# Quadratic Regularization Design for Iterative Reconstruction in 3D multi-slice Axial CT

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ABSTRACT

In X-ray CT, statistical methods for tomographic image reconstruction create images with better noise properties than conventional Filtered Back Projection (FBP) techniques. Penalized-likelihood (PL) image reconstruction methods maximize an objective function based on the log-likelihood of sinogram measurements and on a user defined roughness penalty which controls noise. Penalized-likelihood methods (as well as Penalized Weighted Least Squares methods) based on conventional quadratic regularizers result in nonuniform and anisotropic spatial resolution. We have previously addressed this problem for 2D emission tomography, 2D fan-beam transmission tomography, and 3D cylindrical emission tomography. This paper extends those methods to 3D multi-slice axial CT with small cone angles.

## I. INTRODUCTION

Interactions between the non-quadratic Poisson log-likelihood and a conventional quadratic regularizer lead to nonuniform and anisotropic spatial resolution in CT images reconstructed by penalized-likelihood (or penalized weighted least squares) methods, even for idealized shift-invariant imaging systems [1]. This "problem" could be circumvented by using a quadratically-penalized, *unweighted* least-squares (QPULS) estimation method, but QPULS images have poor noise properties (akin to FBP in fact) because the *weighting*, which is explicit in PWLS methods and implicit in penalized-likelihood methods, is a central advantage of statistical methods over FBP. We previously described methods for designing regularizers to try to improve the uniformity and isotropy of reconstructed images for *2D parallel-beam* systems, *2D fan-beam* systems, and *3D cylindrical PET* [1]–[5]. Here we describe a design method for *3D axial multi-slice CT* that has the goal of providing uniform and isotropic spatial resolution properties by combining the fan-beam approach of [3] and the 3D regularization approach of [5]. Regularization parameters designed with these methods may also be useful for non-quadratic regularizers [6].

## II. THEORY

### A. Local Impulse Response

As shown in [1], the impulse response at the $j$th voxel for a CT image reconstructed using a penalized-likelihood algorithm is:

$$l^j = [\mathbf{A}'\mathbf{W}\mathbf{A} + \beta\mathbf{R}]^{-1}\mathbf{A}'\mathbf{W}\mathbf{A}\delta^j,$$

where $\boldsymbol{A}$ is the system matrix, $\boldsymbol{W} = diag[Y_i]$, and $\boldsymbol{R}$ is the Hessian of our regularizer. Throughout this paper we will use the index $j$ to represent a lexicographical ordering of voxels which can also be indexed with $(n, m, z)$. Assuming that the matrices $\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}$ and $\boldsymbol{R}$ are approximately locally circulant, the Fourier transform of the local impulse response is as follows

$$\boldsymbol{L}^j = \frac{F(\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{\delta}^j)}{F(\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{\delta}^j) + \beta F(\boldsymbol{R}\boldsymbol{\delta}^j)}, \quad (1)$$

where $F()$ denotes a Fourier transform and $\boldsymbol{\delta}^j$ denotes an impulse function at the $j$th voxel. Each element in weighting matrix $\boldsymbol{W}$, which is based on the projection data, can be denoted as $w(s, \beta, t)$, where $s$ is arc length along the detector, and $\beta$ is the source angle, and $t$ is the index to a row of the detector. Prior to [2], design methods were based on discrete Fourier transforms, e.g., in [4]. We show in [5] that the local frequency response for parallel beam cylindrical geometries is

$$L^j(\varrho, \Phi, \Theta) = \frac{\tilde{w}^j(\Phi)}{\tilde{w}^j(\Phi) + \beta|\varrho|R(\varrho, \Phi, \Theta)}. \quad (2)$$

Using analysis from [3], we can apply (2) to multi-slice fan beam axial CT geometries if we replace $\tilde{w}^j(\Phi)$ with new weightings derived from a change of variables to shift from a parallel beam coordinate space to a fan-beam coordinate space. In space, the new weightings are

$$w'^j(\phi) = w(s', \beta', t')\, J(s')\Big|_{\phi=\phi'} + w(s', \beta', t')\, J(s')\Big|_{\phi=\phi'-\pi}. \quad (3)$$

where $s'$, $t'$, and $\beta'$, which are dependent on $\phi$ and $j$, represent the change of variables and $J(s')$ is the Jacobian corresponding to such a change of variables.

