ELSEVIER

Nuclear Instruments and Methods in Physics Research A 488 (2002) 362–374

NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH Section A

www.elsevier.com/locate/nima

Mean and variance of coincidence counting with deadtime

Daniel F. Yu, Jeffrey A. Fessler*

Electrical Engineering and Computer Science, The University of Michigan, 4240 EECS Bldg, 1301 Beal Aver., Ann Arbor, MI 48109-2122, USA

Received 21 August 2000; received in revised form 12 September 2001; accepted 2 January 2002

Abstract

We analyze the first and second moments of the coincidence-counting process for a system affected by paralyzable (extendable) deadtime with (possibly unequal) deadtimes in each singles channel. We consider both "accidental" and "genuine" coincidences, and derive exact analytical expressions for the first and second moments of the number of recorded coincidence events under various scenarios. The results include an exact form for the coincidence rate under the combined effects of decay, background, and deadtime. The analysis confirms that coincidence counts are not exactly Poisson, but suggests that the Poisson statistical model that is used for positron emission tomography image reconstruction is a reasonable approximation since the mean and variance are nearly equal. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 29.40.−n; 29.40.Mc; 02.50.Cw; 87.59.Vb

Keywords: Deadtime; Coincidence counting; Counting processes

1. Introduction

When counting ionizing particles, such as annihilation photons in positron emission tomography (PET) imaging, every measurement system exhibits a characteristic called *deadtime*. Since the pulses produced by a radiation detector have finite time duration, when a second pulse occurs before the first has disappeared, the two pulses will overlap to form a single distorted pulse [1,2]. Depending on the system, one or both particle arrivals will go unrecorded. This loss of counts changes the measurement statistics, in general, and the statistical moments in particular. The statistics of the coincidence-counting process for detectors affected by deadtime is of fundamental importance to the problem of statistical image reconstruction, as elaborated in Ref. [3] for single-photon counting systems.

Counting systems are often characterized as either non-paralyzable or paralyzable (extendable). In a paralyzable system, each particle arrival, whether recorded or not, produces a deadtime of length τ; if there is an arrival at time t, then any arrival from t to $t+\tau$ will go unrecorded. This paper focuses on the *paralyzable* case. Various cases are considered in previous work on counting processes affected by

*Corresponding author.

E-mail address: fessler@umich.edu (J.A. Fessler).

0168-9002/02/$ - see front matter
PII: S0168-9002(02)00460-6

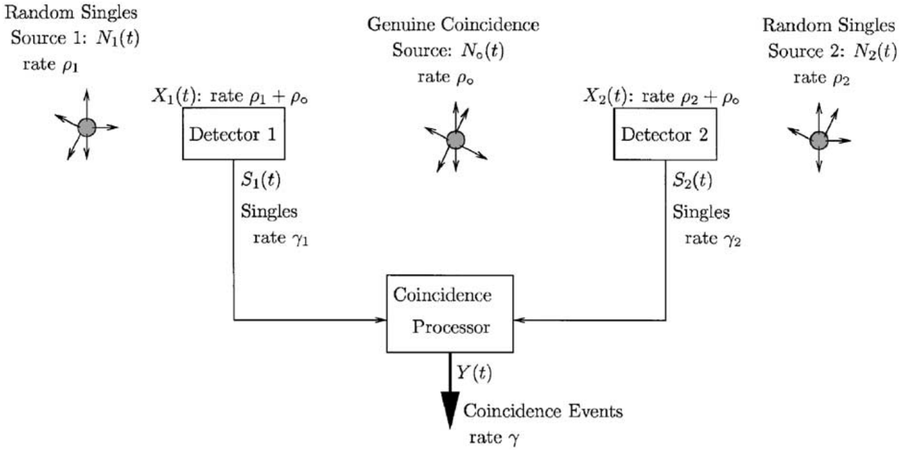

Fig. 1. Model for coincidence counting.

deadtime, e.g., Refs. [4–8]. We mostly follow the terminology used in the recent ICRU report on particle counting [2]. This paper presents rigorous derivations of the mean and variance of coincidence-counting processes, first for accidental coincidences, then for total coincidences (both accidental and genuine), also known as *prompt* coincidences in the PET literature. Our approach is sufficiently general to allow consideration of the combined effects of decay, background, and deadtime, a previously unsolved problem in particle counting [2, p. 27].

2. Preliminaries

We consider the scenario illustrated in Fig. 1. The random process $N_0(t)$ counts the number of particle pairs emitted simultaneously from the ‘‘genuine-coincidence source’’ and arriving at the two detectors in the time interval [0, t]. For simplicity, we assume that the particles in each N_0 pair arrive simultaneously at the two detectors.[1] The random processes $N_1(t)$ and $N_2(t)$ count particles originating from the random singles sources as they *arrive* at the two detectors. We assume that $N_0(t)$, $N_1(t)$, and $N_2(t)$ are statistically independent. To preclude the possibility of multiple simultaneous arrivals, we assume that any arrival process $N_i(t)$ considered in this paper has the property $P[\Delta N_i(t) \geqslant 1] = 0$ for any time t, where $\Delta N_i(t)$ denotes the increment in $N_i(t)$ at time t. For some of the results, but not all, we also assume that the arrival processes are Poisson.

2.1. Measurement model

We assume that the arrival processes at the two detectors satisfy the following model:

$$X_1(t) = N_0(t) + N_1(t)$$
$$X_2(t) = N_0(t) + N_2(t). \tag{1}$$

We let ρ_0, ρ_1, and ρ_2 denote the counting rates of $N_0(t)$, $N_1(t)$, and $N_2(t)$, respectively. For notational simplicity, we absorb the detector counting efficiencies into these rates, following Ref. [2, Eq. (5.1)]. Due to

[1] With additional effort, one can apply the analytical methods described in this paper to derive results for non-simultaneous arrivals, such as when ‘‘time of flight’’ is considered in PET imaging.

the common source N_0, the processes X_1 and X_2 are statistically dependent (unless $\rho_0 = 0$, i.e., the pure "accidental"-coincidence case).

