FOURIER-BASED FORWARD AND BACK-PROJECTORS IN ITERATIVE FAN-BEAM TOMOGRAPHIC IMAGE RECONSTRUCTION

Yingying Zhang, Jeffrey A. Fessler

EECS Dept., The University of Michigan
yzz@eecs.umich.edu,fessler@umich.edu

ABSTRACT

Fourier-based forward and back-projection methods have the potential to reduce computation demands in iterative tomographic image reconstruction. Interpolation errors are a limitation of conventional Fourier-based projectors. Recently, the min-max optimized Kaiser-Bessel interpolation within the nonuniform Fast Fourier transform (NUFFT) approach has been applied in *parallel-beam* image reconstruction, whose results show lower approximation errors than conventional interpolation methods. However, the extension of min-max NUFFT approach to fan-beam data has not been investigated. We have extended the min-max NUFFT framework to the fan-beam tomography case, using the relationship between the fan-beam projections and corresponding projections in parallel-beam geometry. Our studies show that the fan-beam Fourier-based forward and back-projection methods can significantly reduce computation time while still providing comparable accuracy to their space-based counterparts.

1. INTRODUCTION

The classical approach to reconstructing fan-beam tomography images is filtered back-projection. Iterative image reconstruction methods are based on measurement statistics and physics models, and offer numerous advantages, such as the potential for improved bias-variance performance. The primary computation burden of iterative image reconstruction method is forward and back-projections. Recently, Matej *et al.* evaluated Fourier-based forward and back projectors [1] in parallel-beam geometry, with min-max optimized Kaiser-Bessel NUFFT. Their results showed low interpolation error and computation efficiency.

The fan-beam case is more complicated because Fourier slice theorem is not valid for fan-beam geometry. To apply Fourier-based forward and back projectors in fan-beam tomography, we need to associate fan-beam projections with parallel beam projections. Then Fourier slice theorem can be used to relate the 1D FT of each projection (on a polar grid) to samples of the 2D FT of the object on a Cartesian grid.

The key steps influencing the image quality and computation time are the interpolation between polar and Cartesian frequency samples, and between parallel- and fan-beam projection samples. The min-max NUFFT is still applied in the frequency space interpolation. For the interpolation of projections samples, Peng *et al.* [2] have applied interpolation procedure proposed by Clark, Palmer and Lawrence [3] (CPL theorem). We have accomplished this step by applying a 1D min-max NUFFT and a one-dimensional Dirichlet-like ”periodic sinc” interpolation done by FFT fractional shift.

The organization of this paper is as follows. Section 1 briefly reviews the fan-beam geometry. Section 2 describes the framework of the proposed NUFFT method. Section 3 gives simulation results, including an accuracy comparison of NUFFT-based, space-based and analytically computed forward and back-projections, as stand alone modules and within iterations. Finally, discussion and conclusions are in Section 4.

2. THEORY

2.1. Fourier slice theorem

The Fourier slice theorem is the basis of the Fourier-based forward and back-projections in parallel beam CT image reconstruction. Let $g(x, y)$ denote the 2D image and its 2D FT is $G(f_X, f_Y)$. In polar coordinates,

$$G_\theta(\rho) = G(\rho\cos\theta, \rho\sin\theta) \tag{1}$$

By Fourier slice theorem [4], the projection at angle θ as a function of radial distance r is given by

$$p_\theta(r) = \int_{L(r,\theta)} g(x, y) d\ell = \int G_\theta(\rho)\, \mathrm{e}^{i2\pi\rho r}\,, \tag{2}$$

where $L(r, \theta)$ denotes the line at angle θ taken from y axis counter-clockwise, at distance r from the origin. The back-projection operator is the adjoint of the forward projection operator.

2.2. Non-uniform FFT

Fourier-based projectors are based on discretized versions of the Fourier slice theorem. The 2D FT of a discretized object gives the discrete Fourier data on Cartesian lattice, while the 1D inverse FT of $G_\theta(\rho)$ requires Fourier samples on a polar grid. The min-max NUFFT [5] method allows us to retain computational advantages of FFT with lower interpolation errors and is directly applicable where nonuniform Cartesian Fourier samples are required.

Basic steps of the min-max NUFFT are:

1. Calculation of a K/N times scaled and oversampled FFT.

2. Interpolation onto the desired, nonuniformly-spaced frequency locations using the interpolator that minimizes the worst approximation error.

2.3. Min-max NUFFT applied to fan-beam tomography

Fan-beam projection data is indexed by two angular coordinates (β, σ) where β is the angular position of source taken from y axis counter-clockwise, and σ is the angular position, relative to source, of the line.

