# ALTERNATING MINIMIZATION APPROACH FOR MULTI-FRAME IMAGE RECONSTRUCTION

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## ABSTRACT

There are a variety of imaging modalities that record a sequence of measurements where the sensor and/or the objects in the scene are moving and the goal is to reconstruct an image without motion blur. Examples include multi-frame super-resolution problems and motion-compensated image reconstruction problems in medical imaging. Various methods have been proposed for such applications, often in the context of specific imaging modalities. However, many such methods can be formulated in a common framework and thus solved by the same optimization method. To solve the reconstruction problem efficiently, the optimization method must be designed carefully.

This paper proposes a novel approach to solve multi-frame image reconstruction problems more efficiently. We use a variable-splitting technique to dissociate the original problem into a few simpler problems that are then solved individually using an alternating minimization method. The proposed method is amenable to preconditioning, parallelization, and application of block iterative algorithms to the sub-problems. Simulation results demonstrate that even with simple diagonal or circulant preconditioners, the proposed method converges faster than the conjugate gradient (CG) method.

## 1. INTRODUCTION

Every imaging device has limitations in its achievable spatial and temporal resolutions due to physical or practical constraints. The presence of rapid object or sensor motion during data acquisition can lead to images that are corrupted by motion artifacts such as blurring and streaks. For different imaging modalities, various methods have been developed to address such problems, such as super-resolution reconstruction for digital cameras [1] and motion-compensated reconstruction for medical imaging modalities [2, 3]. Many of these methods can be formulated similarly so an efficient optimization method developed for one multi-frame image reconstruction problem may also be useful for other problems.

This work is supported by GE Healthcare.

To solve such problems efficiently, the optimization method must be selected carefully. The formulation of the problem and the characteristics of the system model have critical roles in determining the which optimization methods are efficient. In cardiac computed tomography (CT) problems, the system model in motion-compensated image reconstruction (MCIR) methods includes both the tomographic forward-projector and warp matrices that describe the (nonrigid) object motion. These warp matrices make it computationally very expensive to use iterative algorithms for MCIR. Compared to conventional CT image reconstruction problems, designing a proper preconditioner for MCIR is more difficult due to the complexity of the system model. The popular ordered-subset (OS) type of algorithms [4] are inefficient for MCIR due to the computationally expensive warp matrices.

This paper proposes a novel approach to solving multi-frame image reconstruction problems more efficiently. We use a variable-splitting technique to dissociate the original problem into a few simpler problems that are then solved individually using alternating mnimization. The proposed method is illustrated with a simulation of cardiac CT, which is very important for diagnosing heart disease.

## 2. MULTI-FRAME IMAGE RECONSTRUCTION

### 2.1. Measurement Model

Let $\boldsymbol{x}(\boldsymbol{r}, t)$ denote the time-dependent intensity distribution of the unknown object, where $\boldsymbol{r}$ is the spatial location and $t$ is time. Let $t_m$ be the time of $m$th frame at which the measurements, $\boldsymbol{y}_m$, corresponding to the motion-free state of the objects are acquired. We assume that $\boldsymbol{y}$ is the $M \times 1$ measurement vector consist of $N_f$ scans, $\boldsymbol{y} = [\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{N_f}]$. The measurements are assumed to be linearly related to the object $\boldsymbol{x}_m = \boldsymbol{x}(\cdot, t_m)$ as follows:

$$\boldsymbol{y}_m = \boldsymbol{A}_m \boldsymbol{x}_m + \epsilon_m, \quad m = 1, \cdots, N_f, \qquad (1)$$

where $\boldsymbol{A}_m$ is the system model for $m$th frame and $\epsilon_m$ is the noise. The goal is to reconstruct $\{\boldsymbol{x}_m\}$ from $\{\boldsymbol{y}_m\}$ using a motion model. Here we assume $\boldsymbol{x}_m = \boldsymbol{T}_m \boldsymbol{x}$ where $\boldsymbol{T}_m$ is a

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warp matrix based on motion estimates that are determined separately.