The arrival processes X_1 and X_2 are recorded by two respective singles detectors according to the extendable deadtime model. Let $S_1(t)$ and $S_2(t)$ denote the number of recorded singles at detectors 1 and 2, respectively. We assume that the deadtimes τ_1 and τ_2 are known and deterministic for each singles detector. We ignore uncertainty in the time-stamping of recorded singles [9]. Mathematically

$$\Delta S_i(t) = 1 \text{ if and only if } \Delta X_i(t) \geqslant 1 \text{ and } X_i(t-\tau_i, t) - \Delta X_i(t) = 0$$

where $X(s,t) \triangleq X(t) - X(s)$ denotes the number of increments in the half-open interval $(s,t]$.

Of principal interest, here, is the number of coincidence events recorded in the interval $(0,t]$, denoted by $Y(t)$. We define a "genuine"-coincidence event to be one in which both particles involved originated from the "genuine"-coincidence source. All other coincidence events are defined as "accidental" ones. Let r denote the length of the coincidence timing window (resolving time). We assume $2r < \min\{\tau_1, \tau_2\}$. For a pair of particles to be recorded as a coincidence event, both particles must first be recorded by their respective detectors. If one particle is recorded by detector 1 at time t_1, and another particle is recorded by detector 2 at time t_2, and if $|t_1 - t_2| < r$, then this pair of particles will be recorded as a coincidence event. To avoid ambiguity, we define the time of coincidence to be the arrival time of the later particle. If one particle is recorded by detector 1 at time t_1 and no particle is recorded by detector 2 at time t_1, then the number of coincidences recorded at time t_1 is $S_2(t_1 - r, t_1)$, i.e., the number of particles recorded by detector 2 during $(t_1 - r, t_1]$. If each detector records a particle at the same time t_1, then the number of coincidences recorded at time t_1 is $S_1(t_1 - r, t_1) + S_2(t_1 - r, t_1) - 1$. Mathematically

$$\Delta Y(t) = \Delta S_1(t) S_2(t-r,t) + \Delta S_2(t) S_1(t-r,t) - \Delta S_1(t) \cdot \Delta S_2(t).$$

These conventions define $Y(t)$, and our goal is to study its statistics.

2.2. Analysis tools

Our analysis method is based on integrating the "instantaneous" properties of the counting process $Y(t)$ following Ref. [3]. We define the instantaneous rate $\gamma : \mathbb{R} \to [0, \infty)$ as

$$\gamma(s) \triangleq \lim_{\delta \to 0} \frac{E[Y(s+\delta) - Y(s)]}{\delta}, \tag{2}$$

the instantaneous second moment $\alpha : \mathbb{R} \to [0, \infty)$ as

$$\alpha(s) \triangleq \lim_{\delta \to 0} \frac{E[(Y(s+\delta) - Y(s))^2]}{\delta}, \tag{3}$$

and the correlation function $\beta : \mathbb{R}^2 \to [0, \infty)$ as

$$\beta(s_1, s_2) \triangleq \lim_{\delta_1, \delta_2 \to 0} \frac{E[(Y(s_1+\delta_1) - Y(s_1))(Y(s_2+\delta_2) - Y(s_2))]}{\delta_1 \delta_2}. \tag{4}$$

Consider counting processes satisfying the following assumptions:

(i) $\gamma(\cdot)$ and $\alpha(\cdot)$ are well-defined and finite everywhere, $\beta(s_1, s_2)$ is well-defined and finite everywhere $s_1 \neq s_2$;

(ii) $E[Y(s, s+\delta)]/\delta$ and $E[Y^2(s, s+\delta)]/\delta$ are uniformly bounded for all s and $\delta \in (0,1)$;

(iii) $E[Y(s_1, s_1+\delta_1) Y(s_2, s_2+\delta_2)]/(\delta_1 \delta_2)$ is uniformly bounded for all s_1, s_2, and $\delta_1, \delta_2 \in (0,1)$ such that $(s_1, s_1+\delta_1) \bigcap (s_2, s_2+\delta_2) = \emptyset$.

In Ref. [3], we derived the following general result for the first and second moments:

$$E[Y(t)] = \int_0^t \gamma(s)\,\mathrm{d}s \tag{5}$$

$$E[Y^2(t)] = \int_0^t \alpha(s)\,\mathrm{d}s + 2\int_0^t \int_{s_1}^t \beta(s_1, s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{6}$$

If $Y(t)$ has stationary increments, then we have the following simplification:

$$E[Y(t)] = \gamma t \tag{7}$$

$$E[Y^2(t)] = \alpha t + 2\int_0^t (t-s)\beta(0,s)\,\mathrm{d}s \tag{8}$$

where $\gamma = \gamma(0)$ and $\alpha = \alpha(0)$. For counting processes with deadtime that satisfy this additional assumption:

(iv) there exists a positive δ_0 such that $\forall \delta \in (0, \delta_0),\ Y(s, s+\delta) \leqslant 1$,
we obtained the following corollary [3]:

$$\alpha(s) = \gamma(s)$$
$$E[Y^2(t)] = E[Y(t)] + 2\int_0^t \int_{s_1}^t \beta(s_1, s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{9}$$

All random processes considered in this paper have stationary increments and satisfy assumptions (i)–(iv), except for the accidental-coincidence process with ideal detectors considered in Section 3.2 which does not satisfy assumption (iv); hence we use the more general result (6) for its second moment. Using the above expressions, we can find the mean and variance of a counting process $Y(t)$ by finding the instantaneous functions $\alpha(s)$, $\gamma(s)$, and $\beta(s_1, s_2)$ and then integrating.We also make the following assumptions about the singles processes S_i:

