Homework #1, EECS 755, W13. Due **Wed. Sep. 25** by 5:00 PM

Hand in solutions to any 5 of the following problems.

Image restoration

1. [10]

An image restoration method uses object model (1.3.6) with a 2D rectangular basis function: $\beta_0(x,y) = \mathrm{rect}_2(x/\triangle_X, y/\triangle_Y)$. The sensor is shift invariant with a rectangular blur: $b(x,y) = \frac{1}{4\triangle_X\triangle_Y}\mathrm{rect}_2\left(\frac{x}{2\triangle_X}, \frac{y}{2\triangle_Y}\right)$. Assuming zero end conditions, determine the values of elements a_{ij} of the system matrix $\boldsymbol{A}$. Assume both the spacing of the sensor elements and the spacing of the object basis functions are $(\triangle_X, \triangle_Y)$.

2. [10] If a matrix $\boldsymbol{M}$ is square and circulant, then computing $\boldsymbol{Q}\boldsymbol{M}\boldsymbol{Q}^{-1}$ will yield an exactly diagonal matrix, where $\boldsymbol{Q}$ is the DFT matrix defined in (1.4.27). Consider the following four representations of a system matrix $\boldsymbol{A}$: (1.4.7), (1.4.9), (1.4.10), and (1.4.13). For each representation, using MATLAB to compute $\boldsymbol{D} = \boldsymbol{Q}\boldsymbol{A}'\boldsymbol{A}\boldsymbol{Q}^{-1}$ for the impulse response $b[n] = \delta[n-1] + 2\,\delta[n] + \delta[n+1]$ and for $N = 64$. (Hint: you can create each of the $\boldsymbol{A}$ matrices needed in one or two lines of MATLAB using `convmtx`.) Display for yourself the $\boldsymbol{D}$ matrices to visualize how close to diagonal they are. Compute the fractional off-diagonal "energy" as follows:

$$\frac{\sqrt{\sum_{k\neq j} |d_{kj}|^2}}{\sqrt{\sum_{k,j} |d_{kj}|^2}}.$$

Compare the four models using this quantitative measure of "non-circulant-ness."

3. [10] For any $n_p \times n_p$ unitary matrix $\boldsymbol{U}$, consider the penalized-likelihood cost function

$$\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{U}\boldsymbol{x}\|^2 + \beta \sum_{j=1}^{n_p} \psi(x_j).$$

- Defining $\tilde{\boldsymbol{y}} = \boldsymbol{U}'\boldsymbol{y}$, and using the fact that

$$\|\boldsymbol{y} - \boldsymbol{U}\boldsymbol{x}\|^2 = \|\boldsymbol{U}'\boldsymbol{y} - \boldsymbol{x}\|^2 = \sum_{j=1}^{n_p} |\tilde{y}_j - x_j|^2,$$

  find an *analytical* solution for $\hat{\boldsymbol{x}}$ in terms of $\tilde{\boldsymbol{y}}$, for the l_1 regularized case where $\psi(t) = |t|$. Sketch $\hat{x}_j$ vs $\tilde{y}_j$.
- Compare to the solution when an l_2 penalty is used where $\psi(t) = \frac{1}{2}|t|^2$.
  Repeat for at least one more of the following potential functions.
- l_0 potential: $\psi(t) = \mathbb{1}_{\{t\neq 0\}}$.
- The truncated absolute value potential: $\psi(t) = \min(|t|, \delta)$.
- The broken parabola potential: $\psi(t) = \min\left(\frac{1}{2}|t|^2, \frac{1}{2}\delta^2\right)$.
- Huber potential (1.10.9).
- Generalized-gaussian potential (challenging!): $\psi(t) = |t|^p$, for $p \neq 1$. (Focus on $p \in \{1/2, 4/3, 3/2, 2, 3, 4\}$ [2].)
- The hyperbola potential (challenging!): $\psi(t) = \delta^2(\sqrt{1 + |t/\delta|^2} - 1)$.