### 2.2. Problem Formulation

Consider a penalized-likelihood least squares (PWLS) formulation of multi-frame image reconstruction [1, 3]:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \{\Psi(\boldsymbol{x}) \triangleq \mathsf{L}(\boldsymbol{x}) + \mathsf{R}(\boldsymbol{C}\boldsymbol{x})\}, \tag{2}$$

$$\mathsf{L}(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{T}\boldsymbol{x}\|_{\boldsymbol{W}}^2, \quad \mathsf{R}(\boldsymbol{C}\boldsymbol{x}) = \beta \sum_{k=1}^{K} \psi_k([\boldsymbol{C}\boldsymbol{x}]_k),$$
$$\boldsymbol{A} = \mathrm{diag}\{\boldsymbol{A}_1, \cdots, \boldsymbol{A}_{N_f}\}, \quad \boldsymbol{T} = [\boldsymbol{T}_1' \ldots \boldsymbol{T}_{N_f}']',$$

where $\boldsymbol{A}$ is the system matrix, $\boldsymbol{x} \in \mathbb{R}^N$ is the discretized version of the object being reconstructed, $\boldsymbol{W} = \mathrm{diag}\{w_i\}$ is a statistical weighting matrix, $\beta$ is the regularization parameter, $\psi_k$ is the potential function, $\boldsymbol{C}$ is a matrix that performs finite differences between neighboring voxels, $K$ is the number of neighbors, and $\boldsymbol{T}$ is the warp matrix. The minimization problem (2) is challenging due to the warp matrix $\boldsymbol{T}$ in the system model.

## 3. PROPOSED METHOD

We apply a variable splitting approach to the problem. The basic idea of variable splitting method is to introduce auxiliary constraint variables so that coupled parts in the cost function can be separated [5]. The original problem is transformed into an equivalent constrained optimization problem, and then alternating minimization methods are applied to efficiently solve the problem. Previous works have focussed on splitting the regularization term and also the statistical weighting [6]. In this work, in addition to those splittings, we focus on splitting the warp matrix from the forward-projector in the system matrix.

### 3.1. Equivalent Constrained Optimization Problem

We introduce auxiliary constraint variables $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}$, and $\boldsymbol{s}$, and write (2) as the following equivalent constrained problem:

$$\arg\min_{\boldsymbol{x},\boldsymbol{u},\boldsymbol{v},\boldsymbol{z},\boldsymbol{s}} \Psi(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}, \boldsymbol{s}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{v}\|_{\boldsymbol{W}}^2 + \mathsf{R}(\boldsymbol{z}),$$
$$s.t. \quad \boldsymbol{u} = \boldsymbol{T}\boldsymbol{x}, \quad \boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}, \quad \boldsymbol{z} = \boldsymbol{C}\boldsymbol{s}, \quad \boldsymbol{s} = \boldsymbol{x}, \tag{3}$$

where $\boldsymbol{u} \in \mathbb{R}^{NN_f}$ separates the system matrix from the warp matrix, $\boldsymbol{v} \in \mathbb{R}^M$ separates the effect of the weighting matrix, $\boldsymbol{W}$, on $\boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{z} \in \mathbb{R}^{NK}$ and $\boldsymbol{s} \in \mathbb{R}^N$ detach the warp matrix from the regularizer.

