Homework #2, EECS 755, W13. Due **Wed. Oct. 9** by 5:00 PM

Hand in solutions to at least 3 of the first 7 problems and do problem 8.

Regularization

1. [10] Often it is assumed that the constrained minimization problem

$$\hat{\boldsymbol{x}}_k \triangleq \arg\min_{\boldsymbol{x}} \mathsf{Ł}(\boldsymbol{x}) \text{ sub. to } \mathsf{R}(\boldsymbol{x}) \leq k \tag{E1}$$

is equivalent, for some choice of regularization parameter β, to the following regularized problem:

$$\hat{\boldsymbol{x}}(\beta) \triangleq \arg\min_{\boldsymbol{x}} \mathsf{Ł}(\boldsymbol{x}) + \beta\, \mathsf{R}(\boldsymbol{x}). \tag{E2}$$

Consider the Poisson denoising problem where $\boldsymbol{y} \sim \mathsf{Poisson}\{\boldsymbol{x}+\boldsymbol{r}\}$, where $\boldsymbol{r}$ is a known nonnegative vector, with *counting measure* regularizer $\mathsf{R}(\boldsymbol{x}) = \|\boldsymbol{x}\|_0$. Find *analytical* solutions to $\hat{\boldsymbol{x}}_k$ and $\hat{\boldsymbol{x}}(\beta)$ above and determine if they are equal for some choices of β and k [1, 2].

2. [10] Use 2D FT properties to prove that the thin-plate regularizer (2.4.2) is rotation invariant.

3. [10] Use (2.5.6) to describe how to determine the value of β that minimizes the *worst case* MSE over all signals with $\|\boldsymbol{x}\| \leq 1$. This is a *min-max regularization parameter* selection method.

4. [10] Choose an image $\boldsymbol{x}_{\text{true}}$ and a shift-invariant blur $b[n,m]$ with circulant end conditions and create a noisy, blurry image $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}$. Apply the image restoration method of Example 2.5.1 with quadratic regularization based on 1st-order finite differences for a range of values of β. Plot MSE_β and locate β_{MSE}. Plot at least one of $|\mathsf{RSS}(\hat{\boldsymbol{x}}_\beta) - n_{\mathrm{d}}|$ or $|\mathsf{RSS}(\hat{\boldsymbol{x}}_\beta) - \mathsf{REDF}(\beta)|$ or $\Phi_{\text{CV}}(\beta)$ or $\Phi_{\text{GCV}}(\beta)$ and indicate the corresponding “optimized” β values to compare to β_{MSE}. Examine the restored images $\hat{\boldsymbol{x}}_\beta$ at β_{MSE} and the optimized value of β select by the criterion you chose. Hint: no iterations are needed; this can be done using FFT operations.

5. [10] Extend Problem 1.12 to the case of the generalized Fair potential in §2.7.4.

6. [10] This problem generalizes (2.8.2) and outlines the derivation of (2.8.3). (It also relates to certain *half quadratic* methods in the literature.) Let ψ be any differentiable, symmetric potential function for which (see Theorem 12.5.5) the *potential weighting function* $\omega_\psi(t) = \dot{\psi}(t)/t$ is finite at $t=0$ and monotone decreasing for $t > 0$. Let $g(l) \triangleq \omega_\psi^{-1}(l)$ denote the inverse of ω_ψ and, motivated by (12.5.15), define the function

$$u(l) = \psi(g(l)) - \frac{1}{2} l g^2(l). \tag{E3}$$

Show that minimizing (2.8.2) over l_k yields $l_k = \omega_\psi\left(\sqrt{\sum_{m=1}^M \left|\left[\boldsymbol{C}\boldsymbol{x}_m^{(n)}\right]_k\right|^2}\right)$. Determine which potential function ψ corresponds to (E3).

7. [10] Consider a trapezoid defined by $f(x) = \begin{cases} h, & |x| < a \\ h\left(1 - \frac{|x|-a}{b-a}\right), & a \leq |x| < b \\ 0, & \text{otherwise,} \end{cases}$ for $0 \leq a < b$ and $h > 0$. Solve the optimization problem $\min_{a,b,h} \mathrm{TV}(f)$ subject to $\int f(x)\,\mathrm{d}x = 1$ and $f(x_0) = 0$ for a given $x_0 > 0$.

8. [0] Please do the mid-term course evaluation online. Your feedback is very important to me. Thanks.

[1] D. J. Lingenfelter, J. A. Fessler, and Z. He. Sparsity regularization for image reconstruction with Poisson data. In *Proc. SPIE 7246 Computational Imaging VII*, page 72460F, 2009.

[2] M. Nikolova. Description of the minimizers of least squares regularized with ℓ_0-norm. Uniqueness of the global minimizer, 2013.