Channelized Hotelling Observer Performance for Penalized-Likelihood Image Reconstruction

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Abstract— **What type of regularization method is optimal for penalized-likelihood image reconstruction when the imaging task is signal detection based on a channelized Hotelling (CHO) observer? To answer such questions, one would like to have a simple analytical expression (even if approximate) for the performance (SNR) of the CHO observer given different reconstruction methods. Bonetto, Qi, and Leahy (IEEE T-NS, Aug. 2000) derived and validated one such expression for penalized-likelihood (aka MAP) reconstruction and the Signal Known Exactly (SKE) problem using linearizations and local shift-invariance approximations. This paper describes a further simplification of the analytical SNR expression for the more general case of a Gaussian-distributed signal. This simplification, based on frequency-domain decompositions, greatly reduces computation time and thus can facilitate analytical comparisons between reconstruction methods in the context of detection tasks. It also leads to the very interesting result that regularization is not essential in this context for a large family of linear observers.**

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I. Introduction

In the design of imaging systems, there are fundamental tradeoffs between spatial resolution and sensitivity. Similarly, in the design of image reconstruction methods, there are tradeoffs between bias and variance. For example, the type of regularization method and the value of the regularization parameter influence this tradeoff for regularized image reconstruction methods. One would like to use relevant and quantitative figures of merit to guide the design of imaging systems and image reconstruction methods.

There has been steadily growing interest in the use of signal detection tasks to optimize design parameters of imaging systems and image reconstruction algorithms. As recent examples, Gifford *et al.* used ROC studies to choose the type and amount of filtering for FBP reconstruction of SPECT images [1] and to compare FBP to OSEM reconstructions with detector response compensation [2]. These studies involved both human observers and mathematical observers. The large time expense of using human observers limits one to small sets of comparisons. When using mathematical observers, if the detection performance has an easily-evaluated mathematical expression, then one can quickly compare imaging systems or reconstruction methods. If no such expression is available, then one must resort to Monte Carlo evaluation of the mathematical observer, which can require very substantial computing time.

Bonetto *et al.* [3] recently presented an accurate approximate mathematical expression for the performance (SNR) of the channelized Hotelling (CHO) observer applied to MAP (aka penalized-likelihood) reconstructed images from emission tomography measurements. This expression was derived using adaptations of several recent publications that analyze the statistical properties of MAP / penalized-likelihood reconstruction methods [4–8]. They used this expression to evaluate the SNR as a function of signal contrast and lesion size. They reported that this evaluation required "on the order of a couple of days" to complete, whereas a Monte Carlo evaluation would have required over 200 days! This is an impressive speedup. However, because it requires the expense of two iterative image reconstructions per SNR calculation, the approach described in [3] would still be inconvenient to use systematically for design in a high-dimensional parameter space, such as when designing a shift-variant regularization method.

Consider the question: "what is the best quadratic regularization method in terms of detection performance?" We know that to provide uniform spatial resolution with a penalized-likelihood reconstruction method, one must use a shift-variant penalty function [5], [9], *i.e.*, a different set of regularization parameters for every pixel. Similarly, to optimize pixel contrast-to-noise ratio, one must use shift-variant regularization [7]. It seems plausible thus that the "best" regularization method for detection performance will also be shift-variant. So the design parameter space is enormous (one or more regularization parameters per pixel). To make progress towards answering these types of questions, we must have a *simple analytical approximation* for detection performance. Towards this end, we developed a frequency-domain approximation suitable for CHO observers. After this work was orginally submitted, we learned that Xing *et al.* [10] had also developed this simplification.

We focus below on the task of detecting a signal characterized by a known Gaussian distribution since it is amenable to analysis. The Signal Known Exactly (SKE) task can be treated as a special case. For simplicity of presentation, we consider the case where the measurement noise is zero-mean and Gaussian with a known covariance matrix that is independent of whether the target signal is present or not. Readers familiar with previous analysis of image reconstruction methods will recognize that it is fairly straightforward to extend analysis of non-white Gaussian noise to the case of Poisson noise using approximations that are quite accurate [4–8].