This problem relates to wavelet-based *denoising* using *shrinkage* [3] and *soft thresholding* [4, 5].

4. [10] Let $\boldsymbol{D}_N$ denote the $(N-1)\times N$ one-dimensional finite-differencing matrix shown in (1.8.4), and $\boldsymbol{I}_N$ denote the $N \times N$ identity matrix. Show that the simple quadratic penalty (1.10.1) that uses only horizontal and vertical differences can be written in the form (1.10.7), where $\boldsymbol{C}$ is the following $[M(N-1) + N(M-1)] \times NM$ matrix:

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_M \otimes \boldsymbol{D}_N \\ \boldsymbol{D}_M \otimes \boldsymbol{I}_N \end{bmatrix}, \qquad \text{(E1)}$$

and "$\otimes$" denotes the *Kronecker product* defined in (25.1.12).

5. [10]

For regularized restoration of a $N \times M$ image using a penalty function $\mathsf{R}(\boldsymbol{x}) = \frac{1}{2} \|\boldsymbol{C}\boldsymbol{x}\|^2$, one option is to use

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{D}_{NM} \\ \boldsymbol{T} \end{bmatrix}, \tag{E2}$$

where $\boldsymbol{T}$ is an $N(M-1) \times NM$ Toeplitz matrix with first row $[-1 \; \mathbf{0}'_{N-1} \; 1 \; \mathbf{0}'_{NM-N-1}]$, where $\mathbf{0}'_N$ denotes the row vector of N zeros. Another option is to use $\boldsymbol{C}$ defined in (E1). Using (E2) may be slightly faster (in ANSI C). Explain the advantage of using (E1).

6. [10] The 1D regularizer Hessian matrix in (1.8.6) has eigenvalues given in footnote 12.

Consider the 2D regularizer $\boldsymbol{C}$ for a $N \times M$ image given in (E1), and define the Hessian matrix $\boldsymbol{R} = \boldsymbol{C}'\boldsymbol{C}$. Determine analytically the eigenvalues of $\boldsymbol{R}$.

7. [10] This problem considers whether the penalized least-squares cost function $\Psi(\boldsymbol{x})$ in (1.10.11) has a *unique minimizer* in the usual cases where $\boldsymbol{A}$ and $\boldsymbol{C}$ have *disjoint null spaces*.

- Prove that if the potential function ψ used in (1.10.11) is twice differentiable with a positive second derivative, then Ψ is strictly convex (and thus has a unique minimizer).
- What if ψ is strictly convex, but does not necessarily have a positive second derivative? An example would be $\psi(t) = t^4$.
- What if ψ is merely convex, like the Huber function? Hint: see Fig. 1.

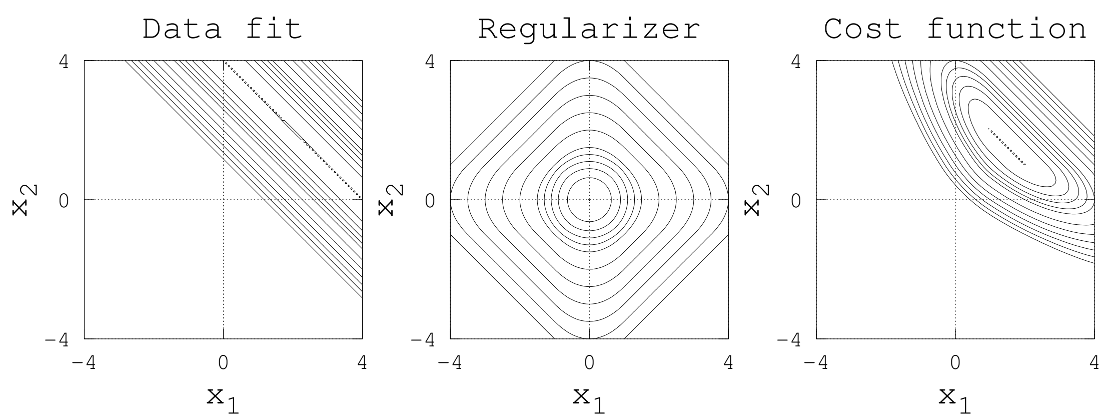

Figure 1: Contours of data fit term, regularizer, and cost function $\Psi(\boldsymbol{x})$ for Problem 1.25.