(v) $\lim_{\delta \to 0} \frac{1}{\delta} P[S_i(s, s+\delta) = k] = 0,\ \forall s,\ \forall k \geqslant 2$,
(vi) $\exists \delta_0 > 0, \sum_{k=2}^{\infty} k^i \bar{p}(k, \delta_0) < \infty$ for $i = 1, 2$, or 3,
where $\bar{p}(k, \delta_0) \triangleq \sup\{\frac{1}{\delta} P[S_i(s, s+\delta) = k] : s \in [0, t), \delta \in (0, \delta_0)\}$. These assumptions hold for a wide variety of (singles) counting processes, including all (singles) processes considered hereafter. Specifically with regard to assumption (vi), if $S(t)$ is a Poisson process that satisfies assumption (ii) with $\frac{1}{\delta} E[S(s, s+\delta)] \leqslant \rho_{\max}$, then

$$\frac{1}{\delta} P[S(s, s+\delta) = k] = \frac{1}{\delta} \mathrm{e}^{-E[S(s,s+\delta)]} (E[S(s, s+\delta)])^k / k! \leqslant \rho_{\max}^k \delta^{k-1} / k! < \rho_{\max}^k / k!$$

for $\delta < 1$, and $\sum_{k=2}^{\infty} k^n \rho_{\max}^k / k! < \infty$ for any integer n, establishing (vi). This argument applies to all Poisson processes with bounded intensity. A process $S_i(t)$ that satisfies (v) and (vi) has the following property for its instantaneous rate:

$$\begin{aligned} \gamma_i(s) &= \lim_{\delta \to 0} \frac{1}{\delta} E[S_i(s, s+\delta)] \\ &= \lim_{\delta \to 0} \frac{1}{\delta} \left[0 + P[S_i(s, s+\delta) = 1] + \sum_{k=2}^{\infty} k P[S_i(s, s+\delta) = k] \right] \\ &= \lim_{\delta \to 0} \frac{1}{\delta} P[S_i(s, s+\delta) = 1]. \end{aligned} \tag{10}$$

3. Moments of the accidental-coincidence process

This section derives the mean and variance of the accidental-coincidence process (assuming $\rho_0 = 0$), first for the case of ideal detectors (no deadtime), and then for realistic detectors (with deadtime). The next section gives the mean and variance (bounds) of the counting process having both genuine and accidental coincidences. Let $Y_a(t)$ and $Y_b(t)$ denote the number of recorded coincidence events during (0, t] that have the later singles event arriving at detectors 1 and 2, respectively. Then the number of coincidence events recorded during (0, t] is

$$Y(t) = Y_a(t) + Y_b(t) + Y_0(t) \tag{11}$$

where $Y_0(t)$ denotes the number of recorded coincidence events corresponding to simultaneous singles arrivals at the two detectors. Since the singles processes $S_1(t)$ and $S_2(t)$ are independent when $\rho_0 = 0$, one can show $Y_0(t)$ is zero with probability one. Thus, we ignore $Y_0(t)$ hereafter since it does not affect the moments of $Y(t)$.

3.1. Mean of accidental coincidences

For the purely accidental-coincidence process, we derive the mean of $Y(t)$ for a general class of recorded singles processes, i.e., we do not assume any particular statistical model such as Poisson. For this derivation, we assume only that the singles-counting processes $S_1(t)$ and $S_2(t)$ are independent and satisfy assumptions (i)–(iii), (v), and (vi). Let $\gamma_i(s) \triangleq \lim_{\delta\to 0}(1/\delta)E[S_i(s, s+\delta)]$, $\quad i = 1, 2$ denote the instantaneous rates of the singles-counting processes.

For a time interval $(s, s+\delta]$, by total probability, we have

$$E[Y_a(s, s+\delta)] = M_1(s, \delta) + M_2(s, \delta)$$

where

$$M_1(s, \delta) \triangleq E[Y_a(s, s+\delta) | S_1(s, s+\delta) = 1] P[S_1(s, s+\delta) = 1]$$

$$M_2(s, \delta) \triangleq \sum_{k=2}^{\infty} E[Y_a(s, s+\delta) | S_1(s, s+\delta) = k] P[S_1(s, s+\delta) = k].$$

If detector 1 records $k \geqslant 0$ singles during $(s, s+\delta]$, then the number of coincidence events recorded during that interval with the later singles event recorded by detector 1 can be no more than $kS_2(s-r, s+\delta)$. Hence, for any $\delta > 0$

$$\begin{aligned} 0 \leqslant M_2(s, \delta) &\leqslant \sum_{k=2}^{\infty} kE|S_2(s-r, s+\delta)] P[S_1(s, s+\delta) = k] \\ &= \sum_{k=2}^{\infty} k \left(\int_{s-r}^{s+\delta} \gamma_2(u)\, \mathrm{d}u \right) P[S_1(s, s+\delta) = k]. \end{aligned} \tag{12}$$

Using assumption (vi) and the Lebesgue dominated convergence theorem [10] allows exchanging the summation and limits in δ, yielding

$$0 \leqslant \lim_{\delta\to 0} \frac{1}{\delta} M_2(s, \delta) \leqslant \sum_{k=2}^{\infty} k \left(\int_{s-r}^{s} \gamma_2(u)\, \mathrm{d}u \right) \lim_{\delta\to 0} \frac{1}{\delta} P[S_1(s, s+\delta) = k] = 0$$

using assumption (v). Hence, $\lim_{\delta\to 0}(1/\delta) M_2(s, \delta) = 0$, so

$$\lim_{\delta\to 0} \frac{1}{\delta} E[Y_a(s, s+\delta)] = \lim_{\delta\to 0} \frac{1}{\delta} M_1(s, \delta).$$

If detector 1 records *one* single during $(s, s+\delta]$, then the number of coincidence events recorded during that interval with the later singles event recorded by detector 1 is the number of singles recorded by detector 2 during the time interval of width r that precedes the arrival. That number must lie between $S_2(s-r+\delta, s)$ and $S_2(s-r, s+\delta)$ inclusively. Thus

$$M_1(s,\delta) \leqslant E[S_1(s-r, s+\delta)]P[S_1(s,s+\delta)=1] = P[S_1(s,s+\delta)=1]\int_{s-r}^{s+\delta} \gamma_2(u)\,\mathrm{d}u$$

so using Eq. (10) yields

$$\lim_{\delta\to 0}\frac{1}{\delta}M_1(s,\delta) \leqslant \gamma_1(s)\int_{s-r}^{s}\gamma_2(u)\,\mathrm{d}u.$$

Similarly

$$M_1(s,\delta) \geqslant \mathrm{P}[S_1(s,s+\delta)=1]E[S_1(s-r+\delta,s)]$$

so by similar arguments

$$\lim_{\delta\to 0}\frac{1}{\delta}M_1(s,\delta) \geqslant \gamma_1(s)\int_{s-r}^{s}\gamma_2(u)\,\mathrm{d}u.$$