We analyzed the SNR of four linear mathematical observers, all well-known from the literature [11], [12]: the region-of-interest (ROI) observer, the non-prewhitening (NPW) observer, the Hotelling observer (equivalent to the prewhitening observer only in the SKE case), and the channelized Hotelling observer (CHO), all operating in the reconstructed image domain. We showed that the first three observers (ROI, NPW, and Hotelling) belong to a broader family of observers that we call *Fisher observers* since they involve a Fisher information matrix. We also found that for each observer in this family, there exists an *unregularized* reconstruction method that, when combined with that observer, achieves the ideal SNR among all linear reconstructor and observer pairs for this detection problem. Therefore, in this specific detection problem regularization is apparently inessential for this entire family of observers, so optimizing regularization would not lead to fruitful results. We also showed this to be true under certain circumstances and given certain approximations for the CHO observer.

We will focus hereafter mainly on the CHO observer, since it has been found to better model human observer performance. We will present the main points of our theoretical results and some preliminary experimental testing of our CHO analysis and approximations.

II. Fisher observers and reconstructors

The detection task at hand is to decide between the following pair of hypotheses:

$$\begin{aligned} H_0 &: \quad \boldsymbol{y} = \boldsymbol{\varepsilon} \\ H_1 &: \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{n_d \times n_p}$ is the system model, $\boldsymbol{\varepsilon}$ is zero-mean Gaussian noise with non-white covariance matrix $\boldsymbol{\Pi}$, and $\boldsymbol{x} \in \mathbb{R}^{n_p}$ is the small target signal, also Gaussian with known mean $\bar{\boldsymbol{x}}$ and covariance matrix $\boldsymbol{\Pi}_x$. The SKE task corresponds to a deterministic signal with $\boldsymbol{x} = \bar{\boldsymbol{x}}$ and $\boldsymbol{\Pi}_x = \boldsymbol{0}$. The a priori probabilities of H_1 (signal present) and H_0 (signal absent) are known and equal to p_1 and $1 - p_1$ respectively. A detection method is a map of $\boldsymbol{y}$ into the set $\{H_0, H_1\}$.

The ideal test statistic (aka observer) for this detection task is quadratic in $\boldsymbol{y}$ (in the general case where $\boldsymbol{\Pi}_x \neq \boldsymbol{0}$) and thus difficult to analyze. Nevertheless, the observers most commonly studied in the literature are all linear and we will focus here on linear observers as well. Also, we will focus on observers that are applied not in the sinogram domain but in the reconstructed image domain, as this is the common practice of human observers.

We consider linear reconstruction methods of the form:

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\boldsymbol{y}) = \boldsymbol{B}\boldsymbol{\Pi}^{-1/2}\boldsymbol{y}$$

where $\boldsymbol{B} \in \mathbb{R}^{n_p \times n_d}$. We consider linear observers of the form:

$$\hat{\lambda} = \hat{\lambda}(\hat{\boldsymbol{x}}) = \boldsymbol{w}'\hat{\boldsymbol{x}}$$

for some *template* $\boldsymbol{w} \in \mathbb{R}^{n_p}$.

As a figure of merit we use the Signal-to-Noise Ratio (SNR):

$$\mathrm{SNR}^2 = \frac{\left(E[\hat{\lambda}|H_1] - E[\hat{\lambda}|H_0]\right)^2}{p_1 \operatorname{Var}\left\{\hat{\lambda}|H_1\right\} + (1 - p_1) \operatorname{Var}\left\{\hat{\lambda}|H_0\right\}} \quad (1)$$

It can be shown that for any $(\boldsymbol{w}, \boldsymbol{B})$ pair, the SNR of the linear observer satisfies $\mathrm{SNR}^2 \leq \bar{\boldsymbol{x}}'\check{\boldsymbol{F}}\bar{\boldsymbol{x}} \triangleq \mathrm{SNR}_\circ^2$, where $\check{\boldsymbol{F}} \triangleq \boldsymbol{A}'\check{\boldsymbol{\Pi}}^{-1}\boldsymbol{A}$ has the form of a Fisher inrormation matrix and $\check{\boldsymbol{\Pi}} \triangleq \boldsymbol{\Pi} + p_1\boldsymbol{A}\boldsymbol{\Pi}_x\boldsymbol{A}'$, is the unconditional covariance of $\boldsymbol{y}$, also known as the *intra-class scatter matrix* in pattern classification literature. For the SKE case this reduces to $\check{\boldsymbol{\Pi}} = \boldsymbol{\Pi}$ and $\check{\boldsymbol{F}} = \boldsymbol{F} \triangleq \boldsymbol{A}'\boldsymbol{\Pi}^{-1}\boldsymbol{A}$.

