Fast, Iterative Subsampled Spiral Reconstruction via Circulant Majorizers

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Synopsis

Majorize-minimize algorithms are a powerful tool for solving image reconstruction problems with sparsity-promoting regularization; however, when non-Cartesian trajectories are used it becomes challenging to design a suitable majorizer for these methods due to the high density of samples near the center of k-space. We derive a new circulant majorizer that is related to the density compensation function of the k-space trajectory. We then use the frequency localization properties of wavelets and the circulant majorizer to design an algorithm that converges faster than conventional FISTA for reconstructing images from undersampled, non-Cartesian k-space data.

Purpose

To accelerate iterative, non-Cartesian image reconstruction with a circulant majorizer designed based on the k-space trajectory.

Methods

MRI scans can be accelerated by using non-Cartesian trajectories. The scans can be further accelerated by subsampling k-space and reconstructing images by solving an optimization problem of the form,[1]

$$\hat{x} = \operatorname*{argmin}_{x} \frac{1}{2}\|y - Ax\|_2^2 + \beta\|Rx\|_1,$$

where A is the system matrix incorporating the non-Cartesian trajectory, x is the object being reconstructed, R is a sparsity-promoting transform (e.g., undecimated Haar wavelets), and β is a regularization parameter. Majorize-minimize algorithms require upper bounding $A'A$ of the data fit term with a majorizing matrix, M_f (i.e., $M_f \succeq A'A$ where $N \succeq 0$ implies that N is positive semidefinite). The standard approach is to use $M_f = LI$, where L is the maximum eigenvalue of $A'A$.[2] This is a loose bound for $A'A$ due to the high density of samples near the center of k-space. Previous methods accelerated convergence by running power iterations to get step sizes for each frequency subband of the wavelet transform,[3] but these methods used orthogonal wavelets and a moving target cost function with weaker convergence guarantees.

We derive a generalization of the previous work (BARISTA/FISTA[2,4]) for non-Cartesian sampling by calculating a circulant majorizing matrix, M_f. We assume that we are working in the SENSE setting (i.e., $A = FS$ where F is the non-Cartesian Fourier encoding operator and S is a matrix with sensitivity maps) where R is an undecimated wavelet transform. We first majorize $F'F$ with a circulant matrix, M_f, by computing $D_f = \operatorname{diag}(QFe_0)$, where Q is a DFT matrix and e_0 is an impulse. We then increase the minimum entry in D_f until $M_f = Q^{-1}D_fQ \succeq F'F$ as determined via power iteration.[5] We found that this was also a majorizing matrix for $A'A$ in numerical experiments when the sensitivity maps are calculated with a square root sum of squares normalization. Then, we design a diagonal matrix, D_R, such that $D_R \succeq RM_f^{-1}R^T$. D_R governs the algorithm step sizes in the regularizer domain. Note that $R^T = [R_1^T \ldots R_B^T]$ for an undecimated wavelet transform with B subbands. For each subband, we have $R_b = Q^{-1}\Lambda_b Q$, where Λ_b is the frequency response for the filter in the bth subband. Inserting this into $RM_f^{-1}R^T$ reveals that we can apply Gershgorin's Theorem to build $D_R = \operatorname{diag}(d_{R,b}I_b)$, where I_b is an identity matrix of the size corresponding to the bth subband and $d_{R,b} = \max_n(\sum_{p=1}^{B} |\lambda_{n,b}\lambda_{n,p}^*/d_{n,f}|)$, where $\lambda_{n,b}$ is the nth entry in Λ_b and $d_{n,f}$ is the nth entry in D_f. Within the BARISTA[4] analysis algorithm (a FISTA variant), this gives the analysis denoising steps of

$$q^{(j+1)} = P_{P^M}(v^{(j)} - \beta^{-1}D_R^{-1}Rx^{(k,j+1)}),$$

$$x^{(k,j+1)} = c^{(k)} - \beta M_f^{-1}R^T v^{(j)},$$

$$c^{(k)} = x^{(k)} - M_f^{-1} A'(Ax^{(k)} - y).$$

We compared the speed of this new algorithm to that of FISTA with the Lipschitz constant in numerical experiments. The experiments simulated an interleaved spiral acquisition with 48 leaves for a 128 by 128 image and 8 receive coils. 12 of the 48 leaves were used for reconstruction. The regularization parameter was chosen manually.

