Homework #4, EECS 755, W13. Due **Wed. Nov. 13** by 5:00 PM

Solve #1 and at least two others. Problems labeled "(Solve?)" are ones where I have not yet worked out a solution.

Image restoration revisited

1. [10] Use circulant end conditions to synthesize a gaussian random field image like that in Fig. 1.7.1, for the 2D finite differencing matrix $\boldsymbol{C}$ defined in (1.10.8).

   Hint. The goal is to draw $\boldsymbol{x} \sim \mathsf{N}(\mathbf{0}, \boldsymbol{K}_{\boldsymbol{x}})$, so let $\boldsymbol{x} = \boldsymbol{K}_{\boldsymbol{x}}^{1/2} \boldsymbol{w}$ where $\boldsymbol{w} \sim \mathsf{N}(\mathbf{0}, \boldsymbol{I})$. In this case, $\boldsymbol{K}_{\boldsymbol{x}} = [\boldsymbol{C}'\boldsymbol{C}]^{-1}$. Because $\boldsymbol{K}_{\boldsymbol{x}}$ is not invertible, use its pseudo inverse. Do *not* use the `pinv` command; use FFTs.

Optimization transfer / majorization

2. [10] From the perspective of algorithm development, is condition (12.1.3) more general, less general, or equivalent to condition (12.1.2)?

3. [10] Analyze the asymptotic convergence rate of a version of the steepest descent algorithm that uses just one subiteration of the line-search method described in §12.5.6. (Solve?)

4. [10] As described in §2.4.2, generalize the algorithm derived in Example 12.5.1 to a 2D problem with a rotationally-invariant penalty function of the form (2.4.3). Hint: use (12.4.15). (Using optimization transfer, this generalization is much simpler than in the half-quadratic approach [1].)

5. [10] Generalize the iterative soft-thresholding algorithm of Example 12.5.2 to the case of non-quadratic data-fit terms

   $$\Psi(\boldsymbol{x}) = \sum_{i=1}^{n_{\mathrm{d}}} \mathsf{h}_i([\boldsymbol{A}\boldsymbol{x}]_i) + \beta \, \|\boldsymbol{U}'\boldsymbol{x}\|_1 .$$

   Make appropriate reasonable assumptions about the h_i functions.

6. [10] Apply the IST algorithm of Example 12.5.2 to a 2D image restoration problem with shift-invariant blur. Hint. One can modify `mri_cs_ist_example.m` to use `Gblur` instead of `Gdft`.

7. [10]

   The standard *iterative soft thresholding* (*IST*) algorithm described in §12.5.7.3 is not applicable to a standard *total variation* type of regularizer $\|\boldsymbol{C}\boldsymbol{x}\|_1$ because the usual choices for $\boldsymbol{C}$ like $\boldsymbol{D}_N$ in (1.8.4) are not invertible. However if we add one more row to $\boldsymbol{D}_N$ as follows: $\boldsymbol{C} = \begin{bmatrix} 1 \; 0 \; \ldots \; 0 \\ \boldsymbol{D}_N \end{bmatrix}$ then this matrix is invertible and its inverse is lower-triangular. Develop a IST algorithm for 1D problems with this regularizer. Does the principle generalize to 2D?

[1] G. Aubert and L. Vese. A variational method in image recovery. *SIAM J. Numer. Anal.*, 34(5):1948–97, October 1997.