J} \text{ (Kaiser–Bessel width)} \]

\[ J=5 \]
\[ J=6 \]
\[ J=7 \]
Kaiser-Bessel: Optimizing Scaling Factors

Kaiser–Bessel Error for K/N=2, $\alpha=2.34 \cdot J$, and $m=0$

Numerical FT scaling factors
Analytical FT scaling factors
Kaiser-Bessel: Scaling Factors Tradeoff

Kaiser–Bessel Error for $K/N=2$, $\alpha=2.34 \cdot J$, and $m=0$

- Numerical FT scaling factors
- Analytical FT scaling factors

$\omega / (2\pi/K)$

$E_{\text{max}}$
Kaiser-Bessel vs Min-Max Interpolators

Maximum error for $K/N=2$

- Min–Max (uniform)
- Gaussian (best)
- Min–Max (best $L=2$)
- Kaiser–Bessel (best)
- Min–Max ($L=13$, $\beta=1$ fit)
Fourier-Based Tomographic Projection
(Radon Transform)

1. Compute $2 \times$ oversampled 2D FFT of object
2. Min-max interpolation onto polar coordinates ($5 \times 5$ neighborhood)
3. Multiply spectrum by effects of
   - shift-invariant detector blur
   - and (square) pixel basis.
4. 1D inverse FFT for each sinogram row
Forward Projector Simulation

- $128 \times 128$ Shepp-Logan digital phantom
- $160 \text{ bins} \times 192 \text{ angles sinogram}$
- 1-bin rectangular detector PSF
- Exact DSFT-based Fourier projector (no interpolation) vs NUFFT based on min-max interpolator
- 6.3s precompute time on 1GHz Pentium III / Linux

<table>
<thead>
<tr>
<th>Exact DSFT</th>
<th>NUFFT/KB($J=4, K/N=2$)</th>
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</thead>
<tbody>
<tr>
<td>cpu = 101.5 s</td>
<td>cpu = 0.15 s</td>
</tr>
<tr>
<td>max diff = 0.04%</td>
<td></td>
</tr>
</tbody>
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Shepp-Logan image
Bilinear Interpolation ("Gridding") Comparison

- **Exact DSFT**
  - Shepp–Logan
  - CPU: 101.5 s

- **NUFFT/Bilinear**
  - K/N=2
  - CPU: 0.11 s
  - Bilinear |Error|: 3.2% max

- **NUFFT/KB(J=4)**
  - K/N=2
  - CPU: 0.15 s
  - KaiserB |Error|: 0.04% max


Back-projector (Adjoint) Test

Sinogram

Exact DSFT

NUFFT/KB

cpu = 144.0 s

cpu = 0.34 s

max |error| = 0.08%

J=4, K/N=2
QPWLS Iterative Reconstruction

Phantom

Exact DSFT

NUFFT(J=5)

20 iter of CG

20 iter of CG

4799.4 sec

20.2 sec

FBP

Exact – NUFFT

4 \times 10^{-3}

-3

0

5
Summary


Future Applications
- MRI with field inhomogeneity
- MRI with multiple coils
- 3D PET

Limitations / Challenges
- Slightly negative $a_{ij}$’s (in tomography)
- Shift-invariant PSF
- Parallel-beam geometry
- Non-uniform radial sampling in ring PET geometry
- Numerical conditioning for large $J$
- Ordered-subsets
Iterative MRI Reconstruction

Spin Echo

Iterative NUFFT with min-max

Uncorrected

Conjugate Phase

Field Map in Hz

SPHERE
References


