Regularization Parameter Selection for Nonlinear Iterative Image Restoration and MRI Reconstruction Using GCV and SURE-Based Methods—Supplementary Material

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We present a proof for Lemma 1 and show additional illustrations that we could not accommodate in the paper due to space constraints. References to equations, figures, etc. are within this material only unless specified otherwise.

I. LEMMA 1 IN THE PAPER

Lemma 1: Let \( f_{\lambda} : \Omega^M \rightarrow \Omega^N \) be differentiable (for \( \Omega = \mathbb{R} \)) or individually analytic (for \( \Omega = \mathbb{C} \) with respect to real and imaginary parts of its argument), respectively, in the weak sense. Then, for any deterministic \( T \in \Omega^{M \times N} \) satisfying \( E_x \{ ||T f_{\lambda}(y)||_m || \} < \infty \), \( m = 1 \ldots M \), we have for \( \Omega = \mathbb{R} \) or \( \mathbb{C} \) that

\[
E_x \{ \xi^H T f_{\lambda}(y) \} = \sigma^2 E_x \{ \text{tr} \{ T J_{f_{\lambda}}^\Omega(y) \} \}. \tag{1}
\]

Proof: The proof for \( \Omega = \mathbb{R} \) is very similar to those in [1] Th. 1, [2] Lemma 1, so we focus on the case of \( \Omega = \mathbb{C} \). The density function \( g_{\mathbb{C}}(\xi) = (\pi \sigma^2)^{-M} \exp(-\xi^H \xi / \sigma^2) \) is analytic with respect to \( \xi \) and \( \xi^* \) individually and satisfies the identity

\[
\xi^H g_{\mathbb{C}}(\xi) = -\frac{\sigma^2}{2} [\nabla_{\xi^x} - i\nabla_{\xi^y}] g_{\mathbb{C}}(\xi), \tag{2}
\]

where \( \nabla_{\xi^x} \) and \( \nabla_{\xi^y} \) denote \( 1 \times M \) gradient operators consisting partial derivatives with respect to
\[
\{\xi_{\pi m}\}_{m=1}^{M}, \{\xi_{\pi m}\}_{m=1}^{M}.
\]
Expanding the LHS of (1) and using (2), we get that
\[
E_\xi\{\xi^H T_f(y)\} = \int g_C(\xi) \xi^H T_f(y) \, d\xi_{\pi} \, d\xi_x = -\frac{g^2}{2} \int \left( \nabla_{\xi_{\pi}} - \iota \nabla_{\xi_{\pi}} \right) g_C(\xi) \, T_f(y) \, d\xi_{\pi} \, d\xi_x. \tag{3}
\]
Integrating-by-parts the term involving \( \nabla_{\xi_{\pi}} \) in (3) and using the fact that \( \lim_{|\xi_{\pi m}| \to \infty} g_C(\xi) [T_f(y)]_{m} = 0 \), when \( E_\xi\{|[T_f(y)]_m|\} < +\infty \) \( \text{1, 2} \), we get that
\[
\int \nabla_{\xi_{\pi}} g_C(\xi) \, T_f(y) \, d\xi_{\pi} \, d\xi_x = \sum_{m=1}^{M} \sum_{n=1}^{N} \int \frac{\partial g_C(\xi)}{\partial \xi_{\pi m}} \, T_{mn} \, f_{\lambda,n}(y) \, d\xi_{\pi} \, d\xi_x
\]
\[
= -\sum_{m=1}^{M} \sum_{n=1}^{N} \int g_C(\xi) \, T_{mn} \, \frac{\partial f_{\lambda,n}(y)}{\partial \xi_{\pi m}} \, d\xi_{\pi} \, d\xi_x
\]
\[
= -\sum_{m=1}^{M} \sum_{n=1}^{N} \int g_C(\xi) \, T_{mn} \, \frac{\partial f_{\lambda,n}(y)}{\partial y_{\pi m}} \, d\xi_{\pi} \, d\xi_x, \tag{4}
\]
where we have set \( \partial/\partial \xi_{\pi m} = \partial/\partial y_{\pi m} \) since \( Ax \) (Eq. (1) in the paper) is a deterministic constant. Going through a similar derivation for the integral involving \( -\iota \nabla_{\xi_{\pi}} \) and using \( \partial/\partial \xi_{\pi m} = \partial/\partial y_{\pi m} \), we get that
\[
-\iota \int \nabla_{\xi_{\pi}} g_C(\xi) \, T_f(y) \, d\xi_{\pi} \, d\xi_x = \iota \sum_{m=1}^{M} \sum_{n=1}^{N} \int g_C(\xi) \, T_{mn} \, \frac{\partial f_{\lambda,n}(y)}{\partial y_{\pi m}} \, d\xi_{\pi} \, d\xi_x. \tag{5}
\]
Combining (3)–(5) with the definition of \( J_{\lambda}^\Omega(y) \) (Eq. (4) in the paper) yields the desired result (1).
II. ADDITIONAL ILLUSTRATIONS FOR IMAGE RESTORATION

