Recursive CR-Type Bounds and the EM Algorithm: Applications to ECT Image Reconstruction[1]

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ABSTRACT

We give a class of iterative algorithms to monotonically approximate submatrices of the CR matrix bound on the covariance of any estimator of a vector parameter $\underline{\theta}$. A natural implementation of the iterative algorithm employs a "complete data - incomplete data" formulation similar to that underlying the EM parameter estimation algorithm. Our results make it feasible to compute CR-type bounds for previously intractible problems involving a large number of "nuisance parameters," such as arise in image reconstruction.

Summary

The Cramer-Rao (CR) bound on estimator covariance is an important tool for predicting fundamental limits on best achievable parameter estimation performance [5], predicting the impact of side information and constraints on estimation performance [3], and obtaining optimal experimental designs [1]. For a vector parameter $\underline{\theta} \in \Theta \subset \mathbb{R}^n$ the upper left $p \times p$ matrix of the inverse of the $n \times n$ Fisher information matrix provides the CR lower bound on the minimum achievable covariance of any unbiased estimator of $\theta_1, \ldots, \theta_p$, $p \leq n$. Equivalently, the first p rows of F_Y^{-1} provide the CR bound. The method of sequential partitioning [4] for computing the upper left $p \times p$ submatrix of F_Y^{-1} and Cholesky based Gaussian elimination techniques [2] for computing the p first rows of F_Y^{-1} are efficient direct methods for obtaining the CR bound but require $O(n^3)$ floating point operations and $O(n^2)$ memory storage. Unfortunately, in many practical cases of interest, e.g. when there are a large number of nuisance parameters, high computation and memory requirements make direct implementation of the CR bound impractical. For example, in the case of image reconstruction for a 256×256 pixelated image F_Y is $256^2 \times 256^2$ so that direct computation of the CR bound on estimation errors in a small region of the image requires on the order of 256^6 or 10^{19} floating point operations and on the order of 4GByte of memory storage!

In this paper we give a class of iterative algorithms for computing columns of the CR bound which requires only $O(n^2)$ floating point operations per column of F_Y^{-1}. These algorithms fall into the class of "splitting matrix iterations" [2]. The inverse of this splitting matrix should be sparse and simply determined. The splitting matrix is chosen based on purely algebraic or purely statistical considerations to ensure that a valid lower bound results at each iteration of the algorithm. By embedding the parameter estimation problem into a particular complete data - incomplete data setting, and applying a version of the "data processing theorem" for Fisher matrices, the Fisher matrix F_X for the complete data set can frequently be used as a splitting matrix. This complete-incomplete data setting is similar to that which underlies the classical formulation of the EM algorithm. The EM algorithm generates a sequence of estimates $\{\hat{\underline{\theta}}^k\}_k$ for $\underline{\theta}$ which successively increase the likelihood function and converge to the maximum likelihood estimator. In a similar manner, our algorithm generates a sequence of tighter and tighter lower bounds on estimator covariance which converge to the actual CR matrix bound. The iterative algorithm allows one to compute the CR bound for estimation problems that would have been intractible by exact methods due to the large dimension of F_Y.

We conclude with an implementation of the recursive algorithm for bounding the minimum achiev-

[1]This research was supported in part by the National Science Foundation under grant BCS-9024370, the National Cancer Institute under grant R01-CA-54362-02, and a DOE Alexander Hollaender Postdoctoral Fellowship.

able estimator error covariance for problems arising in emission computed tomography (ECT). For the case where the complete data is selected as the set of image pixel emission counts in each of d "detector tubes", which is the standard choice of complete data for the EM image reconstruction algorithm, F_X is diagonal. Furthermore, due to the sparseness of the tomographic system response matrix the computation of each column of the CR bound matrix recursion only requires $O(n)$ memory storage as compared to $O(n^2)$ for the general algorithm. We show that in general the rate of convergence depends on the image intensity and the tomographic system response matrix. We have applied the iterative algorithm to compute the CR bound for practical estimation tasks including: reconstruction of a small region-of-interest (ROI), estimation of total uptake in a ROI, estimation of dose distribution heterogeneity in a ROI, impact of anatomical side information on ROI reconstruction.

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