Sufficient Conditions for Norm Convergence of the EM Algorithm[1]

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Abstract — **In this paper we provide sufficient conditions for convergence of a general class of alternating estimation-maximization (EM) type continuous-parameter estimation algorithms with respect to a given norm.**

I. Introduction

Let $\theta = [\theta_1, \ldots, \theta_p]^T$ be a real parameter residing in an open subset Θ of the p-dimensional space $I\!R^p$. Given a general function $Q : \Theta \times \Theta \to I\!R$ and an initial point $\theta^0 \in \Theta$, consider the following recursive algorithm, called the A-algorithm:

$$\textbf{A-algorithm:} \qquad \theta^{i+1} = \mathrm{argmax}_{\theta \in \Theta} Q(\theta, \theta^i). \tag{1}$$

If there are multiple maxima, then θ^{i+1} can be taken to be any one of them. Let $\theta^* \in \Theta$ be a fixed point of (1).

The A-algorithm contains a large number of popular iterative estimation algorithms such as: ML-EM algorithms (e.g. Dempster, Laird, and Rubin (1977), the penalized EM algorithm (e.g. Hebert and Leahy (1989)) , and EM-type algorithms implemented with E-step or M-step approximations (e.g., Antoniadis and Hero (1994), Green (1990)).

II. Convergence Theorem

A *region of monotone convergence* relative to the vector norm $\|\cdot\|$ of the A-algorithm (1) is defined as any open ball $B(\theta^*, \delta) = \{\theta : \|\theta - \theta^*\| < \delta\}$ centered at $\theta = \theta^*$ with radius $\delta > 0$ such that if the initial point θ^0 is in this region then $\|\theta^i - \theta^*\|$, $i = 1, 2, \ldots$, converges monotonically to zero. Note that as defined, the shape in $I\!R^p$ of the region of monotone convergence depends on the norm used. However in $I\!R^p$ monotone convergence in a given norm implies convergence, however possibly non-monotone, in any other norm.

Define the $p \times p$ matrices obtained by averaging $\nabla^{20} Q(u, \overline{u})$ and $\nabla^{11} Q(u, \overline{u})$ over the line segments $u \in \overrightarrow{\theta\theta^*}$ and $\overline{u} \in \overrightarrow{\overline{\theta}\theta^*}$:

$$A_1(\theta, \overline{\theta}) = -\int_0^1 \nabla^{20} Q(t\theta + (1-t)\theta^*, t\overline{\theta} + (1-t)\theta^*) dt$$

$$A_2(\theta, \overline{\theta}) = \int_0^1 \nabla^{11} Q(t\theta + (1-t)\theta^*, t\overline{\theta} + (1-t)\theta^*) dt.$$

Also, define the following set:

$$\mathcal{S}(\overline{\theta}) = \{\theta \in \Theta \ : \ Q(\theta, \overline{\theta}) \geq Q(\overline{\theta}, \overline{\theta})\}.$$

By construction of the A-algorithm (1), we have $\theta^{i+1} \in \mathcal{S}(\theta^i)$.

Definition 1 *For a given vector norm $\|\cdot\|$ and induced matrix norm $|\!|\!| \cdot |\!|\!|$ define $\mathcal{R}_+ \subset \Theta$ as the largest open ball $B(\theta^*, \delta) = \{\theta : \|\theta - \theta^*\| < \delta\}$ such that for each $\overline{\theta} \in B(\theta^*, \delta)$:*

$$A_1(\theta, \overline{\theta}) > 0, \qquad \textit{for all } \theta \in \mathcal{S}(\overline{\theta}) \tag{2}$$

and for some $0 \leq \alpha < 1$

$$\left|\!\left|\!\left| \left[A_1(\theta, \overline{\theta})\right]^{-1} \cdot A_2(\theta, \overline{\theta}) \right|\!\right|\!\right| \leq \alpha, \qquad \textit{for all } \theta \in \mathcal{S}(\overline{\theta}). \tag{3}$$

[1]This work was partially supported by grants: NSF-BCS-9024370, DOE-FG02-87ER60561, NIH-CA-60711

The following convergence theorem establishes that, if $\mathcal{R}_+$ is not empty, the region in Definition 1 is a region of monotone convergence in the norm $\|\cdot\|$ for an algorithm of the form (1). It can be shown that $\mathcal{R}_+$ is non-empty for sufficiently regular problems (Hero and Fessler (1995)).

