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Complex Quasi-Newton Proximal Methods for the Image Reconstruction in Compressed Sensing MRI

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Synopsis

Keywords: Image Reconstruction, Data Processing, Fast reconstruction algorithm

This work studies a complex quasi-Newton proximal method (CQNPM) for MRI reconstruction using wavelets or total variation (TV) based regularization. Our experiments show that our method is faster than the accelerated proximal method [1,2] in terms of iteration and CPU time.

Introduction

The reconstruction of compressed sensing MRI can be formulated as the following minimization problem:

$$\min_{x\in\mathbb{C}^N} F(x),\ F(x) \equiv \underbrace{\frac{1}{2}\|Ax - y\|_2^2}_{f(x)} + \lambda\, h(x), \qquad (1)$$

where $A \in \mathbb{C}^{M\times N}$ refers to the forward model describing a mapping from the signal x to the acquired data y and $\lambda > 0$ is the trade-off parameter. Here, we focus on $h(x) = \|Tx\|_1$ for a wavelet transform T, or $h(x) = TV(x)$. Traditionally, one can use the accelerated proximal method (APM) [1,2] to solve (1). Here we propose a complex quasi-Newton proximal method to solve (1) even faster.

Methods

Denote a weighted proximal operator by

$$\text{Prox}_{\lambda h}^{W}(v) \equiv \arg\min_{x\in\mathbb{C}^N} \frac{1}{2}\|x - v\|_W^2 + \lambda h(x) \qquad (2)$$

where $W \in \mathbb{C}^{N\times N}$ is a Hermitian positive definite matrix and $\|x\|_W = \sqrt{x'Wx}$ denotes the W-weighted Euclidean norm. When $W = I$, (2) becomes the well-known proximal operator. At the kth iteration, the CQNPM update is:

$$x_{k+1} = \text{Prox}_{a_k\lambda h}^{B_k}\left(x_k - a_k B_k^{-1}\nabla_x f(x_k)\right)$$

where a_k denotes the step-size. Here, the symmetric rank-1 method is used to compute B_k [3] so that $B_k \in \mathbb{C}^{N\times N} \equiv D_k \pm u_k u_k'$ with D_k a diagonal matrix and $u_k \in \mathbb{C}^N$.

For $h(\tilde{x}) = \|\tilde{x}\|_1$,one can solve (2) efficiently through the following lemma:
Lemma 1 [4]:
Let $W = D \pm uu'$. Then,

$$\text{Prox}_{\lambda h}^{W}(x) = \text{Prox}_{\lambda h}^{D}(x \mp D^{-1}u\alpha^*),$$

where $\alpha^* \in \mathbb{C}$ is the unique zero of the following nonlinear equation $\mathbb{J}(\alpha) : u'\left(x - \text{Prox}_{\lambda h}^{D}(x \mp D^{-1}u\alpha)\right) + \alpha$.
We solve $\mathbb{J}(\alpha) = 0$ using ``SciPy'' library in Python. When $h(x) = \|Tx\|_1$ where T is an invertible transform, we can rewrite (1) as $\frac{1}{2}\|AT^{-1}x - y\|_2^2 + \lambda\,\|x\|_1$ that Lemma 1 is still appliable.

For $h(x) = TV(x)$, we transform (2) to the following dual problem that is differentiable

$$(P_1^*, Q_1^*, P_2^*, Q_2^*) = \arg\min_{\substack{(P_1,Q_1)\in\square \\ (P_2,Q_2)\in\square}} \|\phi(P_1,Q_1,P_2,Q_2)\|_{\tilde{W}}^2, \qquad (4)$$

where $\tilde{W} = \begin{bmatrix} \Re(W) & -\Im(W) \\ \Im(W) & \Re(W), \end{bmatrix}$, $\square$ denotes a set of real matrix-pairs (P,Q) that satisfy

$$\begin{aligned} P_{i,j}^2 + Q_{i,j}^2 &\le 1 \qquad \text{isotropic TV}, \\ |P_{i,j}| \le 1,\ |Q_{i,j}| &\le 1 \qquad \text{anisotropic TV}, \end{aligned}$$

