Mean and Variance of Photon Counting with Deadtime

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Abstract

The statistics of photon counting by systems affected by deadtime are potentially important for statistical image reconstruction methods. We present a new way of analyzing the moments of the counting process for a counter system affected by various models of deadtime related to PET and SPECT imaging. We derive simple and exact expressions for the first and second moments of the number of recorded events under various models. From our mean expression for a SPECT deadtime model, we derive a simple estimator for the actual intensity of the underlying Poisson process; simulations show that our estimator is unbiased even for extremely high count rates.

I. INTRODUCTION

Every photon counting system exhibits a characteristic called *deadtime*. Since the pulses produced by a detector have finite time duration, if a second pulse occurs before the first has disappeared, the two pulses will overlap to form a single distorted pulse [1]. Depending on the system, one or both arrivals will be lost. In PET or SPECT scanners, the length of pulse resolving time, often just called "deadtime", denoted τ, is around 2μs. Counting systems are usually classified into two categories: nonparalyzable (type I) or paralyzable (type II). In a nonparalyzable system, each recorded photon produces a deadtime of length τ; if an arrival is recorded at t, then any arrival from t to $t+\tau$ will not be recorded. In a paralyzable system, each photon arrival, whether recorded or not, produces a deadtime of length τ; if there is an arrival at t, then any arrival from t to $t+\tau$ will not be recorded. In some SPECT systems [2], we encounter a third model that is similar to the paralyzable model: if two photons arrive within τ of each other, then neither photon will be recorded (*e.g.*, due to pulse pile-up); we call this the type III model. The asymptotic moments of the nonparalyzable model are well known [3]. For the paralyzable model, the exact expression for the mean of the number of recorded events from time 0 to t, denoted $Y(t)$, has been derived previously [4]. However, for the type III model, only an approximate expression for the mean number of recorded events has been derived [2]. In this paper, we derive the exact mean and variance expressions of $Y(t)$ for both type II and type III models.

II. STATISTICAL ANALYSIS OF DEADTIME

We define a "photon arrival" to mean a photon interacting with the scintillator with sufficient deposited energy to trigger detection. The photon arrival process $N(t)$ counts the number of arrivals during the time interval $(0,t]$, and the photon recording process $Y(t)$ counts the number of recorded events. We assume that $N(t)$ is a homogeneous Poisson process with rate λ (photon arrivals per unit time) which stays constant with time. We also assume, for the sake of simplicity, that τ is known and deterministic.

A. Asymptotic Analysis via Renewal Theory

The counting processes in all three types of systems discussed above are examples of "renewal processes" [3], and renewal theory has been the classical basis for deadtime analysis [5]. A renewal process involves recurrent patterns connected with repeated trials. Roughly speaking, if after each occurrence of a pattern $\mathcal{E}$, the random process starts from scratch in the sense that the trials following an occurrence of $\mathcal{E}$ form a replica of the whole process, then the process qualifies as a renewal process. If we define $\mathcal{E}$ to be the state[1] of "the counter is ready to record the next photon arrival", then after each occurrence of $\mathcal{E}$, the counting process is statistically equivalent. A very useful random variable to define is $T_{\mathcal{E}}$, the waiting time between one renewal and the next (renewal here means return to $\mathcal{E}$). Note that in the context of photon counting system, with $\mathcal{E}$ defined as above, the number of renewals from 0 to t is almost exactly the number of recorded events from 0 to t. If $T_{\mathcal{E}}$ has ensemble mean $\mu_{\mathcal{E}}$ and variance $\sigma_{\mathcal{E}}^2$, then the number of renewals from 0 to t, $\tilde{Y}(t)$, is asymptotically Gaussian distributed [6] [3] with the following moments:

$$E[\tilde{Y}(t)] \sim t/\mu_{\mathcal{E}}, \mathrm{Var}[\tilde{Y}(t)] \sim t\sigma_{\mathcal{E}}^2/\mu_{\mathcal{E}}^3, \qquad (1)$$

where $\sim$ indicates that the ratio of the two sides tends to unity as $t/\mu_{\mathcal{E}} \to \infty$. Hence asymptotically, the mean and

[1] For type III deadtime, we define renewal as "return to $\mathcal{E}$ after recording an event".

variance of the waiting time between renewals forms a sort of "duality" relationship with the mean and variance of the number of renewals.