### B. Target Impulse Response

In 2D, using a target LIR associated with penalized unweighted reconstruction is reasonable because it is isotropic. However in 3D, the corresponding LIR is

$$L^j(\varrho, \Phi, \Theta) \approx \frac{1}{1 + \beta\cos(\Theta)\,|\varrho|R_0(\varrho, \Phi, \Theta)}, \quad (4)$$

which is not isotropic due to the $\cos(\Theta)$ term. We pick a value for

$$R_0(\varrho, \Phi, \Theta) = \frac{|2\pi\varrho|^2}{\cos(\Theta)}, \quad (5)$$

which results in

$$L^j(\varrho, \Phi, \Theta) = \frac{1}{1 + \beta(2\pi)^2|\varrho|^3}. \quad (6)$$

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### C. Regularization Structure

In this section we will derive a minimization problem which when solved, will yield the desired regularizer. Regularizers penalize roughness by penalizing differences amongst neighboring pixels. We define a basis function for a regularizer which penalizes the $l$th neighbor as

$$c_l = \frac{1}{\sqrt{n_l^2 + m_l^2 + z_l^2}}(\delta(n,m,z) - \delta(n-n_l, m-m_l, z-z_l)),$$

where $n_l, m_l, z_l$ denote the offset of the neighbor whose difference is being penalized. The conventional regularizer can be expressed as

$$R(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}'\mathbf{R}\boldsymbol{x} = \sum_{n,m,z}\sum_{l=1}^{L}\frac{1}{2}((c_l *** x)(n,m,z))^2. \quad (7)$$

In this conventional regularizer, the difference between each neighbor receives the same penalty. We make the regularizer spatially adaptive with the addition of coefficients $r_l^j$, yielding

$$R(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}'\mathbf{R}\boldsymbol{x} = \sum_{n,m,z}\sum_{l=1}^{L} r_l^{j(n,m,z)}\frac{1}{2}((c_l *** x)(n,m,z))^2. \quad (8)$$

Designing our regularizer reduces to choosing $L$ coefficients $r_l^j$, $l = 1, 2, ..., L$, per voxel, where $L$ is the number of neighbors we are penalizing. If we take the Fourier transform, we get the following expression for $|C_l(\omega_1, \omega_2, \omega_3)|^2$,

$$\begin{aligned}
&= \frac{1}{n_l^2 + m_l^2 + z_l^2}|1 - e^{-i(\omega_1 n_l + \omega_2 m_l + \omega_3 z_l)}|^2 \\
&= \frac{1}{n_l^2 + m_l^2 + z_l^2}(2 - 2\cos(\omega_1 n_l + \omega_2 m_l + \omega_3 z_l)) \\
&\approx \frac{1}{n_l^2 + m_l^2 + z_l^2}\big(2 - 2\big(1 - \frac{1}{2}(\omega_1 n_l + \omega_2 m_l + \omega_3 z_l)^2\big)\big) \\
&\approx \frac{1}{n_l^2 + m_l^2 + z_l^2}(\omega_1 n_l + \omega_2 m_l + \omega_3 z_l)^2. \quad (9)
\end{aligned}$$