Combining, we have

$$\lim_{\delta\to 0}\frac{1}{\delta}E[Y_a(s,s+\delta)] = \lim_{\delta\to 0}\frac{1}{\delta}M_1(s,\delta) = \gamma_1(s)\int_{s-r}^{s}\gamma_2(u)\,\mathrm{d}u.$$

By symmetry, $\lim_{\delta\to 0}(1/\delta)E[Y_b(s,s+\delta)] = \gamma_2(s)\int_{s-\gamma}^{s}\gamma_1(u)\,\mathrm{d}u$, so from Eq. (11) we conclude that the instantaneous rate of the accidental-coincidence process $Y(t)$ is

$$\gamma(s) \triangleq \lim_{\delta\to 0}\frac{1}{\delta}E[Y(s,s+\delta)] = \gamma_1(s)\int_{s-r}^{s}\gamma_2(u)\,\mathrm{d}u + \gamma_2(s)\int_{s-r}^{s}\gamma_1(u)\,\mathrm{d}u. \tag{13}$$

One can verify using Eq. (12) that $E[Y(s,s+\delta)]/\delta$ is uniformly bounded for all $\delta \in (0,1)$. Thus, applying Eq. (5) yields the exact first moment of the accidental-coincidence process:

$$E[Y(t)] = \int_0^t \left[\gamma_1(s)\int_{s-r}^{s}\gamma_2(u)\,\mathrm{d}u + \gamma_2(s)\int_{s-r}^{s}\gamma_1(u)\,\mathrm{d}u\right]\mathrm{d}s. \tag{14}$$

This is a general result that has not been previously derived to our knowledge.

3.1.1. Stationary increments case

One well-known special case of Eq. (14) is when the singles-counting processes have stationary increments, for which Eqs. (13) and (14) simplify to

$$E[Y(t)] = \gamma t, \quad \gamma = 2r\gamma_1\gamma_2. \tag{15}$$

This special case is well known (when the recorded singles process is Poisson, i.e., recorded with no deadtime) [11] but has not been derived formally to our knowledge. This result is consistent with[2] Ref. [2, Eq. (5.37)]. Remarkably, our general result holds for a fairly broad class of singles processes, as we have shown, not just for Poisson processes. This generality is important because deadtime causes non-Poisson singles processes.

[2] Identify $\gamma_1 = \rho_\beta e^{-\rho_\beta\tau_\beta}$ and $\gamma_2 = \rho_\gamma e^{-\rho_\gamma\tau_\gamma}$ and $\rho_{\beta_\gamma} = 0$.

3.1.2. Inhomogeneous Poisson case

Now suppose that the arrival processes $X_1(t)$ and $X_2(t)$ are Poisson with instantaneous rates $\rho_1(t)$ and $\rho_2(t)$, respectively. By a slight generalization of the argument leading to Ref. [3, Eq. (23)], one can show that the instantaneous rates of the singles-counting processes are

$$\gamma_i(s) = \rho_i(s)\exp\left(-\int_{s-\tau_i}^{s} \rho_i(u)\,\mathrm{d}u\right), \quad i = 1, 2 \tag{16}$$

which reduces to the familiar $\rho_i e^{-\rho_i \tau_i}$ in the homogeneous case [2, Eq. (2.9b)]. One can substitute Eq. (16) into Eq. (13) to determine the rate of the accidental-coincidence process.

3.1.3. Decay case

Result (14) is sufficiently general to apply to the general case where one considers a *decaying* source and a time-independent *background* arrival rate, and deadtime. This general case has not been previously solved, according to Ref. [2, p. 29]. Suppose the arrival processes are Poisson with instantaneous rates $\rho_i(t) = \rho_i e^{-\lambda_i t} + b_i$, where λ_i denotes the source decay rate. Substituting into Eq. (16) yields

$$\gamma_i(s) = (\rho_i e^{-\lambda_i t} + b_i)\exp\left(-\left[\tau_i b_i + \rho_i \frac{e^{-\lambda_i(s-\tau_i)} - e^{-\lambda_i s}}{\lambda_i}\right]\right), \quad i = 1, 2. \tag{17}$$

Substituting Eq. (17) into Eq. (14) yields an exact analytical expression for the mean of the accidental-coincidence process for decaying Poisson arrivals with extendable deadtime.

3.2. Variance of accidental coincidences for ideal detectors (no deadtime)

Turning now to the variance of $Y(t)$, we show that the accidental-coincidence process is not exactly Poisson[3] even when the recorded singles processes are homogeneous and Poisson (i.e., in the hypothetical case of no deadtime losses) by showing that $\mathrm{Var}\{Y(t)\} \neq E[Y(t)]$ using Eq. (6) for Poisson arrival processes. We first find the instantaneous correlation function $\beta(0,s)$. For $s > r$, the increments $Y(0,\delta)$ and $Y(s, s+\delta)$ are independent for $\delta < \min(s, s-r)$, hence

$$\beta(0,s) = (2r\gamma_1\gamma_2)^2. \tag{18}$$

For $0 < s < r$, we show in Ref. [12, p. 139] that

$$\beta(0,s) = (2r\gamma_1\gamma_2)^2 + (\gamma_1 + \gamma_2)\gamma_1\gamma_2(2r - s). \tag{19}$$

One can verify that $Y(t)$ satisfies assumption (iii) following Ref. [12, Eqs. (D.8) and (D.12)]. However, $Y(t)$ does not satisfy assumption (iv) in the absence of deadtime, hence we must use Eq. (9) to derive its second moment. We have

$$\begin{aligned} E[Y^2(\delta)] &= E[(Y_a(\delta) + Y_b(\delta))^2] \\ &= E[Y_a^2(\delta)] + E[Y_b^2(\delta)] + 2E[Y_a(\delta)Y_b(\delta)] \end{aligned} \tag{20}$$

[3] In the absence of deadtime, the accidental-coincidence process will be Poisson if the two coincidence particles always arrive at the two detectors at exactly the same time and there is no uncertainty in the time-stamping of recorded particles. It appears that coincidence processes are exactly Poisson only in this *highly* idealized case.