The Fourier slice theorem does not apply to the fan-beam geometry directly. However, the unique mapping between parallel-beam and fan-beam coordinates allows us to apply Fourier slice theorem and therefore Fourier-based forward and back-projections. The well-known relation between (β, σ) and (r, θ) is [2]

$$\begin{aligned} r &= R \sin \sigma \\ \theta &= \beta + \sigma \end{aligned} \tag{3}$$

For uniformly incremented (β, σ), the corresponding parallel-beam data are nonuniformly-spaced in r for any given θ and uniformly-spaced in θ for any given r, but shifted by $n\triangle_\sigma$.

Fig. 1 shows the basic steps of the application of the min-max NUFFT in fan-beam tomography:

1. 2D NUFFT on image of size N to obtain desired frequency samples that are uniformly-spaced on polar grid.

2. 1D nonuniform IFFT (by means of 1D NUFFT) on the samples of each central section to obtain desired projection samples with nonuniform radial sampling suitable for fan-beam geometry.

3. 1D Dirichlet-like "periodic sinc" interpolation (by means of FFT and modulated IFFT) on θ direction to shift each column of the sinogram appropriately.

For completeness, steps of the application in parallel-beam tomography are also depicted in Fig. 1.

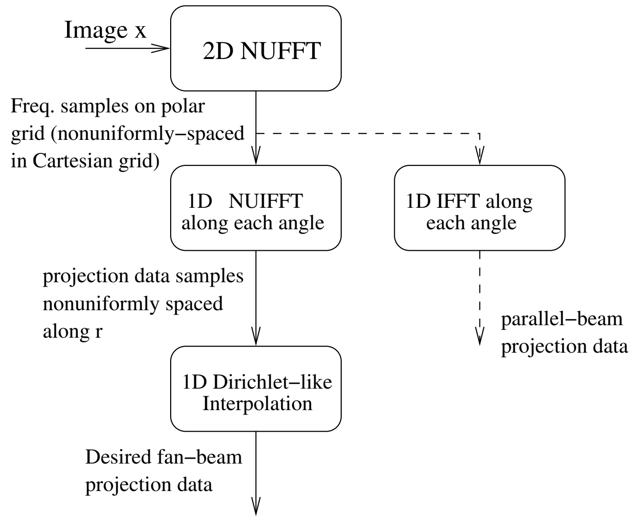

Fig. 1. Basic steps of NUFFT forward projection in fan-beam CT

For iterative algorithms, both forward and back-projection (adjoint of forward projection) are needed. Our approach provides an exact adjoint.

3. SIMULATIONS

3.1. Forward and back-projector as single modules

We evaluated the NUFFT-based fan-beam forward and back-projectors using the Shepp-Logan digital phantom with various image sizes, $N_1 \times N_2 = 128 \times 128, 256 \times 256$ and 384×384. We simulated a 3rd-generation fan-beam X-ray CT system with sinogram size of approximately $2N_1$ radial bins by N_1 views over 360^o. Source to detector distance is 949.075mm and rotation center to detector distance is 408.075mm. Adjoint source positions were incremented by $\triangle_\beta = 2.81$ degrees.

For the Shepp-Logan object shown in Fig. 2, we first computed the exact fan-beam projections analytically to serve as a gold standard. We computed the exact projections by analytically computing the length of each ray's path through the ellipses, each is of constant attenuation coefficients. Then we computed analytically space-based and NUFFT-based projections. Maximum percentage error and root mean square error have been evaluated. For 1D and 2D NUFFT, we used the optimized Kaiser-Bessel interpolation with $K/N = 2$ and $J = 6$. For space-based projections, we computed the area of intersection between each ray beam and each pixel.

Fig. 2 shows the simulation results for forward projector of image size 512×512. We can see that the projections of three different methods are all visually indistinguishable. The accuracy of NUFFT-based method is comparable with space-based method, while the computation