In previous work, we have assumed that we were using square (in 2D) or cubic voxels. While square pixels are realistic in in 2D, real slice thicknesses in imaging systems often never result in non-cubic voxels. Therefore we generalize the analysis of the 3D roughness penalty in [5]. To convert the above expression to polar frequency coordinates, we use our knowledge of frequency and sampling relationships to derive $w_1 = 2\pi\Delta_x\varrho\cos(\Phi)\cos(\Theta)$, $w_2 = 2\pi\Delta_y\varrho\sin(\Phi)\cos(\Theta)$, and $w_3 = 2\pi\Delta_z\varrho\sin(\Theta)$. Substituting the following into (9), we get the following expression for $|C_l(w_1, w_2, w_3)|^2$,

$$\begin{aligned}
&\approx \frac{1}{n_l^2 + m_l^2 + z_l^2}(n_l 2\pi\Delta_x\varrho\cos(\Phi)\cos(\Theta) \\
&\qquad + m_l 2\pi\Delta_y\varrho\sin(\Phi)\cos(\Theta) + z_l 2\pi\Delta_z\varrho\sin(\Theta))^2 \\
&= \frac{1}{n_l^2 + m_l^2 + z_l^2}(2\pi\rho)^2(n_l\Delta_x\cos(\Phi)\cos(\Theta) \\
&\qquad + m_l\Delta_y\sin(\Phi)\cos(\Theta) + z_l\Delta_z\sin(\Theta))^2.
\end{aligned}$$

The resulting local frequency response (using polar coordinates) of this regularizer is:

$$R^j(\varrho, \Phi, \Theta) = (2\pi\varrho)^2\sum_{l=1} r_l^j(f(\Phi_l, \Theta_l)\cdot e(\Phi, \Theta))^2, \quad (10)$$

where $e(\Phi, \Theta) \triangleq (\cos\Theta\cos\Phi, cos\Theta\sin\Phi, \sin\Theta)$ and $f(\Phi_l, \Theta_l) \triangleq (\Delta_x\cos\Theta_l\cos\Phi_l, \Delta_y\cos\Theta_l\sin\Phi_l, \Delta_z\sin\Theta_l)$.

### D. Matching the LIR to the Target Impulse Response

We would like to design $\boldsymbol{R}$ so that the LIR matches the target as close as possible, i.e., such that

$$\begin{aligned}
\frac{w'^j(\Phi)}{w'^j(\Phi) + \beta|\varrho|\cos(\Theta)\,R^j(\varrho,\Phi,\Theta)} &\approx \frac{1}{1 + \beta(2\pi)^2|\varrho|^3} \\
\beta|\varrho|\cos(\Theta)\,R^j(\varrho,\Phi,\Theta) &\approx \beta w'^j(\Phi)(2\pi)^2|\varrho|^3 \\
\cos(\Theta)\,R^j(\varrho,\Phi,\Theta) &\approx w'^j(\Phi)(2\pi|\varrho|)^2 \\
\sum_{l=1} r_l^j\cos(\Theta)(f(\Phi_l,\Theta_l)\cdot e(\Phi,\Theta))^2 &\approx w'^j(\Phi). \quad (11)
\end{aligned}$$

To select $\{r_l^j\}$, we solve the following minimization problem

$$\begin{aligned}
\hat{\boldsymbol{r}}^j = \operatorname*{arg\,min}_{\mathbf{r}^j \geq 0} \int_0^{\pi}\int_{-\pi/2}^{\pi/2} &|w(\Phi) \\
&- \sum_{l=1} r_l\cos(\Theta)(f(\Phi_l,\Theta_l)\cdot e(\Phi,\Theta))^2|^2 d\Theta, d\Phi. \quad (12)
\end{aligned}$$

We can rewrite the term $\sum_{l=1} r_l\cos(\Theta)(f(\Phi_l,\Theta_l)\cdot e(\Phi,\Theta))^2$ as $\boldsymbol{BCr}$ where $\mathbf{r}$ is a $L \times 1$ vector of penalty coefficients. $\cos(\Theta)(f(\Phi_l,\Theta_l)\cdot e(\Phi,\Theta))^2$ will expand into 6 orthonormal basis functions that form the columns of $\boldsymbol{B}$. $\boldsymbol{C}$ is a matrix of linear combination coefficients such that

$$\sum_{l=1} r_l\cos(\Theta)(f(\Phi_l,\Theta_l)\cdot e(\Phi,\Theta))^2 = \boldsymbol{BCr}. \quad (13)$$

