Homework #8, EECS 556, W21. Due **Thu. Apr. 01**, by 9:00AM

This HW is optional for 498-556 students.

Skills and Concepts

- random processes
- image denoising / Wiener filter
- image restoration

Problems

1. [20] Work with your team to design at least one exam question. This should be a problem-solving question, not just a multiple choice or true/false question.
There is a template in the `exam-template` folder on google drive where you get the annotated course notes. In that folder is a file `main.tex` that you should copy but *not* modify, and another file `body.tex` that you must edit.
Apply LaTeX to `main.tex` (*e.g.*, using overleaf.com) to verify that your `body.tex` compiles correctly.
Prof. Fessler will grade your question holistically, considering the following criteria:
   - originality (not a trivial variation of an existing HW problem)
   - clarity of problem statement
   - correctness of solution
   - depth of understanding (*e.g.*, integrates ideas from more than one chapter)
   - suitability (*i.e.*, not too trivial, but not requiring tedious integration)
   - scope (based on the non-optional 556 material),
   - proper LaTeX formatting.

   Problems that involve some JULIA/MATLAB computations are certainly welcome.
One member of your team should upload your `body.tex` file to Canvas.
Optional: have a second member of your team submit a second question. Your team will earn the *maximum* of the scores of the two problems you submit.
This problem will be graded separately from the rest of this HW and earn up to 20 points, *i.e.*, comparable to a regular HW set, so it is worth designing carefully. (You do not need to submit this one to gradescope.)

2. [10] The autocorrelation function $\mathsf{R}_x[m,n]$ of a zero-mean WSS random process is thought to be given by the expression:

$$\mathsf{R}_x[m,n] = \delta_2[m,n] + \alpha\, \delta_2[m-1,n] + \alpha\, \delta_2[m+1,n] + \alpha\, \delta_2[m,n-1] + \alpha\, \delta_2[m,n+1] \,.$$

   For what values of α is this a valid autocorrelation function?

3. [15] A WSS random field $x[m,n]$ has autocorrelation function $\mathsf{R}_x[m,n]$. This field is downsampled as follows:

$$y[m,n] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} x[3m-k, 3n-l] \,.$$

   (a) [5] Determine whether $y[m,n]$ is a WSS random process. Explain.

   (b) [10] If it is, express $\mathsf{R}_y[m,n]$ in terms of $\mathsf{R}_x[m,n]$.

4. [20] Consider the combined blur plus noise model $g[m, n] = b[m, n] ** f[m, n] + v[m, n]$ where $b[m, n]$ is a known blur PSF, $f[m, n]$ and $v[m, n]$ are a zero-mean WSS fields with autocorrelations $\mathsf{R}_f[m, n]$ and $\mathsf{R}_v[m, n]$ respectively, and that $f[m, n]$ and $v[m, n]$ are uncorrelated. Derive the optimal linear estimator $\hat{f}[m, n] = h[m, n] ** g[m, n]$, where optimal here means minimum mean-squared error. Instead of applying the orthogonality principle, use the first-principles Wiener derivation in the notes. You may start after any suitable point in that derivation.

5. [15] An image $f[m, n]$ is degraded by two additive noise processes as follows:

$$\begin{array}{ccccc} f[m,n] \to & \bigoplus & \xrightarrow{x[m,n]} & \bigoplus & \to g[m,n]\,. \\ & \uparrow & & \uparrow & \\ & v[m,n] & & w[m,n] & \end{array}$$

We wish to estimate $f[m, n]$ from $g[m, n]$ by a linear method that minimizes MSE, assuming that $f[m, n]$, $v[m, n]$, and $w[m, n]$ are WSS zero-mean random processes, independent of each other, with known power spectra.

(a) [5] One approach is to use the model $g[m, n] = f[m, n] + u[m, n]$, where $u[m, n] = v[m, n] + w[m, n]$, and then design on LSI filter $h[m, n]$ for estimating $f[m, n]$ from $g[m, n]$ directly. Determine $H(\Omega_1, \Omega_2)$.

(b) [10] An alternative approach is to first estimate $x[m, n] = f[m, n] + v[m, n]$ from $g[m, n]$ and then estimate $f[m, n]$ from $\hat{x}[m, n]$, as follows:

$$g[m,n] \to \boxed{H_1(\Omega_1, \Omega_2)} \xrightarrow{\hat{x}[m,n]} \boxed{H_2(\Omega_1, \Omega_2)} \to \hat{f}[m,n]\,.$$

Here we first design $h_1[m, n]$ by minimizing the MSE between $\hat{x}[m, n]$ and $x[m, n]$. We then design $h_2[m, n]$ by pretending that $\hat{x}[m, n]$ is the true $x[m, n]$, and then choosing $h_2[m, n]$ to minimize the MSE between $\hat{f}[m, n]$ and $f[m, n]$. Determine $H_1(\Omega_1, \Omega_2)$ and $H_2(\Omega_1, \Omega_2)$.