For the other two deadtime models, if we try to derive $E[Y(t)]$ from $E[T_{\mathcal{E}}]$, it is much more difficult to obtain a simple closed form expression because the $E[T_{\mathcal{E}}]$ we get is probably an infinite sum and it is often not easy to obtain every term in this sum; the variance of $T_{\mathcal{E}}$ is even more complicated. Therefore, in the following section, we describe a new approach for deriving the moments of counting processes.

B. Exact Mean and Variance of Counting Processes

We first consider a general counting process Y where $Y(t_1, t_2)$ denotes the number of recorded events during the time interval $(t_1, t_2]$ and $Y(t)$ is a shorthand for $Y(0, t)$. We define the instantaneous rate $\gamma : \mathbb{R} \to [0, \infty)$ of the process $Y(t)$ as:

$$\gamma(s) \triangleq \lim_{\delta \to 0} E[Y(s+\delta) - Y(s)]/\delta, \tag{2}$$

and the instantaneous second moment $\alpha : \mathbb{R} \to [0, \infty)$ as:

$$\alpha(s) \triangleq \lim_{\delta \to 0} E[(Y(s+\delta) - Y(s))^2]/\delta. \tag{3}$$

We also define the correlation function $\beta : \mathbb{R}^2 \to [0, \infty)$ as:

$$\begin{aligned} \beta(s_1, s_2) \quad \triangleq \quad & \lim_{\delta_1, \delta_2 \to 0} E[(Y(s_1+\delta_1) - Y(s_1)) \cdot \\ & (Y(s_2+\delta_2) - Y(s_2))]/(\delta_1\delta_2). \end{aligned} \tag{4}$$

We assume[2]

(i) γ and α are well-defined μ-almost everywhere, and β is well defined μ_2-almost everywhere, and γ and β are integrable with respect to μ and μ_2 over any finite interval and rectangle, respectively;

(ii) $E[Y(s, s+\delta)]/\delta$ and $E[Y^2(s, s+\delta)]/\delta$ are uniformly bounded for all s and $\delta \in (0, 1)$;

(iii) $E[Y(s_1, s_1+\delta_1)Y(s_2, s_2+\delta_2)]/(\delta_1\delta_2)$ is uniformly bounded for all s_1, s_2, and $\delta_1, \delta_2 \in (0, 1)$ such that $(s_1, s_1+\delta_1) \cap (s_2, s_2+\delta_2) = \emptyset$.

These Assumptions hold for a wide variety of counting processes, including any homogeneous Poisson process with finite intensity.

[2] μ and μ_2 denote Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

For analysis purposes, we artificially divide the time interval $[0, t]$ into n segments of length δ each, *i.e.*, $t = n\delta$. We have

$$Y(t) = \sum_{i=0}^{n-1} Y(i\delta, (i+1)\delta), \tag{5}$$

$$E[Y(t)] = \sum_{i=0}^{n-1} E[Y(i\delta, (i+1)\delta)], \tag{6}$$

$$= \int_{\mathbb{R}} f_\delta(s)ds, \tag{7}$$

where we define the following piecewise constant function:

$$f_\delta(s) \triangleq \begin{cases} E[Y(j\delta, (j+1)\delta)]/\delta, & \text{if } s \in (j\delta, (j+1)\delta], \\ & 0 \le j \le n-1 \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Since $\gamma(t)$ is well-defined almost everywhere in the interval $[0, t]$ and $E[Y(s, s+\delta)]/\delta$ is uniformly bounded, by the Lebesgue Dominated Convergence theorem (LDCT) [7],

$$\begin{aligned} \lim_{\delta \to 0} \int_{\mathbb{R}} f_\delta(s) d\mu(s) &= \int_{\mathbb{R}} \lim_{\delta \to 0} f_\delta(s) d\mu(s) \\ &= \int_0^t \gamma(s) ds. \end{aligned} \tag{9}$$