We focus now on the family of *Fisher observers* (for lack of a better term), whose templates have the form:

$$\boldsymbol{w} = \check{\boldsymbol{F}}^p\bar{\boldsymbol{x}}$$

for some $p \in \mathbb{R}$. We allow p to be negative, in which case we interpret $\check{\boldsymbol{F}}^p = (\check{\boldsymbol{F}}^\dagger)^{-p}$, where the superscript "$\dagger$" denotes a pseudo inverse.

We also focus on the family of *Fisher reconstructors*, defined by:

$$\boldsymbol{B} = \check{\boldsymbol{F}}^q\boldsymbol{A}'\check{\boldsymbol{\Pi}}^{-1}\boldsymbol{\Pi}^{1/2}, \quad (2)$$

for some $q \in \mathbb{R}$.

We have shown that when a Fisher observer and a Fisher reconstructor with $p = -q$ are combined, then $\mathrm{SNR} = \mathrm{SNR}_\circ$, *i.e., for any Fisher observer, there is a corresponding Fisher reconstructor that achieves the ideal SNR.* Furthermore, these Fisher reconstructors appear to be devoid of regularization, so it appears that *regularization is inessential* for this large family of linear observers in this specific detection task. The conclusions hold even if the problem has a singular Fisher information matrix, such as when one uses more pixels than sinogram measurements, or when one uses a continuous-to-discrete formulation of the imaging problem.

More specifically, we have shown that for each of the ROI, NPW, and Hotelling observers, there is a Fisher reconstructor that they can be paired with to achieve the maximum $\mathrm{SNR} = \mathrm{SNR}_\circ$ in the detection task at hand. These pairs of observers and reconstruction methods appear in Table I.

TABLE I
Linear Observer and Reconstructor Pairs Achieving Optimal SNR in Gaussian Signal Detection Task

Observer	Best estimator $\hat{\boldsymbol{x}}$	q	Interpretation
ROI	$\boldsymbol{A}'\check{\boldsymbol{\Pi}}^{-1}\boldsymbol{y}$	0	Backprojection
NPW	$(\check{\boldsymbol{F}}^\dagger)^{1/2}\boldsymbol{A}'\check{\boldsymbol{\Pi}}^{-1}\boldsymbol{y}$	-1/2	Partly deconvolved backproj.
Hotelling	$\check{\boldsymbol{F}}^q\boldsymbol{A}'\check{\boldsymbol{\Pi}}^{-1}\boldsymbol{y}$	$\mathbb{R}$	Any Fisher reconstructor

III. Analysis of CHO observer for Fisher reconstructors

The CHO observer passes the reconstructed image $\hat{\boldsymbol{x}}$ through a small set of M bandpass filters that attempt to model the human visual system. Let $\boldsymbol{C} \in \mathbb{C}^{n_p \times M}$ denote the matrix with

mth column equal to the impulse response of the mth bandpass filter centered at the ROI center. The output of this M-channel filter-bank can be expressed:

$$\hat{\boldsymbol{c}} = \hat{\boldsymbol{c}}(\boldsymbol{y}) = \boldsymbol{C}'\hat{\boldsymbol{x}}(\boldsymbol{y}).$$

The CHO observer is essentially the Hotelling observer applied not directly to the reconstructed image $\hat{\boldsymbol{x}}$ but to the channel output vector $\hat{\boldsymbol{c}}$ instead. In other words, the corresponding test statistic is:

$$\begin{aligned}\hat{\lambda}(\boldsymbol{y}) &= \{[p_1 \operatorname{Cov}\{\hat{\boldsymbol{c}}|H_1\} + (1-p_1)\operatorname{Cov}\{\hat{\boldsymbol{c}}|H_0\}]^\dagger \\ &\quad \cdot (E[\hat{\boldsymbol{c}}|H_1] - E[\hat{\boldsymbol{c}}|H_0])\}'\hat{\boldsymbol{c}}(\boldsymbol{y}). \qquad (3)\end{aligned}$$

We now examine the existence of a Fisher reconstructor that would achieve the optimal SNR when paired with the CHO observer. If such a reconstructor existed, it would mean that regularization is not important for this observer either, at least as far as SNR in this detection task is concerned. It can be shown that when a Fisher reconstructor of the form (2) is used, (3) gives:

$$\hat{\lambda}(\boldsymbol{y}) = \bar{\boldsymbol{x}}'\check{\boldsymbol{F}}\check{\boldsymbol{F}}^q\boldsymbol{C}(\boldsymbol{C}'\check{\boldsymbol{F}}^q\check{\boldsymbol{F}}\check{\boldsymbol{F}}^q\boldsymbol{C})^\dagger\hat{\boldsymbol{c}}(\boldsymbol{y}).$$