and

$$\begin{aligned} E[Y_a^2(\delta)] &= \sum_{k=0}^{\infty} E[Y_a^2(0,\delta)|X_1(0,\delta)=k]P[X_1(0,\delta)=k] \\ &= 0 + E[Y_a^2(0,\delta)|X_1(0,\delta)=1]P[X_1(0,\delta)=1] \\ &\quad + \sum_{k=2}^{\infty} E[Y_a^2(0,\delta)|X_1(0,\delta)=k]P[X_1(0,\delta)=k]. \end{aligned} \tag{21}$$

If $X_1(0,\delta)=k$, then $Y_a(0,\delta) \leqslant kX_2(-r,\delta)$, so

$$\begin{aligned} &\lim_{\delta\to 0}\frac{1}{\delta}\sum_{k=2}^{\infty} E[Y_a^2(0,\delta)|X_1(0,\delta)=k]P[X_1(0,\delta)=k] \\ &\quad \leqslant \lim_{\delta\to 0}\frac{1}{\delta}\sum_{k=2}^{\infty}(\gamma_2(r+\delta)+(\gamma_2(r+\delta))^2)k^2P[X_1(0,\delta)=k]=0. \end{aligned} \tag{22}$$

Thus, combining Eqs. (21) and (22)

$$\begin{aligned} \lim_{\delta\to 0}\frac{1}{\delta}E[Y_a^2(\delta)] &= \lim_{\delta\to 0}\frac{1}{\delta}E[Y_a^2(0,\delta)|X_1(0,\delta)=1]P[X_1(0,\delta)=1]+0 \\ &= \lim_{\delta\to 0}\frac{1}{\delta}(r\gamma_2+(r\gamma_2)^2)e^{-\delta\gamma_1}(\delta\gamma_1) \\ &= (r\gamma_2+(r\gamma_2)^2)\gamma_1. \end{aligned} \tag{23}$$

Furthermore

$$\begin{aligned} \lim_{\delta\to 0}\frac{1}{\delta}E[Y_a(\delta)Y_b(\delta)] &= \lim_{\delta\to 0}\frac{1}{\delta}\sum_{k,l=1}^{\infty} E[Y_a(\delta)Y_b(\delta)|X_1(0,\delta)=k, X_2(0,\delta)=l]P[X_1(0,\delta)=k, X_2(0,\delta)=l] \\ &\leqslant \lim_{\delta\to 0}\frac{1}{\delta}\sum_{k,l=1}^{\infty}(k+\gamma_1(\delta+r))(l+\gamma_2(\delta+r))klP[X_1(0,\delta)=k, X_2(0,\delta)=l] \\ &= \lim_{\delta\to 0}\frac{1}{\delta}\left(\sum_{k=1}^{\infty}(k+\gamma_1(\delta+r))kP[X_1(0,\delta)=k]\right)\left(\sum_{l=1}^{\infty}(l+\gamma_1(\delta+r))lP[X_2(0,\delta)=l]\right) \\ &= 0. \end{aligned} \tag{24}$$

Hence, applying Eqs. (23) and (24) to Eq. (20) and using symmetry:

$$\alpha(s) = r\gamma_1\gamma_2(2+r(\gamma_1+\gamma_2)). \tag{25}$$

Using ideas leading to Eqs. (22) and (24), one can easily verify that $E[Y^2(0,\delta)]/\delta$ is uniformly bounded. Hence, from Eqs. (19), (18), (25) and (6), for $t > r$, the variance of $Y(t)$ is

$$\begin{aligned} \mathrm{Var}\{Y(t)\} &= \int_0^t \alpha(s)\,\mathrm{d}s + 2\int_0^t (t-s)(2r\gamma_1\gamma_2)^2\,\mathrm{d}s + 2\int_0^{\gamma}(t-s)(\gamma_1+\gamma_2)\gamma_1\gamma_2(2r-s)\,\mathrm{d}s - (\gamma t)^2 \\ &= r^2\gamma_1\gamma_2(\gamma_1+\gamma_2)t + 2r\gamma_1\gamma_2 t + (2r\gamma_1\gamma_2 t)^2 + r^2(\gamma_1+\gamma_2)\gamma_1\gamma_2(3t-\tfrac{4}{3}r) - (2r\gamma_1\gamma_2 t)^2 \\ &= 2r\gamma_1\gamma_2 t(1+2r(\gamma_1+\gamma_2)(1-r/3t)). \end{aligned} \tag{26}$$

Variance (26) is "inflated" relative to mean (15) by the factor $1+2r(\gamma_1+\gamma_2)(1-r/(3t))$, so the accidental-coincidence process is not Poisson even in the absence of deadtime.

3.3. Variance of accidental coincidences for non-ideal detectors (with deadtime)

We now derive the variance of the accidental-coincidence process $Y(t)$ in the presence of deadtime with $\tau_1 = \tau_2 = \tau > 2r$ (assuming equal deadtimes for simplicity). As in Section 3.2, we first derive $\beta(0,s)$. Since we assume $\tau > 2r$, forming two coincidence events would require two recorded pairs of particles, each pair forming a coincidence event. The minimum time separation for the two latter-recorded particles is at least τ, hence[4] $\beta(0,s) = 0$ for $0 < s < \tau$. If $\tau + r < s < t$, then $Y(0,\delta)$ and $Y(s, s+\delta)$ are independent for $\delta < s - \tau - r$, hence $\beta(0,s) = \gamma^2$. The most complicated case is when $\tau < s < \tau + r$; we show in Ref. [12, p. 144] that

$$\beta(0,s) = \rho_1^2\rho_2^2 e^{-(\rho_1+\rho_2)\tau}(r^2 + (s-\tau)(4r - s + \tau)). \tag{27}$$

We observe that $0 \leqslant (s-\tau)(4r-s+\tau) \leqslant 3r^2$ when $0 < s-\tau < r$, hence $0 < \beta(0,s) < \gamma^2$ for $\tau < s < \tau + r$. Thus, for $t > \tau + r$

$$\begin{aligned}
\mathrm{Var}\{Y(t)\} &= \gamma t + 2\int_{\tau+r}^{t} (t-s)(2r\gamma_1\gamma_2)^2\, ds \\
&\quad + 2\int_{\tau}^{\tau+r} (t-s)(\gamma_1\gamma_2)^2 (r^2 + (s-\tau)(4r-s+\tau))\, ds - (\gamma t)^2 \\
&= 2r\gamma_1\gamma_2 t + (\gamma_1\gamma_2)^2 \frac{1}{6} r^2 (5r^2 - 16r(t-\tau) + 24(t-\tau)^2) - (2r\gamma_1\gamma_2 t)^2 \\
&= \gamma t(1 - \gamma\tau(2 - \tau/t)) + \frac{\gamma^2}{24}(5r^2 - 16r(t-\tau))
\end{aligned} \tag{28}$$

where $\gamma = 2r\gamma_1\gamma_2 = 2r\rho_1\rho_2 e^{-(\rho_1+\rho_2)\tau}$ from Eq. (15). To our knowledge, the exact variance results (26) and (28) are new. From Eq. (7) the mean of $Y(t)$ is

$$E[Y(t)] = \gamma t, \quad \gamma = 2r\rho_1\rho_2 e^{-(\rho_1+\rho_2)\tau}. \tag{29}$$