### 3.2. Method of Multipliers

We use the framework of method of multipliers [7] to solve (3), and construct an augmented Lagrangian function as follows:

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}, \boldsymbol{s}) \triangleq\ & \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{v}\|_{\boldsymbol{W}}^2 + \mathsf{R}(\boldsymbol{z}) \\ & + \frac{\mu_{\boldsymbol{u}}}{2}\|\boldsymbol{u} - \boldsymbol{T}\boldsymbol{x} - \eta_{\boldsymbol{u}}\|^2 + \frac{\mu_{\boldsymbol{v}}}{2}\|\boldsymbol{v} - \boldsymbol{A}\boldsymbol{u} - \eta_{\boldsymbol{v}}\|^2 \\ & + \frac{\mu_{\boldsymbol{z}}}{2}\|\boldsymbol{z} - \boldsymbol{C}\boldsymbol{s} - \eta_{\boldsymbol{z}}\|^2 + \frac{\mu_{\boldsymbol{s}}}{2}\|\boldsymbol{s} - \boldsymbol{x} - \eta_{\boldsymbol{s}}\|^2, \end{aligned} \tag{4}$$

where $\eta$'s are Lagrange-multiplier-like vectors and $\mu$'s are the AL penalty parameters (see [6] for details).

Solving (3) using the AL function would require jointly minimizing (4) with respect to all variables which is computationally expensive. So we apply alternating minimization [6].

### 3.3. Alternating Direction Minimization

At the $j$th iteration, we update each vector in turn as follows:

$$\begin{aligned} \boldsymbol{x}^{(j+1)} &= \arg\min_{\boldsymbol{x}} \frac{\mu_{\boldsymbol{u}}}{2}\left\|\boldsymbol{u}^{(j)} - \boldsymbol{T}\boldsymbol{x} - \eta_{\boldsymbol{u}}^{(j)}\right\|^2 \\ &\quad + \frac{\mu_{\boldsymbol{s}}}{2}\left\|\boldsymbol{s}^{(j)} - \boldsymbol{x} - \eta_{\boldsymbol{s}}^{(j)}\right\|^2, \end{aligned} \tag{5}$$

$$\begin{aligned} \boldsymbol{u}^{(j+1)} &= \arg\min_{\boldsymbol{u}} \frac{\mu_{\boldsymbol{u}}}{2}\left\|\boldsymbol{u} - \boldsymbol{T}\boldsymbol{x}^{(j+1)} - \eta_{\boldsymbol{u}}^{(j)}\right\|^2 \\ &\quad + \frac{\mu_{\boldsymbol{v}}}{2}\left\|\boldsymbol{v}^{(j)} - \boldsymbol{A}\boldsymbol{u} - \eta_{\boldsymbol{v}}^{(j)}\right\|^2, \end{aligned} \tag{6}$$

$$\begin{aligned} \boldsymbol{v}^{(j+1)} &= \arg\min_{\boldsymbol{v}} \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{v}\|_{\boldsymbol{W}}^2 \\ &\quad + \frac{\mu_{\boldsymbol{v}}}{2}\left\|\boldsymbol{v} - \boldsymbol{A}\boldsymbol{u}^{(j+1)} - \eta_{\boldsymbol{v}}^{(j)}\right\|^2, \end{aligned} \tag{7}$$

$$\begin{aligned} \boldsymbol{s}^{(j+1)} &= \arg\min_{\boldsymbol{s}} \frac{\mu_{\boldsymbol{z}}}{2}\left\|\boldsymbol{z}^{(j)} - \boldsymbol{C}\boldsymbol{s} - \eta_{\boldsymbol{z}}^{(j)}\right\|^2 \\ &\quad + \frac{\mu_{\boldsymbol{s}}}{2}\left\|\boldsymbol{s} - \boldsymbol{x}^{(j+1)} - \eta_{\boldsymbol{s}}^{(j)}\right\|^2, \end{aligned} \tag{8}$$

$$\begin{aligned} \boldsymbol{z}^{(j+1)} &= \arg\min_{\boldsymbol{z}} \mathsf{R}(\boldsymbol{z}) \\ &\quad + \frac{\mu_{\boldsymbol{z}}}{2}\left\|\boldsymbol{z} - \boldsymbol{C}\boldsymbol{s}^{(j+1)} - \eta_{\boldsymbol{z}}^{(j)}\right\|^2, \end{aligned} \tag{9}$$