(c) [0] Compare the properties of the final estimated signal $\hat{f}[m, n]$ in each of the two cases.

6. [10] Show that the "filtered Bernoulli point process" signal $z[m, n]$, defined in the section "2nd-order statistic - an incomplete story" in the lecture notes, is WSS, and derive its autocorrelation function.

7. [10] Determine the power spectrum of the "filtered Bernoulli point process" signal $z[m, n]$, defined in the section "2nd-order statistics - an incomplete story" section in the lecture notes, on p. 11.33. Hint. The answer does *not* involve $\mathrm{sinc}^2(\cdot)$.

8. [80] The purpose of this problem is to give you both theoretical and practical experience with Wiener filters. It also integrates many tools from previous chapters (convolution, DSFT, FFT, etc.).

(a) [5] From the web page, download `wiener1_template.jl` template.
(There is also a MATLAB template for 498-556 students who want to try this.)
Run the file and look at the image displayed. Analyze the code and determine the *theoretical* autocorrelation function of the random process $f[m,n]$ that is first generated by this code.

(b) [10] Determine analytically the power spectrum of that random process. Hint. Use a previous HW problem.

(c) [5] Compute the *empirical* autocorrelation function of $f[m,n]$. Display both the empirical estimate and the theoretical autocorrelation function of $f[m,n]$.

(d) [10] Examine the code to understand the signal $g[m,n]$ generated therein. Determine analytically the frequency response $H(\Omega_1,\Omega_2)$ of the Wiener filter that is the minimum MSE linear estimator of $f[m,n]$ given $g[m,n]$. Display it using `jim`, either as a DSFT $H(\Omega_1,\Omega_2)$ or samples like $H[k,l]$.

(e) [30] Apply the Wiener filter to $g[m,n]$ using FFT's (and the *ideal* autocorrelation functions) and display the filtered image $\hat{f}[m,n]$.
Hint. Because power spectra are real, your Wiener filter should be real too.

(f) [5] Download the `npls_sps.jl` from the web site, which is an alternative denoising method called non-quadratically penalized least squares (NPLS). Apply this method to the noisy data $g[m,n]$ by a command like `f_hat = npls_sps(gmn)`. Display the denoised result.

(g) [10] Compare the Wiener filter estimate to the NPLS estimate both qualitatively and quantitatively by computing the (empirical) RMSE. In terms of the RMSE, do the results you get contradict the optimality of the Wiener filter? Explain.

(h) [5] You may have found the Wiener filter results to be disappointing relative the example in the lecture notes. What differences between the two cases explain the differences in performance?

My JULIA solution to this problem has more lines of code for plotting than for computation and uses quite a few packages including `OffsetArrays, FFTW, DSP, Statistics`. Using `OffsetArrays` simplified the code a lot for me, in part because the latest version of `MIRTjim.jim` automatically displays the appropriate image axes based on the `axes` of the `OffsetArrays`. You are not required to use `OffsetArrays`, but it is worth learning. Try this code to see an example:

```
using MIRTjim:  jim
using OffsetArrays
x = OffsetArray(rand(9,6), -4:4, -3:2)
jim(x)
```

Some functions I found useful include:
`ImageFiltering.padarray`,
`ImageFiltering.Fill`,
`ImageFiltering.freqkernel`,
`ImageFiltering.imfilter`,
`ImageFiltering.reflect`,
`DSP.conv`,
`FFTW.fftshift`

There are multiple ways to solve the steps in this problem so you might not use all these functions.
Optional: compare with the (somewhat peculiar) `wiener` function in `DSP/Deconvolution.jl`

Optional problems

9. [0] Consider a signal + noise model $g[m,n] = f[m,n] + v[m,n]$ where $v[m,n]$ is signal-independent zero-mean white noise with variance σ_v^2. To reduce noise, we filter $g[m,n]$ with $h[m,n]$, producing image estimate $\hat{f}[m,n]$, *i.e.*:

$$\hat{f}[m,n] = f'[m,n] + v'[m,n],$$

where $f'[m,n]$ is the filtered original signal and $v'[m,n]$ is the filtered noise.

(a) [0] Determine $\mathsf{Var}\{v'[m,n]\}$.

(b) [0] Determine $P_{v'}(\Omega_1, \Omega_2)$.

(c) [0] Is $v'[m,n]$ white noise?