When $r \ll \tau \ll t$, we can approximate Eq. (28) by $\gamma t(1 - 2\gamma\tau)$, hence the variance is "deflated" relative to the mean by approximately $1 - \xi$, where $\xi \triangleq 2\gamma\tau$ is usually very small. Fig. 2 shows the mean and variance of the coincidence process recorded by non-ideal detectors. The mean and variance are extremely close at all rates ρ.

4. Moments of the coincidence-counting process

We now address the moments of *total* recorded coincidences (both accidental and genuine). This case is more complicated than the case of pure accidental coincidences considered in the previous section, so we focus on Poisson arrival processes. For analyzing the mean, we allow the arrival processes to be inhomogeneous, to allow for decay or other temporal variations. Recall that for an inhomogeneous Poisson process with intensity $\lambda(t)$, the waiting time of the first arrival after time s is distributed, for $v \geqslant s$, as $\lambda(v) e^{-\int_s^v \lambda(u)\, du}$.

Hereafter, we assume $2r < \min\{\tau_1, \tau_2\}$, such that the event $[Y(s, s+\delta) \geqslant 2]$ can never occur when $\delta < r$.

4.1. Mean of recorded coincidences

In the presence of deadtime, the event that there is one recorded coincidence event during $(s, s+\delta]$ consists of one of the following four mutually exclusive events:

[4] Note that if $r < \tau < 2r$, then $\beta(0,s) \neq 0$ if $s < r$.

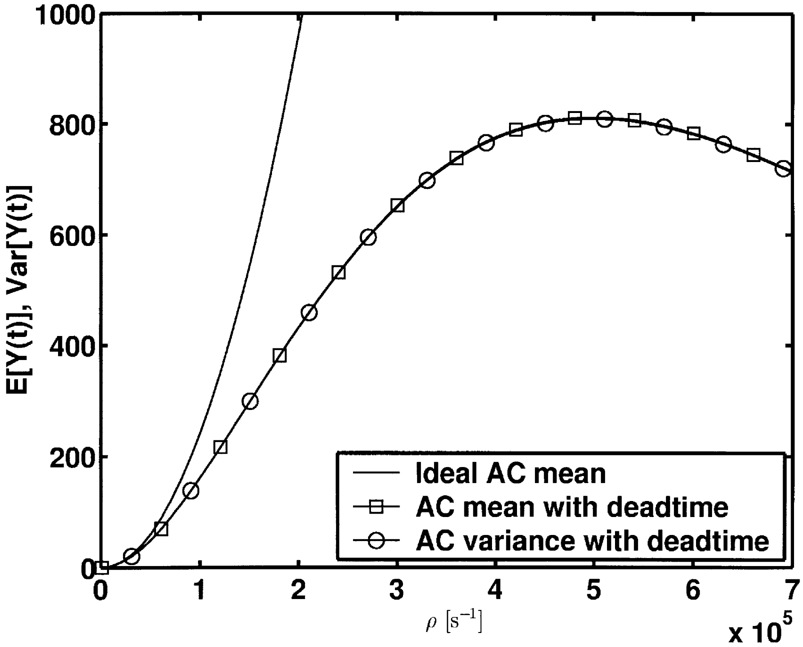

Fig. 2. Mean and variance of accidental-coincidence (AC) counts $Y(t)$ in a paralyzable system with $t = 1\,\mathrm{s}$, $\tau = 2\,\mu\mathrm{s}$, $\gamma = 12\,\mathrm{ns}$, and $\rho_1 = \rho_2 = \rho$. Based on Eqs. (15) and (28) and (29) the mean and variance curves are nearly indistinguishable.

(E_1) one N_1 particle and one N_2 particle form a coincidence event;
(E_2) one N_0 particle at detector 1, and one N_2 particle at detector 2;
(E_3) one N_0 particle at detector 2, and one N_1 particle at detector 1;
(E_4) a pair of N_0 particles is recorded.

We will derive the probability for each of these four events.

We split E_1 into two disjoint sub-events: E_{1a} and E_{1b}, where E_{1a} denotes the event that one N_1 particle and one N_2 particle form a coincidence event and furthermore that the later particle[5] is recorded by detector 1. Let T_1 denote the time of the first N_1 particle arrival after time s, and $T_2|T_1$ denote the time of the first N_2 particle arrival after $T_1 - r$:

$$\begin{aligned} P[E_{1a}] &= \int_s^{s+\delta} \int_{s_1-r}^{s_1} P[N_1(s_1-\tau_1, s) = 0, N_2(s_2-\tau_2, s_1-r) = 0, N_0(\min\{s_1-\tau_1, s_2-\tau_2\}, s_1) = 0 \\ &\quad |T_1 = s_1, T_2 = s_2] f_{T_2|T_1}(s_2|s_1) f_{T_1}(s_1)\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &= \int_s^{s+\delta}\int_{s_1-r}^{s_1} \mathrm{e}^{-\int_{s_1-\tau_1}^{s} \rho_1(u)\,\mathrm{d}u}\, \mathrm{e}^{-\int_{s_2-r_1}^{s_1-r} \rho_2(u)\,\mathrm{d}u}\, \mathrm{e}^{-\int_{\min\{s_1-\tau_1, s_2-\tau_2\}}^{s_1} \rho_0(u)\,\mathrm{d}u} \rho_2(s_2) \mathrm{e}^{-\int_{s_1-r}^{s_2} \rho_2(u)\,\mathrm{d}u} \rho_1(\mathrm{s}_1) \mathrm{e}^{-\int_s^{s_1} \rho_1(u)\,\mathrm{d}u}\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &= \int_s^{s+\delta} \int_{s_1-\gamma}^{s_1} \rho_1(s_1)\rho_2(s_2) \mathrm{e}^{-\int_{s_1-\tau_1}^{r} \rho_1(u)\,\mathrm{d}u}\, \mathrm{e}^{-\int_{s_2-r_2}^{s_2} \rho_2(u)\,\mathrm{d}u}\, \mathrm{e}^{-\int_{\min\{s_1-\tau_1, s_2-\tau_2\}}^{s_1} \rho_0(u)\,\mathrm{d}u}\, \mathrm{d}s_2 \mathrm{d}s_1 \\ &= \int_s^{s+\delta} \int_{s_1-r}^{s_1} \rho_1(s_1)\rho_2(s_2) q_1(s_1) q_2(s_2) q_0(\min\{s_1-\tau_1, s_2-\tau_2\}, s_1)\, \mathrm{d}s_2\, \mathrm{d}s_1, \end{aligned}$$