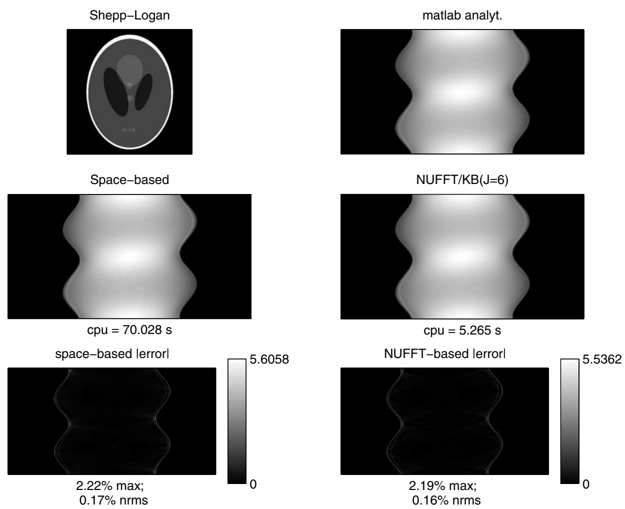

Fig. 2. Simulation results for forward projectors

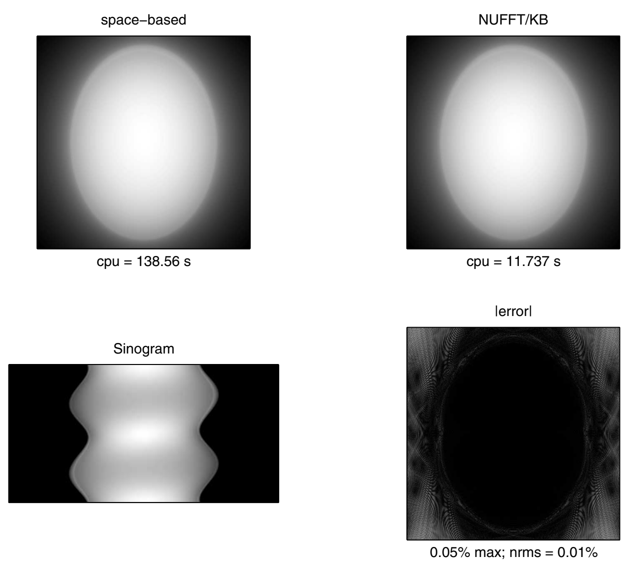

Fig. 3. Simulation results for back-projectors

time (using matlab `tic` command) is about 10 times faster. Fig. 3 shows the simulation results of back-projectors; these are not conventional pixel driven back projectors but rather the exact adjoint of forward projectors mentioned above. We applied both space-based and NUFFT-based methods on the sinogram obtained from the analytical projections. Again, the results are visually indistinguishable and agree with each other with high accuracy. We compared the results only within the field of view.

The computation efficiency of NUFFT-based method is more advantageous with bigger image sizes.

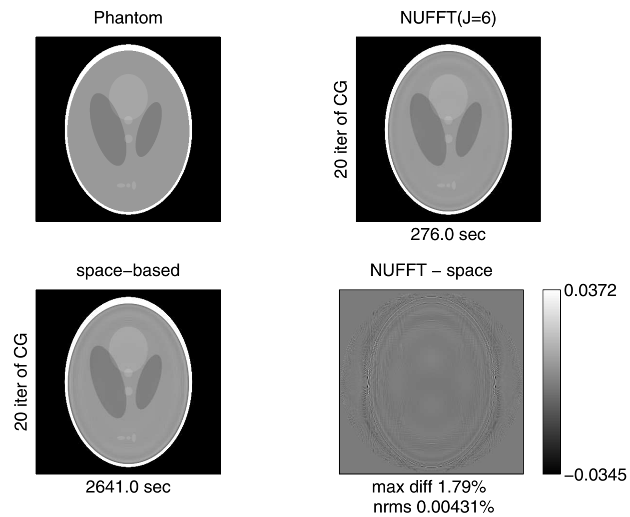

Fig. 4. QPWLS-CG reconstruction (20 iterations) for Shepp-Logan phantom

3.2. Forward and back-projectors within iterative reconstruction

We ran 20 iterations of the conjugate gradient algorithm for a data-weighted least-squares cost function with a quadratic roughness penalty. We ran it with space-based and NUFFT-based forward and back projectors respectively. We used analytical method mentioned above to simulate noiseless sinogram measurements from 384×384 Shepp-Logan object. Sinogram size is of 750 radial bins by 576 views over 360^o.

Fig. 4 shows the preliminary simulation results. The reconstructed images with space-based and NUFFT-based approaches were visually indistinguishable. The max percent difference is less than 2% and normalized rms is about 0.004%, while the computation time significantly reduced for NUFFT approach as compared to space-based method.

For further evaluation, our next step will be to compare these methods within an iterative algorithm, using more realistic phantom and real data, a different roughness penalty such as edge-preserving penalty and a quarter detector offset to reduce aliasing.

4. DISCUSSION

Our results show that the min-max NUFFT approach provides an accurate and efficient method for fan-beam forward and back- projection. The NUFFT-based forward and back-projectors with optimized (in min-max sense) Kaiser-Bessel interpolation kernel are computation efficient and reasonably accurate. The approximation error is low as compared to analytical and space-based projectors.

5. REFERENCES

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