Using (1), the SNR of this test statistic is shown to be:

$$\mathrm{SNR}^2_{\text{CHO,F}} = \bar{\boldsymbol{x}}'\check{\boldsymbol{F}}\check{\boldsymbol{F}}^q\boldsymbol{C}(\boldsymbol{C}'\check{\boldsymbol{F}}^q\check{\boldsymbol{F}}\check{\boldsymbol{F}}^q\boldsymbol{C})^\dagger\boldsymbol{C}'\check{\boldsymbol{F}}^q\check{\boldsymbol{F}}\bar{\boldsymbol{x}}. \qquad (4)$$

Evaluating the SNR from (4) would be computationally expensive. However, using local shift-invariance approximations, we can derive a much more computationally tractable, Fourier-domain expression for the SNR. As described in [5], we can often find a matrix $\boldsymbol{G}$ for which $\boldsymbol{G}'\boldsymbol{G}$ is approximately shift invariant and $\boldsymbol{F} \approx \boldsymbol{DQ\Lambda Q'D}$ where $\boldsymbol{\Lambda} = \operatorname{diag}\{\lambda_k\}$, λ_k is the frequency response of $\boldsymbol{G}'\boldsymbol{G}$ local to the target signal position, $\boldsymbol{Q}$ is the orthonormal DFT matrix, $\boldsymbol{D} = \operatorname{diag}\{\kappa_k\}$ with:

$$\kappa_k \triangleq \sqrt{\frac{\sum_{i=1}^{n_d}[\boldsymbol{\Pi}^{-1/2}\boldsymbol{A}\boldsymbol{e}_k]_i^2}{\sum_{i=1}^{n_d} g_{ik}^2}},$$

and $\boldsymbol{e}_k$ is the kth unit vector of length n_p. Similarly, we can take $\boldsymbol{\Pi}_x \approx \boldsymbol{QMQ'}$ where $\boldsymbol{M} = \operatorname{diag}\{\mu_k\}$ and μ_k is the frequency response of $\boldsymbol{\Pi}_x$ (*i.e.* the object power spectrum) local to the target signal position. Using these approximations, as well as the probably riskier one $\boldsymbol{Q}'\boldsymbol{D}\boldsymbol{\Pi}_x\boldsymbol{DQ} \approx \boldsymbol{D}^2\boldsymbol{M}$, we can show that:

$$\check{\boldsymbol{F}} \approx \boldsymbol{DQ}\check{\boldsymbol{\Lambda}}\boldsymbol{Q}'\boldsymbol{D}, \qquad (5)$$

where $\check{\boldsymbol{\Lambda}} = \operatorname{diag}\{\check{\lambda}_k\}$ and

$$\check{\lambda}_k \triangleq \frac{\lambda_k}{1 + p_1\kappa_k^2\mu_k\lambda_k},$$

which reduces to $\check{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}$ in the SKE case.

To further simplify this expression, we observe that the channel bandpass filters can also be expressed in terms of the DFT matrix $\boldsymbol{Q}$. Let $\boldsymbol{t}^m \in \mathbb{C}^{n_p}$ denote the frequency response of the mth bandpass filter. Without loss of generality, choosing the target signal center to be the "0" coordinate for the DFT matrix, one can then show that:

$$\boldsymbol{C} = \boldsymbol{QT}, \qquad (6)$$

where the mth column of $\boldsymbol{T}$ is $\boldsymbol{t}^m/\sqrt{n_p}$.

Substituting (5) and (6) in the SNR expression (4) and stretching the approximations even further yields:

$$\begin{aligned}\mathrm{SNR}^2_{\text{CHO,F}} &\approx \boldsymbol{X}'D^{2+2q}\check{\boldsymbol{\Lambda}}^{1+q}\boldsymbol{T}(\boldsymbol{T}'D^{2+4q}\check{\boldsymbol{\Lambda}}^{1+2q}\boldsymbol{T})^\dagger \\ &\quad \cdot \boldsymbol{T}'D^{2+2q}\check{\boldsymbol{\Lambda}}^{1+q}\boldsymbol{X},\end{aligned}$$

where $\boldsymbol{X} = \boldsymbol{Q}'\bar{\boldsymbol{x}}$ is spectrum of the mean target signal.