Hence, we have the following simple general expression for the mean of the counting process in terms of its instantaneous rate:

$$E[Y(t)] = \int_0^t \gamma(s) ds. \tag{10}$$

We consider the second moment by a similar argument:

$$\begin{aligned} E[Y^2(t)] &= E[(\sum_{i=0}^{n-1} Y(i\delta, (i+1)\delta))^2] \\ &= \sum_{i=0}^{n-1} E[Y^2(i\delta, (i+1)\delta))] + 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} \\ & \quad E[Y(i\delta, (i+1)\delta)Y(j\delta, (j+1)\delta)] \qquad (11) \\ &= \int_{\mathbb{R}} g_\delta(s) d\mu(s) \\ & \quad + 2\int_{\mathbb{R}^2} h_\delta(s_1, s_2) d\mu_2(s_1, s_2), \qquad (12) \end{aligned}$$

where we define the following piecewise constant functions:

$$g_\delta(s) \triangleq \begin{cases} E[Y^2(j\delta, (j+1)\delta)]/\delta, & \text{if } s \in (j\delta, (j+1)\delta] \\ & \text{and } 0 \le j \le n-1 \\ 0, & \text{otherwise,} \end{cases} \tag{13}$$

and

$$h_\delta(s_1,s_2) \triangleq \begin{cases} E[Y(i\delta,(i+1)\delta)\cdot & \text{if } s_1 \in (i\delta,(i+1)\delta], \\ Y(j\delta, & s_2 \in (j\delta,(j+1)\delta], \\ (j+1)\delta)]/\delta^2, & 0 \le i \le n-2, \\ & \text{and } i+1 \le j \le n-1 \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

Since β is well-defined almost everywhere in $[0,t]\times[0,t]$ and $E[Y(s_1,s_1+\delta)Y(s_2,s_2+\delta)]/\delta^2$ is uniformly bounded, by LDCT and Fubini's Theorem [7],

$$\begin{aligned} &\lim_{\delta\to 0}\int_{\mathbb{R}^2} h_\delta(s_1,s_2)d\mu_2(s_1,s_2) \\ &= \int_{\mathbb{R}^2}\lim_{\delta\to 0} h_\delta(s_1,s_2)d\mu_2(s_1,s_2) \\ &= \int_0^t\int_{s_2}^t \beta(s_1,s_2)ds_1ds_2. \end{aligned} \quad (15)$$

Similarly, one can show that

$$\lim_{\delta\to 0}\int_{\mathbb{R}} g_\delta(s)d\mu(s) = \int_0^t \alpha(s)ds. \quad (16)$$

Thus using (12), (15), and (16), we have the following general expression for the second moment of $Y(t)$:

$$E[Y^2(t)] = \int_0^t \alpha(s)ds + 2\int_0^t\int_{s_1}^t \beta(s_1,s_2)ds_2ds_1. \quad (17)$$

In the context of counting processes with deadtime, which includes all random processes considered in this paper, the process satisfies this *additional assumption*:

(iv) there exists a positive δ_0 such that $\forall\delta\in(0,\delta_0), Y(s,s+\delta)\le 1$.