$$\eta_{\boldsymbol{u}}^{(j+1)} = \eta_{\boldsymbol{u}}^{(j)} - (\boldsymbol{u}^{(j+1)} - \boldsymbol{T}\boldsymbol{x}^{(j+1)}), \tag{10}$$
$$\eta_{\boldsymbol{v}}^{(j+1)} = \eta_{\boldsymbol{v}}^{(j)} - (\boldsymbol{v}^{(j+1)} - \boldsymbol{A}\boldsymbol{u}^{(j+1)}), \tag{11}$$
$$\eta_{\boldsymbol{s}}^{(j+1)} = \eta_{\boldsymbol{s}}^{(j)} - (\boldsymbol{s}^{(j+1)} - \boldsymbol{x}^{(j+1)}), \tag{12}$$
$$\eta_{\boldsymbol{z}}^{(j+1)} = \eta_{\boldsymbol{z}}^{(j)} - (\boldsymbol{z}^{(j+1)} - \boldsymbol{C}\boldsymbol{s}^{(j+1)}). \tag{13}$$

The sub-problems (5) to (8) are all quadratic problems for which analytical solutions exist. However, (5) and (6) cannot be implemented explicitly due to the enormous sizes of the matrices involved. We employ the iterative CG-solver for these sub-problems.

Sub-problem (5) is an image-registration-type problem, which has the following analytical solution:

$$\boldsymbol{x}^{(j+1)} = \boldsymbol{H}^{-1}(\mu_{\boldsymbol{u}}\boldsymbol{T}'(\boldsymbol{u}^{(j)} - \eta_{\boldsymbol{u}}^{(j)}) + \mu_{\boldsymbol{s}}(\boldsymbol{s}^{(j)} - \eta_{\boldsymbol{s}}^{(j)})), \tag{14}$$

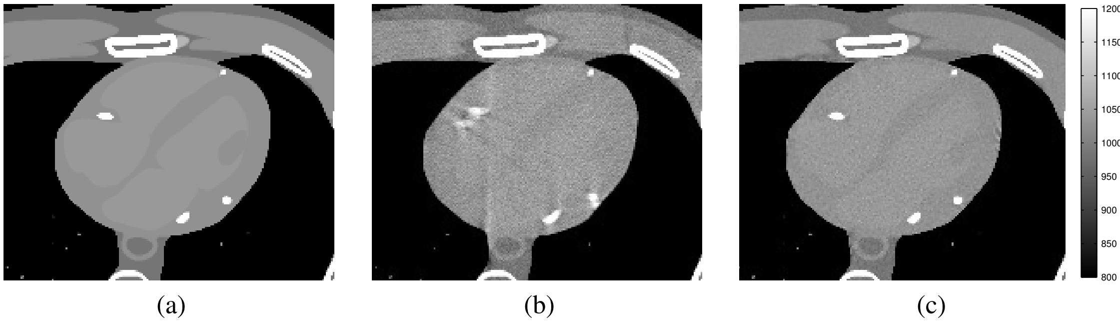


**Fig. 1**. Images in the ROI of (a) XCAT phantom, (b) FBP reconstruction with Hanning filter (also the initial guess $\boldsymbol{x}^{(0)}$), (c) Converged Image $\boldsymbol{x}^{(\infty)}$.

where $\boldsymbol{H} = \mu_{\boldsymbol{u}} \boldsymbol{T}'\boldsymbol{T} + \mu_{\boldsymbol{s}} \boldsymbol{I}_N$. Since $\boldsymbol{H}$ is much simpler than the Hessian of the original data term in (2), it is more amenable to preconditioning. We accelerate the CG-solver for (14) by using a suitable preconditioner for $\boldsymbol{H}$.