Furthermore, note that the bandpass filters used in CHO observers often have non-overlapping passbands. (This simplification is not essential.) In this case, the $M \times M$ matrix $\boldsymbol{T}'D^{2+4q}\check{\boldsymbol{\Lambda}}^{1+2q}\boldsymbol{T}$ is diagonal, and we can prove the following simplified the SNR expression and bound:

$$\begin{aligned}\mathrm{SNR}^2_{\text{CHO,F}} &\approx \sum_{m=1}^{M} \frac{\left|\sum_k X_k t_k^m \kappa_k^{2+2q}\check{\lambda}_k^{1+q}\right|^2}{\sum_k |t_k^m|^2\kappa_k^{2+4q}\check{\lambda}_k^{1+2q}} \qquad (7) \\ &\le \sum_k |X_k|^2\kappa_k^2\check{\lambda}_k \approx \bar{\boldsymbol{x}}'\check{\boldsymbol{F}}\bar{\boldsymbol{x}} = \mathrm{SNR}^2_\circ.\end{aligned}$$

The expression in (7) captures the properties of the imaging system, the reconstruction method, the target signal, and the observer. There are two obvious cases where we have found Fisher reconstructors that achieve the SNR upper bound above. This happens when each channel filter has a flat passband, *i.e.*, $t_k^m = 1_{\{k \in \mathcal{T}_m\}}$ and either of the following is true:

- The X_k's are constant over each passband, in which case $q = 0$ is optimal or
- The κ_k's and $\check{\lambda}_k$'s are constant over each passband, in which case any $q \in \mathbb{R}$ is optimal (regardless of $\bar{\boldsymbol{x}}$).

In practice, it may be unlikely that either the κ_k's and $\check{\lambda}_k$'s or the X_k's are *exactly* uniform over each channel's passband, but if the passbands are reasonably narrow, then it is likely that these spectra will be *approximately* uniform over each channel. So to within the accuracy of the approximations considered above, one or more Fisher reconstructors will nearly achieve the highest SNR obtainable in the specified detection task for the given CHO channels. Once again, it appears that regularization does not seem to have a dominant role, even for the CHO observer.

IV. Analysis of CHO observer for PWLS

Although regularization does not seem to be too significant in maximing the SNR of the CHO observer, it is still interesting to derive expressions for the SNR of this observer when paired with regularized reconstruction. For reasons of simplicity, we focus here on the penalized weighted least-squares (PWLS) reconstruction method; extension to penalized-likelihood methods is fairly straightforward [4], [7].

An unconstrained PWLS estimator has the form:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \frac{1}{2}(\boldsymbol{y} - \boldsymbol{Ax})'\boldsymbol{W}(\boldsymbol{y} - \boldsymbol{Ax}) + \frac{1}{2}\boldsymbol{x}'\boldsymbol{Rx},$$

for some regularization matrix $\boldsymbol{R}$ that we would like to design to optimize detectability. The usual weight matrix $\boldsymbol{W} = \boldsymbol{\Pi}^{-1}$ is appropriate for the SKE case but to accomodate signal variability and for ease of analysis we use $\boldsymbol{W} = \check{\boldsymbol{\Pi}}^{-1}$. In this case, the estimator is easily shown to be:

$$\hat{\boldsymbol{x}} = [\check{\boldsymbol{F}} + \boldsymbol{R}]^{-1} \boldsymbol{A}' \check{\boldsymbol{\Pi}}^{-1} \boldsymbol{y}.$$

Using (1), we can show that the SNR of the CHO observer for the PWLS reconstruction method is:

$$\mathrm{SNR}^2_{\text{CHO,PWLS}} = \bar{\boldsymbol{x}}' \check{\boldsymbol{F}}^{1/2} \boldsymbol{S}' (\boldsymbol{S}\boldsymbol{S}')^{\dagger} \boldsymbol{S} \check{\boldsymbol{F}}^{1/2} \bar{\boldsymbol{x}},$$