This assumption greatly simplifies the derivations for the moments of counting processes affected by deadtime, since for $\delta<\delta_0<\tau$,

$$E[Y^2(s,s+\delta)] = E[Y(s,s+\delta)] \quad (18)$$

using $0^2=0$ and $1^2=1$, so

$$\alpha(s) = \gamma(s). \quad (19)$$

Thus we obtain the following corollary of (17) for random processes satisfying assumptions (i) to (iv):

$$E[Y^2(t)] = E[Y(t)] + 2\int_0^t\int_{s_1}^t \beta(s_1,s_2)ds_2ds_1. \quad (20)$$

Furthermore, if $Y(t)$ has stationary increments, then $\gamma(s)$ is constant and $\beta(s_1,s_2)=\beta(0,s_2-s_1)$ and we can further simplify the results (10) and (20) to the following:

$$\begin{aligned} E[Y(t)] &= \gamma t & (21) \\ E[Y^2(t)] &= \gamma t + 2\int_0^t (t-s)\beta(0,s)ds. & (22) \end{aligned}$$

III. Single Photon Counting

A. Mean and Variance of Recorded Singles Counts, Model Type II

First we consider the paralyzable model in which if the waiting time for a photon arrival is less than τ, then this photon is not recorded. We derive the mean and variance of $Y(t)$, the number of recorded events from time 0 to time t. We observe that $Y(t)$ inherits the stationary increment property of the arrival process $N(t)$. We first derive $E[Y(0,\delta)]$, where we pick $\delta<\tau$ such that the number of recorded events during $(0,\delta]$ is either 0 or 1. Let T_1 denote the time of the first photon arrival after time 0; it is exponentially distributed. If there is an arrival at $T_1=s$, $0<s<\delta$, and there is no arrival between $s-\tau$ and s (in fact, we only need to make sure there is no arrival between $s-\tau$ and 0, *i.e.*, $N(0)-N(s-\tau)=0$, since the first arrival after 0 occurs at s), then there will be a recorded event during the interval $(0,\delta]$. Thus

$$\begin{aligned} E[Y(0,\delta)] &= \int_0^\infty \mathrm{P}[Y(0,\delta)=1|T_1=s]f_{T_1}(s)ds \\ &= \int_0^\delta \mathrm{P}[N(s-\tau,0)=0|T_1=s]f_{T_1}(s)ds \\ &= \int_0^\delta e^{-\lambda(\tau-s)}\lambda e^{-\lambda s}ds \\ &= \lambda\delta e^{-\lambda\tau}. \end{aligned} \quad (23)$$

Hence by the definition given in (2), the instantaneous rate of $Y(t)$ is

$$\gamma = \lambda e^{-\lambda\tau}, \quad (24)$$

and by (21), we easily obtain the following result (*e.g.*, [1]),

$$E[Y(t)] = \lambda t e^{-\lambda\tau}. \quad (25)$$

The variance of $Y(t)$ for the type II model is (see Appendix A):

$$\mathrm{Var}[Y(t)] = \lambda t e^{-\lambda\tau}(1-(2\lambda\tau-\lambda\tau^2/t)e^{-\lambda\tau}). \quad (26)$$

Figure 1 shows the mean and variance of the singles count for a detector affected by deadtime of type II. Since the mean and variance can differ greatly, $Y(t)$ is not Poisson.

B. Mean and Variance of Recorded Singles Counts, Model Type III

Now we turn to the type of system described in [2], in which if the waiting time for a photon arrival is less than τ, then neither this photon nor the previous photon

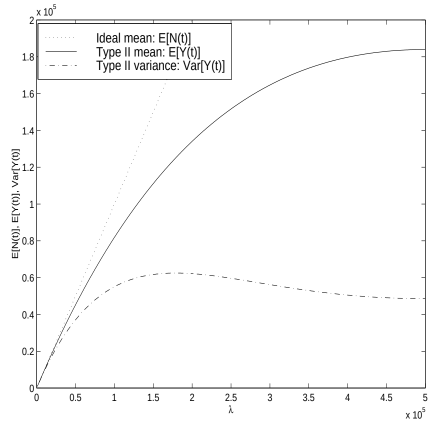

Fig. 1 Mean and variance for paralyzable (type II) systems, with $t = 1\mathrm{s}, \tau = 2\mu\mathrm{s}$.