$\phi(P_1,Q_1,P_2,Q_2) = \begin{bmatrix} \Re(v) \\ \Im(v) \end{bmatrix} - \lambda\left(\tilde{W}\right)^{-1}\begin{bmatrix} \text{vec}(\square(P_1,Q_1)) \\ \text{vec}(\square(P_2,Q_2)) \end{bmatrix}$, and $\square(P,Q)_{i,j} = P_{i,j} + Q_{i,j} - P_{i-1,j} - Q_{i,j-1}$. Note that $\Re(\cdot)$ (respectively, $\Im(\cdot)$) refers to an operator to take the real (respectively, imaginary) partand $\text{vec}(\cdot)$ denotes the vectorization of a matrix. We compute $(\tilde{W})^{-1}$ in $\phi(P_1,Q_1,P_2,Q_2)$ efficiently through the Schur complement since $W = D \pm uu'$. After solving (4), we reach

$$\begin{bmatrix} \Re(\text{Prox}_{\lambda h}^{W}(v)) \\ \Im(\text{Prox}_{\lambda h}^{W}(v)) \end{bmatrix} = \phi(P_1^*, Q_1^*, P_2^*, Q_2^*).$$

Results

All experiments are implemented in SigPy [5]. We used the data from [6]. Figures 1-4 show the results and experimental details.

Conclusion

For a general matrix W, solving (2) would be as hard as the original problem (1). By using the structure of W, i.e., $W = D \pm uu'$, we propose efficient approaches to address (2) when $h(x) = \|Tx\|_1$ or $TV(x)$. Compared with the computational cost in the proximal operator, i.e., $W = I$, the increased computation in (2) is insignificant, as illustrated by our CPU time comparisons.

Acknowledgements

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Figures

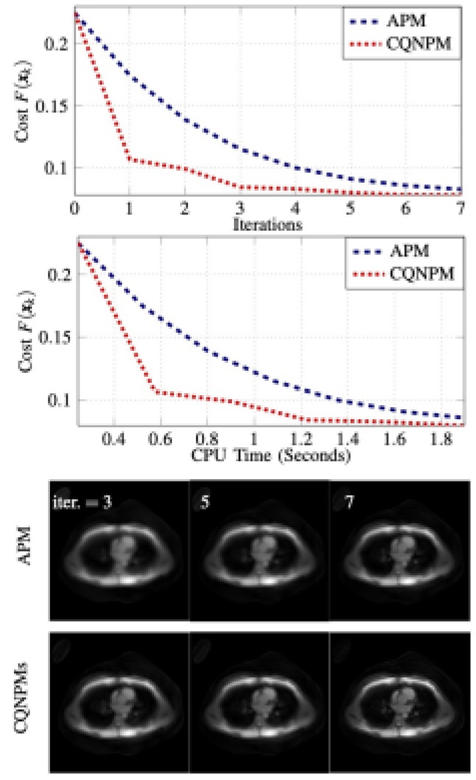

Figure 1 Test on a Cardiac dataset with regularizer $h(x) = \|Tx\|_1$ for an orthonormal wavelet transform T with 5 levels. Acquisition: spiral trajectory with 3 interleaves, 3996 readout points and under-sampling $= 8$; $1.5T$ GE Healthcare scanner with 8-channel cardiac coil. Matrix size $= 320 \times 320$. TR $= 25.8$ms.