8. [10]

Consider the regularized least-squares problem (1.10.11) with regularizer (1.10.10) and the usual 1st-order finite differencing matrix $\boldsymbol{C}$.

- Ken uses the generalized-gaussian potential function $\psi(t) = |t|^q$ with $q = 1.5$, and states that the solution $\hat{\boldsymbol{x}}$ satisfies the recursive expression (1.10.15). Discuss.
- Maria uses the Geman & McClure potential function $\psi(t) = t^2/(1+t^2)$ and also states that the solution $\hat{\boldsymbol{x}}$ satisfies the recursive expression (1.10.15). Discuss.

9. [10] Use circulant end conditions to synthesize a gaussian random field image like that in Fig. 1.7.1, for the 2D finite differencing matrix $\boldsymbol{C}$ defined in (1.10.8).

Hint. The goal is to draw $\boldsymbol{x} \sim \mathsf{N}(\mathbf{0}, \boldsymbol{K}_{\boldsymbol{x}})$, so let $\boldsymbol{x} = \boldsymbol{K}_{\boldsymbol{x}}^{1/2}\boldsymbol{w}$ where $\boldsymbol{w} \sim \mathsf{N}(\mathbf{0}, \boldsymbol{I})$. In this case, $\boldsymbol{K}_{\boldsymbol{x}} = [\boldsymbol{C}'\boldsymbol{C}]^{-1}$. Because $\boldsymbol{K}_{\boldsymbol{x}}$ is not invertible, use its pseudo inverse.

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10. [10] Consider the discrete-space denoising problem with no boundary conditions and zero-mean white noise:

$$g[n] = f[n] + \varepsilon[n], \quad n \in \mathbb{Z}, .$$

Analyze the spatial resolution properties of the following quadratically-regularized denoising estimator for $\beta > 0$:

$$\hat{f} = \underset{f \in \ell_2}{\arg\min} \sum_{n=-\infty}^{\infty} \frac{1}{2} |g[n] - f[n]|^2 + \beta \sum_{n=-\infty}^{\infty} \frac{1}{2} |f[n] - f[n-1]|^2 .$$

Hint. Using the DTFT, first find the frequency-domain relationship between $\mathsf{E}\left[\hat{f}\right]$ and f.

Optional: show that $\mathsf{E}\left[\hat{f}\right] = h * f$, where the impulse response is

$$h[n] = ab^{|n|} = a\, \mathrm{e}^{-|\log b||n|}, \quad b = \frac{1 + 2\beta - \sqrt{1 + 4\beta}}{2\beta} = \frac{2\beta}{1 + 2\beta + \sqrt{1 + 4\beta}},$$

where $a = \frac{1+b^2}{1-b^2} \frac{1}{1+2\beta}$. Note that $0 < b < 1$. Use §26.6.1. This is one of the few cases where we can find an explicit expression for the impulse response of a regularized problem [6, 7].
Determine the FWHM of the impulse response in terms of b.

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11. [10] Find a matrix $\boldsymbol{C}$ such that when $f(t) = \sum_{j=1}^{n_{\mathrm{p}}} x_j \operatorname{tri}(t - j)$, we get equivalent values for the following continuous-space and discrete-space roughness penalty functions:

$$\int \left|\dot{f}\right|^2 \mathrm{d}t = \|\boldsymbol{C}\boldsymbol{x}\|^2 .$$

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