Results

We calculated a "converged" solution, $x^{(\infty)}$, by running many thousands of iterations of FISTA. Figure 1 shows the converged image ($x^{(\infty)}$), while Figures 2 and 3 show $x^{(60)}$ of FISTA and the proposed method, respectively. The proposed method has more substantially reduced subsampling artifacts compared to FISTA after 60 iterations. We plot $\xi(k) = \frac{\|x^{(k)} - x^{(\infty)}\|}{\|x^{(\infty)}\|}$, the norm residual to convergence vs. time in Figure 4 to compare the convergence speed of the algorithms. The proposed method has about a 2-3 factor increase in convergence speed over FISTA.

Conclusion

We have shown accelerated convergence by upper bounding $A'A$ in the non-Cartesian, SENSE parallel MRI setting a circulant majorizer. The circulant majorizer has a frequency response that is related to the density compensation function of the k-space trajectory. We further showed that this property can be exploited in the setting where the regularizing matrix, R, is an undecimated Haar wavelet transform. This gives different step sizes for each frequency subband of the Haar wavelet transform, and the algorithm can be extended to any case where R can be decomposed into B filters. In the future we plan to examine strategies for creating smaller M_f matrices that more tightly bound $A'A$.

Acknowledgements

References

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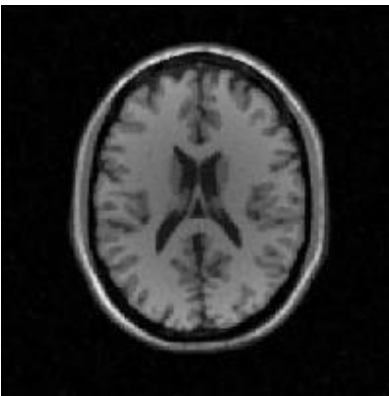
Figures

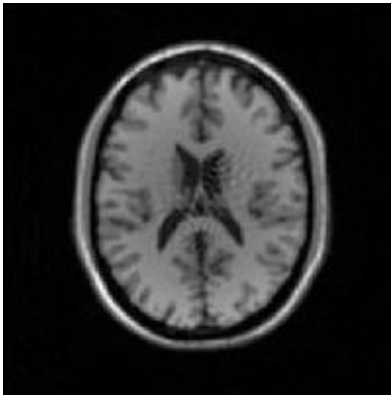Figure 1: $x^{(\infty)}$, the converged solution of the optimization problem.

Figure 2: FISTA solution after 60 iterations, showing spiral subsampling artifacts near the lateral ventricles.
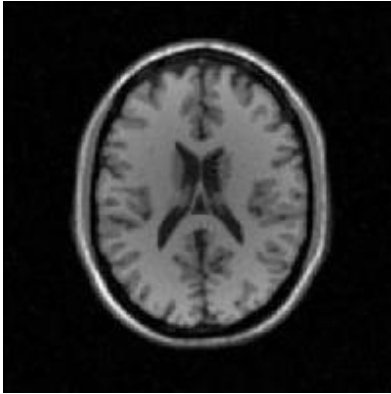

Figure 3: The proposed method solution after 60 iterations, showing reduced spiral subsampling artifacts.
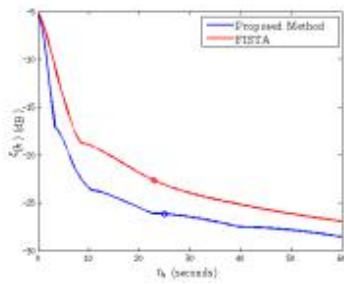

Figure 4: Convergence plot comparing the two methods. The proposed method is about 2-3 times faster than FISTA. The markers indicate the time points at which the images in Figures 2 and 3 were taken.