We undergo this factorization to simplify the problem. Our current problem is therefore framed as

$$\boldsymbol{r}^j = \operatorname*{arg\,min}_{\boldsymbol{r}\geq 0}||\boldsymbol{w}^j - \boldsymbol{BCr}||^2$$

This is equivalent too

$$\begin{aligned}
&= \operatorname*{arg\,min}_{\boldsymbol{r}\geq 0}||\boldsymbol{w}^j - \boldsymbol{BCr}||^2 \\
&= \operatorname*{arg\,min}_{\boldsymbol{r}\geq 0} < \boldsymbol{w}^j - \boldsymbol{BCr}, \boldsymbol{w}^j - \boldsymbol{BCr} > \\
&= \operatorname*{arg\,min}_{\boldsymbol{r}\geq 0}||\boldsymbol{w}^j||^2 - 2 < \boldsymbol{w}^j, \boldsymbol{BCr} > + < \boldsymbol{BCr}, \boldsymbol{BCr} > \\
&= \operatorname*{arg\,min}_{\boldsymbol{r}\geq 0}||\boldsymbol{w}^j||^2 - 2 < \boldsymbol{B}^H\boldsymbol{w}^j, \boldsymbol{Cr} > \\
&\qquad + < \boldsymbol{B}^H\boldsymbol{BCr}, \boldsymbol{Cr} > .
\end{aligned}$$

Since the columns of $\boldsymbol{B}$ are orthonormal functions, $\boldsymbol{B}^H\boldsymbol{B}$ is the Identity matrix. We can replace $||\boldsymbol{w}^j||^2$ with $||\boldsymbol{B}^H\boldsymbol{w}^j||^2$ since it is an irrelevant constant which has no effect on our minimization, leaving us with

$$\begin{aligned}
&= \operatorname*{arg\,min}_{\boldsymbol{r}\geq 0}||\boldsymbol{B}^H\boldsymbol{w}^j||^2 - 2 < \boldsymbol{B}^H\boldsymbol{w}^j, \boldsymbol{Cr} > + ||\boldsymbol{Cr}||^2 \\
&= \operatorname*{arg\,min}_{\boldsymbol{r}\geq 0}||\boldsymbol{B}^H\boldsymbol{w}^j - \boldsymbol{Cr}||^2. \quad (14)
\end{aligned}$$

Because $\boldsymbol{B}^H\boldsymbol{w}$ produces a $6 \times 1$ vector, and $\boldsymbol{C}$ is $6 \times L$, The minimization problem (14) is much smaller than (12) and can be solved with any NNLS algorithm.

### E. Implementation Details

In the 2D case, one can minimize (14) analytically [2] such that the solution is minimum norm. In a NNLS problem that is under-determined there are many solutions that minimize the cost function. Finding the solution with a minimum norm solves for $\boldsymbol{r}^j$ as a continuous function of $\boldsymbol{B}^H\boldsymbol{w}^j$ and leads to good image reconstruction properties. Without the continuous mapping from $\boldsymbol{B}^H\boldsymbol{w}^j$ to $\boldsymbol{r}^j$, neighboring pixels can have drastically different weights, thus violating our assumptions of local spatial invariance. [2] exploits properties of the matrices used to solve the 2D problem to find a minimum norm solution. Unfortunately those properties do not hold in the 3D case. To compensate for this problem, we try to alter the minimization problem from

$$\arg\min_{\boldsymbol{r}\geq 0} ||\mathbf{B}^H\boldsymbol{w}^j - \mathbf{C}\boldsymbol{r}||^2,$$

to

$$\arg\min_{\boldsymbol{r}\geq 0} ||\mathbf{B}^H\boldsymbol{w}^j - \mathbf{C}\boldsymbol{r}||^2 + \epsilon||\boldsymbol{r}||^2,$$