[5] For completeness, we can also include the probability zero events of two simultaneous singles arrivals in this event, which will have no effect on the moments.

where

$$q_0(s,t) \triangleq \exp\left(-\int_s^t \rho_0(u)\,\mathrm{d}u\right), \quad q_i(s) \triangleq \exp\left(-\int_{s-\tau_i}^s \rho_i(u)\,\mathrm{d}u\right), \quad i=1,2.$$

By symmetry

$$P[E_{1b}] = \int_s^{s+\delta}\int_{s_2-r}^{s_2} \rho_1(s_1)\rho_2(s_2)q_1(s_1)q_2(s_2)q_0(\min\{s_1-\tau_1, s_2-\tau_2\}, s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

For E_2, if an N_2 particle arriving at detector 2 and an N_0 particle arriving at detector 1 form a coincidence event, then the N_2 particle arriving at detector 2 must arrive before the N_0 particle because otherwise the N_2 will be lost due to the N_0 particle at detector 2. Hence, the N_0 particle arriving at detector 1 will be the later arriving particle. Let T_0 denote the time of the first N_0 particle arrival after time s, and $T_2|T_0$ denote the time of the first N_2 particle arrival after $T_0 - r$, then

$$\begin{aligned} P[E_2] &= \int_s^{s+\delta}\int_{s_1-r}^{s_1} P[N_0(\min\{s_1-\tau_1, s_2-\tau_2\}, s)=0,\;\; N_1(s_1-\tau_1, s_1)=0,\;\; N_2(s_2-\tau_2, s_1-r)=0 \\ &\quad |T_0=s_1, T_2=s_2] f_{T_2|T_0}(s_2|s_1) f_{T_0}(s_1)\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= \int_s^{s+\delta}\int_{s_1-\gamma}^{s_1} \mathrm{e}^{-\int_{\min\{s_1-\tau_1,s_2-\tau_2\}}^{s}\rho_0(u)\,\mathrm{d}u}\mathrm{e}^{-\int_{s_1-\tau_1}^{s}\rho_1(u)\,\mathrm{d}u}\mathrm{e}^{-\int_{s_2-\tau_2}^{s_1-\tau}\rho_2(u)\,\mathrm{d}u} \\ &\quad \times \rho_2(s_2)\mathrm{e}^{-\int_{s_1-r}^{s_2}\rho_2(u)\,\mathrm{d}u}\rho_0(\mathrm{s}_1)\mathrm{e}^{-\int_s^{s_1}\rho_0(u)\,\mathrm{d}u}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= \int_s^{s+\delta}\int_{s_1-r}^{s_1} \rho_0(s_1)\rho_2(s_2)q_1(s_1)q_2(s_2)q_0(\min\{s_1-\tau_1, s_2-\tau_2\}, s_1)\,\mathrm{d}s_2\,\mathrm{d}s_1. \end{aligned}$$

By symmetry

$$P[E_3] = \int_s^{s+\delta}\int_{s_2-r}^{s_2} \rho_0(s_2)\rho_1(s_1)q_1(s_1)q_2(s_2)q_0(\min\{s_1-\tau_1, s_2-\tau_2\}, s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2.$$

For E_4, the simplest of all four cases, we only need to ensure that there is at least one coincidence arrival during $(s, s+\delta]$, and there is no coincidence or random arrival preceding the coincidence arrival by the deadtimes:

$$\begin{aligned} P[E_4] &= \int_s^{s+\delta} P[N_0(s_0-\tau_{\max}, s)=0, N_1(s_0-\tau_1, s_0)=0, N_2(s_0-\tau_0)=0|T_o=s_0] f_{T_0}(s_0)\,\mathrm{d}s_0 \\ &= \int_s^{s+\delta} \mathrm{e}^{-\int_{s_0-\tau\max}^{s}\rho_0(u)\,\mathrm{d}u}\mathrm{e}^{-\int_{s_0}^{s_0}\rho_1(u)\,\mathrm{d}u}\mathrm{e}^{-\int_{s_0-\tau_2}^{s_0}\rho_2(u)\,\mathrm{d}u}\rho_0(s_0)\mathrm{e}^{-\int_s^{s_0}\rho_2(u)\,\mathrm{d}u}\,\mathrm{d}s_0 \\ &= \int_s^{s+\delta} \rho_0(s_0)q_1(s_0)q_2(s_0)q_0(s_0-\tau_{\max}, s_0)\,\mathrm{d}s_0 \end{aligned}$$

where $\tau_{\max} = \max\{\tau_1, \tau_2\}$.