Theorem 1 *Let $\theta^* \in \Theta$ be a fixed point of the A algorithm (1), where $\theta^{i+1} = \mathrm{argmax}_{\theta\in\Theta} Q(\theta, \theta^i)$, $i = 0, 1, \ldots$. Assume: i) for all $\overline{\theta} \in \Theta$, the maximum $\max_\theta Q(\theta, \overline{\theta})$ is achieved on the interior of the set Θ; ii) $Q(\theta, \overline{\theta})$ is twice continuously differentiable in $\theta \in \Theta$ and $\overline{\theta} \in \Theta$, and iii) the A-algorithm (1) is initialized at a point $\theta^0 \in \mathcal{R}_+$ for a norm $\|\cdot\|$.*

1. *The iterates θ^i, $i = 0, 1, \ldots$ all lie in $\mathcal{R}_+$,*
2. *the successive differences $\Delta\theta^i = \theta^i - \theta^*$ of the A algorithm obey the recursion:*

$$\Delta\theta^{i+1} = [A_1(\theta^{i+1}, \theta^i)]^{-1} A_2(\theta^{i+1}, \theta^i) \cdot \Delta\theta^i, \tag{4}$$

3. *the norm $\|\Delta\theta^i\|$ converges monotonically to zero with at least linear rate, and*
4. *$\Delta\theta^i$ asymptotically converges to zero with root convergence factor*

$$\rho\left(\left[-\nabla^{20} Q(\theta^*, \theta^*)\right]^{-1} \nabla^{11} Q(\theta^*, \theta^*)\right) < 1.$$

III. Tomography Application

In emission computed tomography the objective is to estimate the object intensity vector $\theta = [\theta_1, \ldots, \theta_p]^T$, $\theta_b \geq 0$, from Poisson distributed projection data $\mathbf{Y} = [\mathbf{Y}_1, \ldots, \mathbf{Y}_m]^T$. The Shepp-Vardi implementati^n of the ML-EM algorithm for estimating the intensity θ h he form:

$$\theta_b^{i+1} = \frac{\theta_b^i}{P_b} \sum_{d=1}^m \frac{\mathbf{Y}_d P_{d|b}}{\sum_{j=1}^p P_{d|j} \theta_j^i}, \qquad b = 1, \ldots, p, \tag{5}$$

where $P_{d|b}$ is a full rank matrix of transition probabilities from emission locations to projection locations and $P_b = \sum_{d=1}^m P_{d|b}$.

Using Theorem 1 we obtain

Theorem 2 *Assume that the unpenalized ECT EM algorithm specified by (5) converges to the strictly positive limit θ^*. Then, for some sufficiently large positive integer M:*

$$\|\ln\theta^{i+1} - \ln\theta^*\| \leq \alpha \|\ln\theta^i - \ln\theta^*\|, \qquad i \geq M,$$

where $\alpha = \rho([\boldsymbol{B} + \boldsymbol{C}]^{-1}\boldsymbol{C})$, $\boldsymbol{B} = \boldsymbol{B}(\theta^*)$, $\boldsymbol{C} = \boldsymbol{C}(\theta^*)$, the norm $\|\bullet\|$ is defined as:

$$\|u\|^2 \stackrel{def}{=} \sum_{b=1}^p P_b \theta_b^* \, u_b^2. \tag{6}$$

Lange and Carson (1984) showed that the ECT EM algorithm converges to the maximum likelihood estimate. As long as θ^* is strictly positive, Theorem 2 asserts that in the final iterations of the algorithm the logarithmic differences $\ln\theta^i - \ln\theta^*$ converge monotonically to zero relative to the norm (6).