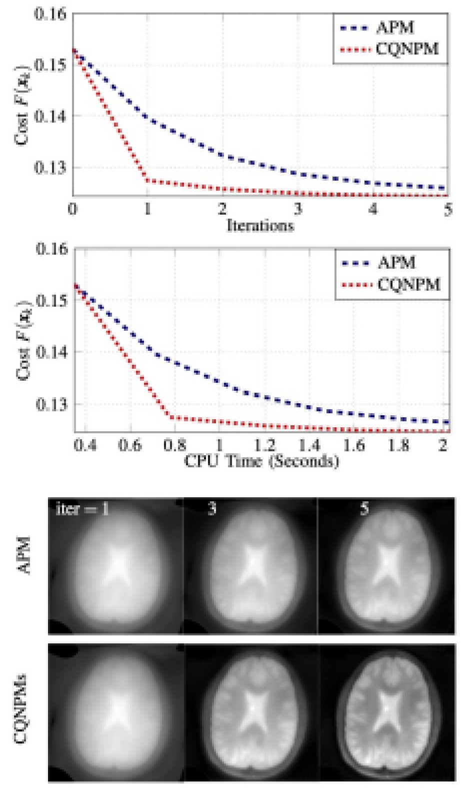

Figure 2 Test on a radial brain dataset (12 coils, 96 radial projections) with regularizer $h(x) = \|Tx\|_1$ an orthonormal wavelet transform T with 5 levels. This data comes from https://github.com/mikgroup/sigpy-mri-tutorial [5].

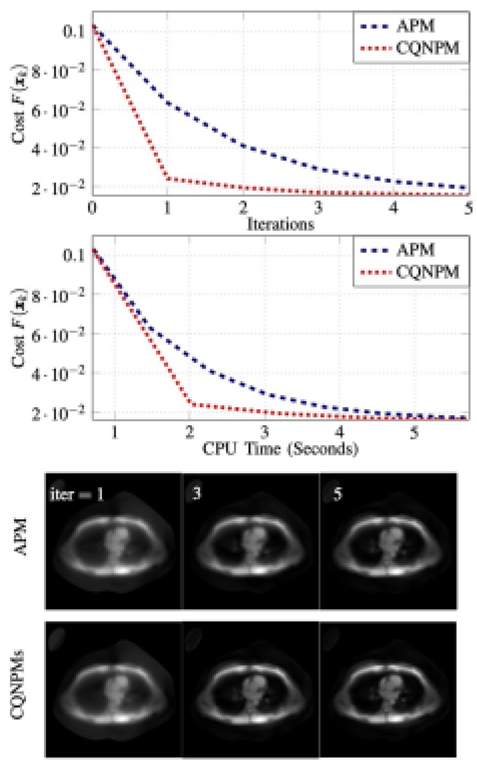

Figure 3 Same data as Figure 1 but with TV regularizer $h(x) = TV(x)$.

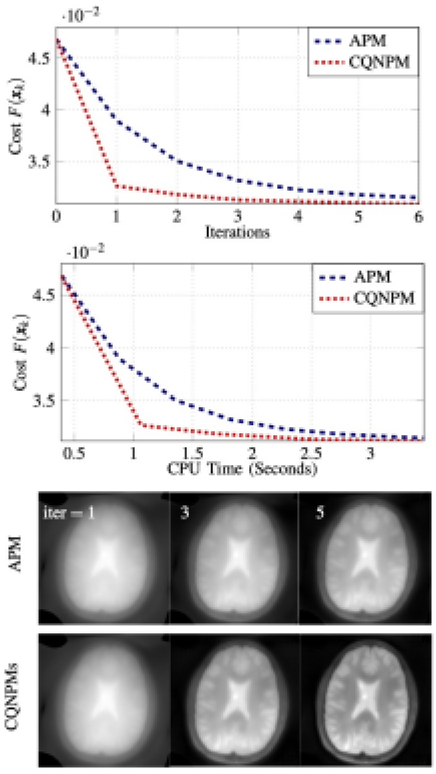

Figure 4 Same data as Figure 2 but with TV regularizer $\mathrm{h(x)} = \mathrm{TV(x)}$.