Combining all four events

$$E[Y(s, s+\delta)] = P[E_4] + (P[E_{1a}] + P[E_2]) + (P[E_{1b}] + P[E_3])$$

and applying $\lim_{\delta\to 0}$ in Eq. (2) yields the instantaneous rate of the coincidence-counting process $Y(t)$

$$\begin{aligned}\gamma(s) &= \rho_0(s)q_1(s)q_2(s)q_0(s-\tau_{\max}, s) \\ &\quad + (\rho_1(s)+\rho_0(s))q_1(s)\int_{s-r}^{s}\rho_2(u)q_2(u)q_0(\min\{s-\tau_1, u-\tau_2\}, s)\,\mathrm{d}u \\ &\quad + (\rho_2(s)+\rho_0(s))q_2(s)\int_{s-r}^{s}\rho_1(u)q_1(u)q_0(\min\{u-\tau_1, s-\tau_2\}, s)\,\mathrm{d}u. \end{aligned} \tag{30}$$

If ρ_0, ρ_1, and ρ_2 are constant in time, and if the two deadtimes are equal, then Eq. (30) simplifies to the following:

$$\gamma = \left[\left(\frac{2\rho_1\rho_2}{\rho_0} + \rho_1 + \rho_2\right)(1 - \mathrm{e}^{-\rho_0 r}) + \rho_0 \mathrm{e}^{-(\rho_0+\rho_1+\rho_2)\tau}\right]. \tag{31}$$

This expression is consistent[6] with Ref. [2, Eq. (5.39)].

One can find the exact mean of $Y(t)$ for the general case where *decay*, *background*, and *deadtime* are considered by substituting $\rho_i(t) = \rho_i \mathrm{e}^{-\lambda_i t} + b_i$ into Eq. (30), and then integrating using Eq. (5). This exact solution solves the open problem described in Ref. [2, p. 27]. One could perform integration (5) numerically to arbitrary accuracy and solve numerically for ρ_0 if the other parameters $(\rho_i, \lambda_i, b_i, \tau_i, r)$ can be determined.

4.2. Variance of recorded coincidences

We now consider the variance of $Y(t)$ for the case of homogeneous arrivals and equal deadtines. From Ref. [12, p. 96], a bound on the variance $Y(t)$ is

$$\gamma t[1 - \gamma(\tau + r)(2 - (\tau + r)/t] \leqslant \mathrm{Var}\{Y(t)\} \leqslant \gamma t[1 - \gamma\tau(2 - \tau/t)] \tag{32}$$

where $\gamma = E[Y(t)]/t$ is given in Eq. (31).

This bound is very tight since usually $\gamma \ll \tau$. Thus, $\mathrm{Var}\{Y(t)\} \approx E[Y(t)](1 - 2\gamma\tau)$, so again the mean and variance are very similar. To the best of our knowledge, the variance result (32) is new.

5. Discussion

We have analyzed the mean and variance of coincidence counting under various scenarios. In each case, the coincidence-counting process is not Poisson, including in the simplest case of purely accidental-coincidence counts recorded by ideal detectors. Nevertheless, we have shown that the variance is very close to the mean for parameters typical of PET imaging where $r \ll \tau_1 = \tau_2$.

For paralyzable detectors, the variance over the mean is approximately $1 - \xi$, where ξ is approximately the expected number of recorded events during the time interval $(0, 2\tau]$. Unless the genuine-coincidence rate is so high that $\xi \gg 0$, the ratio of variance over mean will be very close to unity.

It is interesting to compare the coincidence-counting process to the singles-counting process [3]. Deadtime causes the singles-counting process to be significantly non-Poisson, in the sense that the variance of the process is significantly less than the mean. In contrast, the preceding analysis shows that the variance of the coincidence-counting process is very close to the mean. The reason for this lies in the fact that ξ is primarily determined by γ (in addition to τ), the instantaneous rate of the counting process itself. In a singles-counting process, γ can be relatively large, but in the coincidence-counting process in PET imaging, the coincidence rate γ is usually quite small compared to the singles rate γ_1 and γ_2. Thus deadtime causes

[6] Identify $\rho_{\beta\gamma} = \rho_0, \rho_\beta = \rho_1 + \rho_0$, and $\rho_\gamma = \rho_2 + \rho_0$.

significant loss in the *mean* of the process, necessitating deadtime correction, but leaves the *ratio* of the variance over the mean relatively unaffected.

Extensions to include attenuation effects and delayed coincidences are considered in Ref. [12]. The key to our analysis is the instantaneous moments used in Eqs. (2)–(6). The use of this approach allows one to consider inhomogeneous processes using ''local'' analysis. This has enabled our new results on the mean of the coincidence-counting process for the usual scenario where decay, background, and deadtime are all present [2, p. 27]. Presumably, one could also address further extensions such as random deadtimes within this framework.

References

[1] James A. Sorenson, Michael E. Phelps. Physics in nuclear medicine, 2nd Edition, Sanders, Philadelpyhia, 1987.

[2] ICRU Repot 52. Particle counting in radioactivity measurements, 1994. (International commission on radiation units and measurements.)

[3] D.F. Yu, J.A. Fessler, Mean and variance of singles photon counting with deadtime, Phys. Med. Biol. 45 (7) (2000) 2043.

[4] F. Carloni, A. Corberi, M. Marseguerra, C.M. Porceddu, The asymptotic method of dead-time correction in Poissonian distribution, Nucl. Instr. and Meth. 78 (1970) 70.

[5] J.W. Müller, Dead-time problems, Nucl. Instr. and Meth. 112 (1) (1973) 47.

[6] J.W. Müller, Some formulae for a dead-time-distorted Poisson process, Nucl. Instr. and Meth. 117 (2) (1974) 401.

[7] J. Libert, Statistique de comptage avec distribution unifomre de temps mort, Nucl. Instr. and Meth. 151 (3) (1978) 555.

[8] G. Faraci, A.R. Pennisi, Experimental dead-time distortions of Poisson processes, Nucl. Instr. and Meth. 212 (1) (1983) 307.

[9] N.H. Clinthorne, N.A. Petrick, W.L. Rogers, A.O. Hero, A fundamental limit on timing performance with scintillation detectors, IEEE Trans. Nucl. Sci. 37 (2) (1990) 658.

[10] A.M. Bruckner, J.B. Bruckner, B.S. Thomson, Real Analysis, Prentice Hall, Englewood Cliffs, NJ, 1997.

[11] E.J. Hoffman, S.C. Huang, M.E. Phelps, D.E. Kuhl, Quantitation in positron emission computed tomography: 4 effect of accidental coincidences, J. Comp. Assisted Tomo. 5 (3) (1981) 391.

[12] Daniel Feng Yu. Statistical methods for transmission image reconstruction with nonlocal edge-preserving regularization. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, 48109-2122, Ann Arbor, MI., 2000. http://www.eecs.unich.edu/~fessler/student.