We now consider (6), which is a tomography problem with the following solution:

$$\boldsymbol{u}^{(j+1)} = \boldsymbol{G}^{-1}(\mu_{\boldsymbol{u}}(\boldsymbol{T}\boldsymbol{x}^{(j+1)} + \eta_{\boldsymbol{u}}^{(j)}) + \mu_{\boldsymbol{v}} \boldsymbol{A}'(\boldsymbol{v}^{(j)} - \eta_{\boldsymbol{v}}^{(j)})),$$

where $\boldsymbol{G} = \mu_{\boldsymbol{v}} \boldsymbol{A}'\boldsymbol{A} + \mu_{\boldsymbol{u}} \boldsymbol{I}_{NN_f}$. We preconditioned this term with a circulant matrix to obtain faster convergence [6, 8]. This sub-problem can be further parallelized into $N_f$ problems. Each parallelized problem can be efficiently solved by preconditioned CG or ordered-subsets type algorithms, which are less efficient for the original problem.

Sub-problems (7) - (9) can be solved much more easily compared to above two sub-problems. Sub-problem (7) has a simple analytical solution, and (8) is exactly solvable with Fourier transform if we use $\boldsymbol{C}$ with periodic end condition. Finally, (9) can be solved easily with iterative algorithms or exactly solved for a variety of potential functions. Here, we consider one of the edge-preserving regularization using the Fair potential function. For this regularizer, (9) separates into 1D minimization problems and has an exact solution (See [6] for details). The AL parameters, $\mu$'s, govern the convergence speed of the proposed splitting method [6]; we selected them empirically to achieve good convergence speed.

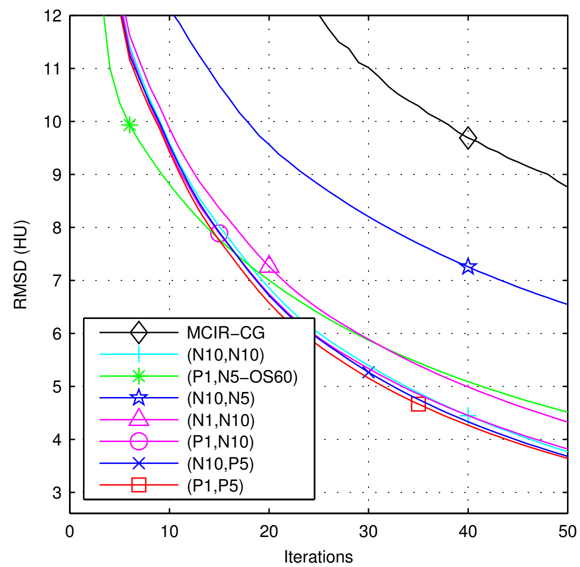


**Fig. 2**. Plot of RMSD versus iteration for various settings of the proposed method compared to the conventional CG method. For the proposed method, (N10,P5) indicates 10 iterations for sub-problem (5) without preconditioner for $\boldsymbol{H}$ and 5 iterations for sub-problem (6) with a preconditioner for $\boldsymbol{G}$. OS60 indicates that ordered-subsets method was used instead of CG to solve sub-problem (6).

## 4. RESULTS

The proposed algorithm was investigated on a 2D CT image reconstruction problem with cardiac motion from simulated data. We simulated a 3rd-generation fan-beam CT system using the separable footprint projector [9]. The simulated system had 888 channels per view spaced 1.0239 mm apart, and 984 evenly spaced view angles over 360°. The image was reconstructed to a 512 $\times$ 512 grid of 0.9766 mm pixels. We generated seven frames of the XCAT phantom for a heart rate of 75 bpm. The motion between the frames was estimated directly from XCAT images using nonrigid image registration. Estimating motion parameters from true images is unrealistic, but our focus is not on obtaining reasonable motion estimates. We only focus on the image reconstruction part of MCIR. For the regularizer, we used a Fair potential function to provide edge-preservation and a certainty-based penalty to obtain more uniform resolution. The sinogram was generated with Poisson noise, and the weights in the data-fit term in (2) were chosen as $w_i = \exp(-[\boldsymbol{A}\boldsymbol{x}]_i)$. We selected the regularization parameter $\beta$ such that the target PSF had a full-width at half-maximum (FWHM) of approximately 1.3 mm.