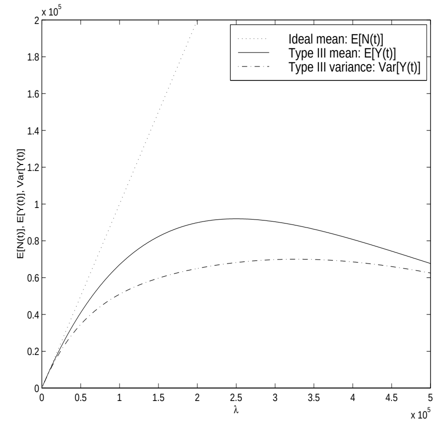

Fig. 2 Mean and variance for type III systems, with $t = 1\mathrm{s}, \tau = 2\mu\mathrm{s}$.

will be recorded. We again observe that $Y(t)$ inherits the stationary increment property of the arrival process $N(t)$. We first derive $E[Y(0, \delta)]$, where we pick $\delta < \tau$ such that the number of recorded events during $(0, \delta]$ is still either 0 or 1. Hence,

$$\begin{aligned} & E[Y(0,\delta)] \\ &= \int_0^\delta \mathrm{P}[Y(0,\delta)=1|T_1=s] f_{T_1}(s)ds \\ &= \int_0^\delta \mathrm{P}[N(s-\tau,0)=0]\mathrm{P}[(s,s+\tau)=0] f_{T_1}(s)ds \\ &= \int_0^\delta e^{-\lambda(\tau-s)} e^{-\lambda\tau} \lambda e^{-\lambda s} ds \\ &= \lambda\delta e^{-\lambda 2\tau}. \end{aligned} \tag{27}$$

Hence for this system, the instantaneous rate as defined in (2) is

$$\gamma = \lambda e^{-\lambda 2\tau}, \tag{28}$$

and by (21), the expected number of recorded events for a type III system is exactly:

$$E[Y(t)] = \lambda t e^{-\lambda 2\tau}. \tag{29}$$

The variance of $Y(t)$ for the type III model is (we omit the derivation due to space constraints):

$$\begin{aligned} \mathrm{Var}[Y(t)] &= \lambda t e^{-\lambda 2\tau} + 2e^{-3\lambda\tau}(\lambda t - \lambda\tau - 1) \\ &+ e^{-4\lambda\tau}(4\lambda^2\tau^2 - 4\lambda^2 t\tau + 2 - 2\lambda t + 4\lambda\tau). \end{aligned} \tag{30}$$

Figure 2 shows the (exact) mean and variance of the singles count $Y(t)$ for type III systems. Again $Y(t)$ is not Poisson.

IV. Count Rate Correction for System Type III

For a quantitatively accurate reconstruction, we must correct for the effect of deadtime to avoid underestimation of source activity. For type III systems, Engeland *et al* [2] proposed the following correction formula,

$$\hat{\lambda} = \frac{Y}{t}\left(1 + \frac{2Y}{t}\tau + \frac{6Y}{t^2}\tau^2\right), \tag{31}$$

which they obtained by solving an approximate mean waiting time expression up to second order in τ by means of the expansion $\lambda = a + b\tau + c\tau^2$. We propose to estimate the true count rate by solving numerically our exact expression (29), *i.e.*, solve

$$\frac{Y}{t} = \hat{\lambda} e^{-2\hat{\lambda}\tau} \tag{32}$$

for $\hat{\lambda}$ given Y and t. One could solve analytically the exact mean waiting time expression up to second order in τ, which yields exactly the same estimator as (31), but this estimator does not the mean waiting time expression exactly. Figure 3 compares our new estimator (32) and the estimator proposed in [2]. It shows that our new estimator is unbiased even at very high count rates. The error bars are not shown in the figure as they are smaller than the plotting symbols. When t is large, the standard deviation is very small when compared to the mean of $Y(t)$, thus these estimates have extremely small standard deviations. By solving (32) numerically, we obtain essentially perfect deadtime correction for a type III system.