where $\boldsymbol{S} \triangleq \boldsymbol{C}' [\check{\boldsymbol{F}} + \boldsymbol{R}]^{-1} \check{\boldsymbol{F}}^{1/2} \in \mathbb{C}^{M \times n_p}$. Using local shift-invariance approximations we can write $\boldsymbol{F} \approx \boldsymbol{D}\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'\boldsymbol{D}$ as before, and also $\boldsymbol{R} \approx \boldsymbol{Q}\boldsymbol{\Omega}\boldsymbol{Q}'$, where $\boldsymbol{\Omega} = \operatorname{diag}\{\omega_k\}$ and ω_k is the frequency response of the regularizer local to the target signal. These approximations were used by Bonetto *et al.* [3] to partially simplify the alternate SNR expression: $\mathrm{SNR}^2 = \hat{\lambda}(E[\boldsymbol{y}|H_1] - E[\boldsymbol{y}|H_0])$.

We further simplify as in the previous section by using $\boldsymbol{C} = \boldsymbol{Q}\boldsymbol{T}$, $\boldsymbol{D}\boldsymbol{Q}'\boldsymbol{\Lambda}\boldsymbol{Q}\boldsymbol{D} \approx \boldsymbol{D}^2\boldsymbol{\Lambda}$ and assuming nonoverlapping passbands. Then we can derive the following form for the SNR:

$$\mathrm{SNR}^2_{\text{CHO,PWLS}} \approx \sum_{m=1}^{M} \frac{\left| \sum_k X_k t_k^m \frac{\kappa_k^2 \check{\lambda}_k}{\kappa_k^2 \check{\lambda}_k + \omega_k} \right|^2}{\sum_k |t_k^m|^2 \frac{\kappa_k^2 \check{\lambda}_k}{|\kappa_k^2 \check{\lambda}_k + \omega_k|^2}}.$$

With this expression it is feasible to begin tackling questions like finding the best regularization parameter in terms of a detection task. The dependence of the SNR on the mean signal spectrum is somewhat unappealing, since in practice the mean target signal is not exactly known so it would seem somewhat unrealistic to optimize the regularization method for a particular signal. Instead, we consider the case where the size of the target signal is very small compared to the system spatial resolution. In such cases, the numerator of the SNR expression above will be dominated by the decay of the $\check{\lambda}_k$ terms. (In the limiting case of a point-source target, the spectrum X_k would be a constant.) Thus we propose to focus on the following *relative SNR*:

$$\mathrm{SNR}^2_{\text{CHO,PWLS,rel}} \approx \left(\frac{1}{\sum_k \check{\lambda}_k} \right) \sum_{m=1}^{M} \frac{\left| \sum_k t_k^m \frac{\kappa_k^2 \check{\lambda}_k}{\kappa_k^2 \check{\lambda}_k + \omega_k} \right|^2}{\sum_k |t_k^m|^2 \frac{\kappa_k^2 \check{\lambda}_k}{|\kappa_k^2 \check{\lambda}_k + \omega_k|^2}}, \quad (8)$$

where the normalization term is the approximate $\mathrm{SNR}^2_{\text{CHO,PWLS}}$ when the CHO observer has all n_p ideal channels. This relative SNR is at most unity.

V. Preliminary results

A. SNR approximation error

We considered the case where $\boldsymbol{A}$ is a 2-D SPECT system model with depth-dependent system blur and image size 32×32, the mean target signal $\bar{\boldsymbol{x}}$ is a small Gaussian bump and and the target signal covariance $\boldsymbol{\Pi}_x$ is generated by a Gaussian autocorrelation fuction (ACF). We generated a measurement noise covariance $\boldsymbol{\Pi} = \operatorname{diag}\{\boldsymbol{A}\boldsymbol{x}^{\text{true}}\}$ by taking the noiseless projection of an anthropomorphic phantom $\boldsymbol{x}^{\text{true}}$. We used dyadic constant-Q bandpass filters as the CHO channels. We evaluated the SNR for the combination of the CHO observer with a Fisher reconstructor for various values of q, using both the exact (4) and the approximate (7) expression. For each different reconstructor we evaluated the SNR while increasing the spread of the target signal ACF (thus increasing the correlation between adjacent pixel values). The relative error of the approximation with respect to the true value is plotted in Figure 1 versus the correlation coefficient of adjacent signal pixels. The plot shows that even though the system model is shift-variant and thus its Fourier expansion is not exact, the approximation error is within 10% for $q = -0.5, 0, 0.5$ for a significant range of correlation coefficients. The accuracy is lower for larger q's but the $q \leq 0$ cases are perhaps of greater interest.