For comparison, we used the (nonlinear) conjugate gradient algorithm to solve the original problem (2). To analyze the convergence speed of the proposed method we computed the root mean squared (RMS) difference between the image estimate at the $n$th iteration, $\boldsymbol{x}^{(n)}$, with the "fully" converged solution, $\boldsymbol{x}^{\infty}$. For the Fair potential, the original MCIR problem is strictly convex and thus has a unique minimizer, $\boldsymbol{x}^{\infty}$. We numerically approximated $\boldsymbol{x}^{\infty}$ as the mean of the images reconstructed (assuming convergence) by running 1000 iterations of CG and 700 iterations of the proposed method with (10,10) sub-iterations.

Fig. 1 illustrates that the conventional filtered backprojection (FBP) method gives a reconstructed image with severe motion artifacts whereas the motion-compensated image contains much less motion artifacts. Some residual motion artifacts still exist due to imperfect motion estimates even though they were obtained directly from the true XCAT images.

Fig. 2 illustrates that the proposed method converges much faster in iterations compared to the conventional CG method when we use enough sub-iterations with obvious computation overhead. This result suggests that if we have a proper preconditioner for each sub-problem, we can still obtain fast convergence. We also investigated different options for the proposed method summarized in Fig. 2. Using a preconditioner for sub-problems helped reduce the number of sub-iterations while achieving fast convergence speed.

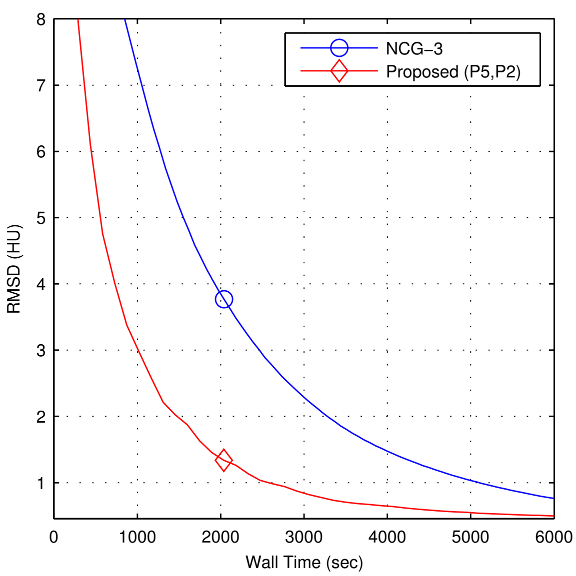


**Fig. 3**. Plot of RMSD versus wall time for the proposed method compared to the conventional CG method with 3 iterations for line-search. A diagonal preconditioner and a circulant preconditioner were used for sub-problems (5) and (6) respectively.

In Fig. 3, we compare the proposed method implemented with sub-optimal preconditioners. We used a simple diagonal preconditioner for (5) based on the diagonal elements of $\boldsymbol{H}$ and a circulant preconditioner for (6) using the fact that $\boldsymbol{G}$ contains $\boldsymbol{A}_m' \boldsymbol{A}_m$, which is approximately shift invariant [6]. The proposed method shows faster convergence compared to CG method. While the proposed method as implemented in MATLAB provides significant improvement in convergence speed over CG, we believe that the proposed method can further be improved using proper code optimization and more efficient implementations.

## 5. DISCUSSION

We applied a variable splitting approach to the motion-compensated image reconstruction problem for cardiac CT. The proposed method converges faster than the conjugate gradient method, and offers the potential for parallelizabilty and preconditioning of sub-problems. Some of the sub-problems can be solved simultaneously or further divided into smaller problems. By using more sophisticated preconditioners for the sub-problems, the performance of the proposed method can be further improved. In this study, we focussed on the image reconstruction part of MCIR for cardiac CT, but our method also can be applied to other multi-frame image reconstruction problems and extended to the problem of jointly estimating the motion model and the image.

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