V. Discussion

We have analyzed the mean and variance of the recorded singles counts for two distinct models of deadtime. In both cases, the variance can be significantly

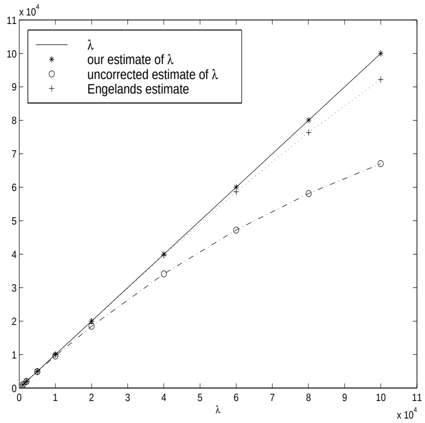

Fig. 3 20 realizations, with $t = 10\text{s}$, $\tau = 2\mu\text{s}$.

less than the mean, indicating that the counting statistics are not Poisson in the presence of deadtime. Deadtime losses can be significant in practical SPECT and PET systems, particularly in fully 3D PET imaging and in SPECT transmission measurements with a scanning line source. The count rates for a detector block (PET) or detector zone (SPECT) can be significant enough to yield non-Poisson statistics for the total counts recorded by the block or zone. However[3], in the practical situations that we are aware of, the count rates for individual detector elements within the block or zone are usually not high enough to correspond to significant deadtime losses. Even though the variance of the counts recorded by a block can be significantly lower than the mean, the variance of the counts recorded by an *individual detector element* is nevertheless quite close to the mean and likely to be well approximated by a Poisson distribution. Furthermore, the correlation between individual detectors will be fairly small. Thus it appears that statistical image reconstruction based on Poisson models, while certainly not optimal, should be adequate in practice even under fairly large deadtime losses, provided the deadtime loss factor is included in the system matrix. We must add one caveat to this conclusion however. Although pairs of individual detectors have small correlation, the correlation coefficient between the *sum* of one group of detectors and the *sum* of all other detectors in a block may not be small in the presence of deadtime. The effect of such correlations on image reconstruction algorithms is unknown and may deserve further investigation. Another natural extension of this work would be to consider systems with random resolving times τ. As long as the minimum resolving time is greater than zero, assumption (iv) would still hold and the derivations would be similar.

[3]Due to space constraints, we omit detailed analysis and only present our conclusions.

VI. Appendix A

We derive the variance of $Y(t)$ for deadtime model II, the paralyzable model. We first derive $\beta(0, s)$. We consider two cases.

<u>Case 1: $0 < s < \tau$</u>

We pick δ such that $0 < \delta < s < s + \delta < \tau$. Two recorded events cannot correspond to photons that arrived within τ of each other. Hence for $0 < s < \tau$, $E[Y(0, \delta)Y(s, s + \delta)] = 0$, and by the definition given in (2): $\beta(0, s) = 0$.

<u>Case 2: $\tau < s < t$</u>

We pick δ such that $\delta < \tau$ and $s+\delta < t$ and $\delta < s-\tau$. For $s > \tau$, $Y(0, \delta)$ and $Y(s, s + \delta)$ are statistically independent, since the event "there is an arrival during $(0, \delta]$" is statistically independent from the event "there is an arrival during $(s, s + \delta]$", because they are at least τ apart in time. Hence by (23),

$$E[Y(0, \delta)Y(s, s + \delta)] = E^2[Y(0, \delta)] = (\lambda\delta e^{-\lambda\tau})^2, \quad (33)$$

and

$$\beta(0, s) = (\lambda e^{-\lambda\tau})^2. \quad (34)$$

Combining the above two cases and using (22) yields

$$\begin{aligned} E[Y^2(t)] &= \gamma t + 2\int_\tau^t (t - s)(\lambda e^{-\lambda\tau})^2 ds \\ &= \lambda t e^{-\lambda\tau} + [(t - \tau)(\lambda e^{-\lambda\tau})]^2. \quad (35) \end{aligned}$$

Using $\text{Var}[Y(t)] = E[Y^2(t)] - E^2[Y(t)]$, with (25) and (35), and simplifying yields (26).

VII. References

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