Homework #6, EECS 556, W21. Due **Thu. Mar. 11**, by 9:00AM

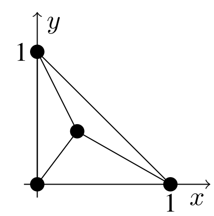
Skills and Concepts

- interpolation

Problems

1. [10] A modified version of **2D nearest-neighbor interpolation** works as follows. The interpolated value is taken from the nearest neighbor, unless there is a 2-way or 4-way tie for which neighbors are the nearest. For such ties, the interpolated value is the *average* of the corresponding samples. Convince yourself that this procedure is integer shift invariant, and then determine the corresponding **interpolation kernel** $h(x, y)$. Hint: your answer will likely need braces for cases. Is your interpolation kernel a **separable function**?

2. [20] The **bilinear interpolation** method uses the four nearest sample points to any point (x, y) of interest. An alternative 2D linear interpolation method is to interpolate by fitting a plane to the three nearest sample points. This process is equivalent to interpolation using a certain **interpolation kernel** $h(x, y)$.
   (a) [10] Determine that kernel; find a mathematical expression for it, and describe its shape in words. For simplicity, you may assume unit sample spacing. The purpose of this problem is to illustrate converting an interpolation procedure into an interpolation kernel. Hint: $h(x, y)$ is nonseparable but has several symmetry properties.
   (b) [5] The three nearest points define a triangle, and the line segments between a point (x, y) within that triangle and each of the corners subdivide it into three smaller triangles. An alternative approach is to weight the samples at the corner of the triangle by the relative area of the sub-triangle opposite of it. What is the interpolation kernel for this approach?



   (c) [5] Determine if these interpolation methods provide a 2D **partition of unity** of the form $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(x - m, y - m) = 1, \; \forall x, y \in \mathbb{R}$.

3. [25] Given samples $\{g_\mathrm{d}[n]\}$ of a signal $g_\mathrm{a}(x)$, we want to find a representation of the form $\hat{g}_\mathrm{a}(x) = \sum_{n=-\infty}^{\infty} c[n]\, b(x - n)$, such that $\hat{g}_\mathrm{a}(x)\Big|_{x=n} = g_\mathrm{d}[n]$, for a trapezoidal kernel where $b(x) = \begin{cases} 1, & |x| \le 1/4 \\ 5/4 - |x|, & 1/4 < |x| < 5/4 \\ 0, & \text{otherwise.} \end{cases}$
   (a) [10] Determine the equivalent frequency response $H(\nu)$ for this interpolation method.
   Hint. A trapezoid can be expressed as the difference of two triangle functions.
   (b) [10] Describe (with equations) how to compute efficiently the $c[n]$ coefficients from the samples $\{g_\mathrm{d}[n]\}$.
   (c) [5] What happens if instead of this trapezoid we used a triangle $b(x) = \mathrm{tri}(x/2)$?

4. [10]
   (a) [0] Determine analytically the equivalent frequency response $H^{(2)}(\nu)$ of the quadratic B-spline interpolator.
   (b) [10] Make a single plot of the frequency responses of the Dodgson interpolation kernel and the equivalent frequency response of the quadratic B-spline interpolator. Use a log scale from -80 to 5 dB for the vertical axis, and a linear scale from 0 to 2 cycles per unit-sample-distance on the horizontal axis. Overlay the ideal frequency response on this plot.
   (c) [0] Discuss which you think is better and why.

The following problem is optional for EECS 498 students.

5. [50] A radial MRI scan samples the frequency content ("k-space") of the imaged object on a polar grid. Angular samples are taken on the interval $[-\pi, \pi]$, and radial samples are taken on the interval $[0, k_{\max}]$, where $k_{\max}$ is the maximum frequency to sample. (Here $k_{\max} = \pi$ radians/sample.) To reconstruct the image, however, one cannot directly use an inverse **FFT** because the data do not lie on a Cartesian grid, so one must first convert the data from polar coordinates to Cartesian coordinates (a process called gridding). Some forms of radar imaging use similar processing.
Given radial samples of the frequency content (**DSFT**) of a brain image, reconstruct the image by first converting (**interpolating**) the polar data into Cartesian coordinates and then applying an inverse **FFT**.
Use the provided template code `gridding_template.jl` on Canvas as a starting point. The provided code reads in the true image, simulates a radial MRI scan using the nonuniform FFT (`nufft`), defines the Cartesian grid (`kx` and `ky`) onto which you should interpolate the radial data, and displays the true and reconstructed images and their frequency content. Use cubic B-spline interpolation; see `CubicSplineInterpolation`.
Points on the Cartesian grid that lie outside the radial sampling (*i.e.*, $\sqrt{k_X^2 + k_Y^2} > k_{\max}$) should be set to 0.
When you first run the template code, the NRMSE will be about 97%. After you provide the proper solution, the NRMSE should be below 2%.
You should submit your code and the figure generated using the template code (which will contain 6 images, including your interpolated frequency data, the reconstruction, and the error image with the NRMSE shown).
Optional: compare cubic B-spline interpolation to classical bilinear interpolation.
Optional: The template code has angular sampling on the interval $[-\pi, \pi]$. In a practical setting, however, the sampling would be on the interval $[-\pi, \pi)$ (to avoid sampling the same data twice). Modify the code to use this more realistic angular sampling. What other changes do you have to make to achieve low reconstruction error?
Hint: `atan` is likely to be useful.
All the "frequencies" here have units radians/sample. So think $k_X = \Omega_1$ and $k_Y = \Omega_2$.

Optional problems

6. [0] The lecture notes describe two quadratic **interpolation kernels**; one that has support $[-1, 1]$, and one proposed by Dodgson [1] that has support [-3/2,3/2].
   (a) Make a single plot of the **frequency responses** of these two kernels. Use a log scale from -80 to 5 dB of the magnitude response for the vertical axis, and a linear scale from 0 to 2 cycles per unit-sample-distance on the horizontal axis. Overlay the ideal frequency response on this plot.
   (b) Compare the advantages and disadvantages of the two choices with respect to all relevant criteria.

[1] N. A. Dodgson. "Quadratic interpolation for image resampling". In: *IEEE Trans. Im. Proc.* 6.9 (Sept. 1997), 1322–6.