so that the norm of $\boldsymbol{r}^j$ becomes a factor in the cost function. We append a scaled identity matrix to the bottom of $\mathbf{C}$ and zero pad $\mathbf{B}^H\boldsymbol{w}^j$

$$\tilde{\mathbf{C}} = \begin{pmatrix} \mathbf{C} \\ \epsilon\boldsymbol{I} \end{pmatrix}, \tilde{\boldsymbol{d}}^j = \begin{pmatrix} \mathbf{B}^H\boldsymbol{w}^j \\ \mathbf{0} \end{pmatrix}$$

so that (14) is changed to

$$\begin{aligned} \boldsymbol{r}^j &= \arg\min_{\boldsymbol{r}\geq 0} ||\tilde{\boldsymbol{d}}^j - \tilde{\boldsymbol{C}}\boldsymbol{r}|| \qquad (15) \\ &= \arg\min_{\boldsymbol{r}\geq 0} ||\mathbf{B}^H\boldsymbol{w}^j - \mathbf{C}\boldsymbol{r}||^2 + \epsilon||\boldsymbol{r}||^2. \end{aligned}$$

This modification makes $\boldsymbol{r}^j$ a continuous function of $\boldsymbol{w}^j$ thereby eliminating the discontinuities.

Since we are minimizing with a non-negative constraint, our design has the potential to yield many $\boldsymbol{r}^j$ values that are zero. If there are too many zeros in $\boldsymbol{r}^j$, there will be zeros in the Hessian, leaving us with bad convergence properties. Instead of using a non-negative constraint, we would like to have $\boldsymbol{r}^j$ be greater than $\epsilon$ for selected $r_l^j$ to ensure that enough $r_l^j$ are non-zero. We select the 3 adjacent neighbors in the immediate x,y, and z directions to be non-zero. We turn to previous work to select $\epsilon$. In [1], Fessler derives a spatially variant $\beta$ which seeks to preserve uniform spatial resolution. So for any given pixel, we take $\epsilon = \alpha(\beta^j)^2$ where this $\beta^j$ is the spatially variant $\beta$ for a pixel $(x^j, y^j, z^j)$. Increasing $\alpha$ improves convergence at the expense of isotropy, while use of the spatially variant $\beta$ helps us preserve uniformity.

Now we must formulate our problem so that NNLS algorithms will accept this new constraint. We can create a vector $\boldsymbol{\epsilon}$ that is zero for most neighbors, and $\alpha(\beta^j)_2$ for immediate x, y and z neighbors. let $\tilde{\boldsymbol{r}}^j = \boldsymbol{r}^j - \boldsymbol{\epsilon}$. Solving with the constraint of $\tilde{\boldsymbol{r}}^j \geq 0$ ensures that $\boldsymbol{r}^j \geq \boldsymbol{\epsilon}$. Plugging $\tilde{\boldsymbol{r}}^j$ into (16) we get $|\tilde{\boldsymbol{C}}\tilde{\boldsymbol{r}}^j - \tilde{\boldsymbol{d}}| = |\tilde{\boldsymbol{C}}(\boldsymbol{r}^j - \boldsymbol{\epsilon}) - \tilde{\boldsymbol{d}}| = |\tilde{\boldsymbol{C}}\boldsymbol{r}^j - (\tilde{\boldsymbol{d}} + \tilde{\boldsymbol{C}}\boldsymbol{\epsilon})|$ which can be plugged into an NNLS algorithm. Using $\tilde{\boldsymbol{r}}^j$ for reconstruction achieves this minimum condition.

## III. Conclusion

Preliminary results are worse than we expected. We believe this to be a result of $w^j(\Phi)$ being more complicated for real CT data than phantoms we have previously worked with which make them harder to approximate. Our system also works hard to compensate for the $\cos(\Theta)$ term in (11) even though completely undoing it is impossible. This results in a tradeoff between isotropy in the z direction for isotropy in the x,y directions which can be undesirable. Future work will consider using 2D regularization with this framework which has achieved positive results in the past, combined with post-filtering in the z direction to preserve isotropy.

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