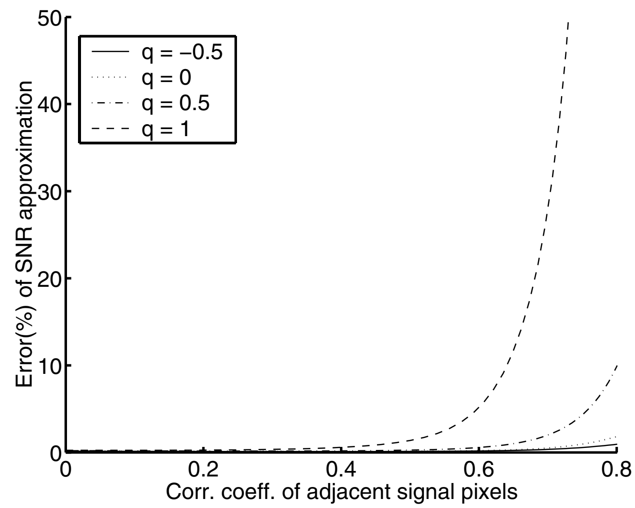

Fig. 1. SNR approximation error versus correlation coefficient of adjacent signal pixels for the CHO observer applied to several Fisher reconstructors.

B. Choosing the regularization parameter

To illustrate using the relative SNR approximation in (8) to choose the regularization parameter β in penalized-likelihood image reconstruction, we used the same system model as above (with image size 64×64), an impulse as the mean target signal, and the same $\boldsymbol{\Pi}_x$ and $\boldsymbol{\Pi}$ as above. We evaluated the expression in (8) for PWLS with first-order quadratic regularization and various values of the regularization parameter β. We considered various bandpass filter combinations for the CHO observer. Figure 2 shows the relative SNR vs the regularization parameter β for two representative cases: dyadic constant-Q bandpass filters, and a single allpass channel. Such plots can be created in seconds using the above simple SNR expression. The plots show that for the single allpass channel the relative SNR varies significantly with the amount of regularization. Nevertheless, for the more useful case of the dyadic bandpass filters the SNR curve stays relatively flat, which reinforces our conclusion that regularization is not important in this detection problem.

VI. Summary

The formulas we derived above should facilitate theoretical comparisons between different reconstruction methods for CHO observers much as Qi and Huesman described for the PW and NPW observers [8].

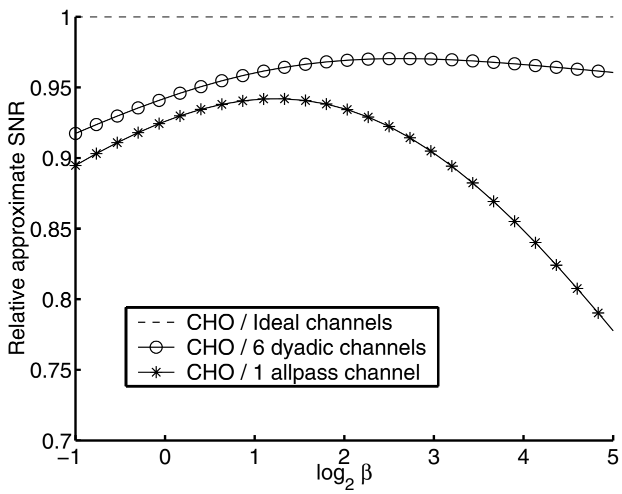

Fig. 2. Relative approximate SNR versus β for the CHO observer applied to PWLS reconstruction.

Our analysis shows that regularization is inconsequential for the ROI, NPW and Hotelling observers and, to within the accuracy of certain approximations, it has limited impact for the CHO observer. Preliminary results from testing these approximations show that they can be trusted for a variety of cases.

This may well mean that that the specific detection task is too simplistic to capture the benefits of regularization in practical clinical diagnosis. For that, the use of more realistic task models such as uncertainty in the signal location might be in order. Additional future work includes examining the usefulness of our approximations when the measurements are Poisson-distributed, as well as the usefulness of regularization when combined with "naive" reconstruction methods that do not have prior knowledge of second-order target signal statistics, as would normally be the case in practice.

Finally, it is worth noting that for the different observers we examined, the corresponding optimal reconstruction methods were also different. Thus, clearly there is no universally optimal reconstruction method: it will depend on both the task *and* the form of the observer under consideration.

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