Fig. 2 plots Projected-MSE(\(\lambda\)), Projected-SURE(\(\lambda\)), Predicted-MSE(\(\lambda\)) and Predicted-SURE(\(\lambda\)) versus \(\lambda\) for specific instances of Experiments IR-A and IR-B (corresponding to Cameraman and House test images, respectively, see Table I in the paper). Projected-SURE- and Predicted-SURE-curves are accurate in capturing the trends of Projected-MSE- and Predicted-MSE-curves and also exhibit minima (indicated by *) close to that of MSE (solid vertical line). The deblurred images (corresponding to an instance of Experiment IR-B) obtained by minimizing Projected-SURE(\(\lambda\)) (Fig. 2d), Predicted-SURE(\(\lambda\)) (Fig. 2e) and NGCV(\(\lambda\)) (Fig. 2f) closely resemble the corresponding minimum-MSE result (Fig. 2c) in terms of
### III. ADDITIONAL ILLUSTRATIONS FOR MRI RECONSTRUCTION

We evaluated Predicted-MSE($\lambda$), Predicted-SURE($\lambda$) as functions of $\lambda$ and plotted them in Figs. 3a and 4 for specific instances of Experiments MRI-A (with synthetic data corresponding to noise-free Shepp-Logan phantom), MRI-B (with synthetic data corresponding to a noise-free $T_2$-weighted MR image)
Fig. 5. Experiment MRI-A with Shepp-Logan phantom (corresponding to second row of Table VI: (a) Noisefree Shepp-Logan (256 x 256) phantom; (b) Retrospective sampling along radial lines (30 lines on a Cartesian grid with 89% undersampling, black lines indicate sample locations); (c) Magnitude of zero-filled iFFT reconstruction from undersampled data; and magnitude of TV-regularized reconstructions with regularization parameter $\lambda$ selected to minimize (d) (oracle) MSE (22.28 dB); (e) Predicted-SURE (22.21 dB); (f) NGCV (21.68 dB). Regularized reconstructions (d)-(f) have reduced artifacts compared to the zero-filled iFFT reconstruction (c). Predicted-SURE-based and NGCV-based results (e), (f), respectively, closely resemble the oracle MSE-based result (d) in this experiment.

and MRI-C (with real GE phantom data) in the paper. Predicted-SURE not only captures the trend of Predicted-MSE in Figs. 3a and 4 but also exhibits minima (indicated by *) close to that of MSE (solid vertical line).

We also plot PSNR($\lambda$) versus $\lambda$ in Fig. 3b and present reconstructions in Fig. 5 for the instance of Experiment MRI-A considered in Figs. 3b: Although NGCV-selection is slightly away from the (oracle) MSE-selection in Fig. 3b, the corresponding reconstruction Fig. 5f is visually similar to the MSE-based one in Fig. 5d in this case. Predicted-SURE-selection is close to the (oracle) MSE-selection in Fig. 3b and therefore naturally leads a reconstruction Fig. 5e that resembles the MSE-based one in Fig. 5d. These illustrations point to (the sub-optimality of NGCV and) the accuracy of Predicted-SURE($\lambda$) for MRI reconstruction from partially sampled Cartesian $k$-space data and are consistent with our results portrayed in the paper.
REFERENCES
